MECHANICS

S. P. Strelkov

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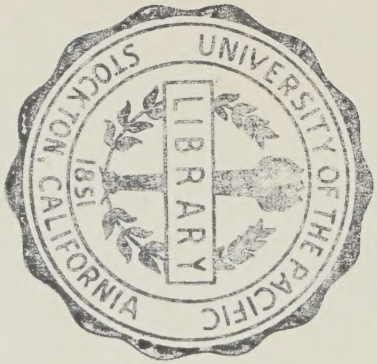
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S. P. Strelkov

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and I. G. VOLOSOVA

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Foreword to the Third Russian Edition

Mechanics by late Professor S. P. Strelkov is the first part of the general course in physics meant for students of Physical and Mathematical Departments of universities and pedagogical colleges.

The book is based on the material of the seminars and lectures given by the author for many years at the Physical Department and at the Professional Courses for college teachers of the Lomonosov State University of Moscow.

The first Russian edition of the book appeared in 1956. In the second edition (1965) the author perfected the course, extended the material of some sections related to the branches of engineering and science whose importance has increased in recent years (such as satellite and spacecraft flight, weightlessness, and motion with supersonic velocity) and introduced into the course the fundamentals of special relativity theory.

The author did not complete the third edition of the book but performed the necessary preparatory work. In accordance with the new extended school programmes in mathematics and physics, the third edition includes a number of additional theoretical questions. For instance, the tensor character of the moment of inertia and the basic tensors dealt with in mechanics of isotropic elastic body are discussed, the chapter devoted to the special relativity theory is revised and extended. However, the visual physical interpretation of the basic physical phenomena and laws characteristic of the previous editions has been retained.

In the preparation of the present edition of the course and its extension and further improvement much valuable work has been done by A. A. Kharlamov who collaborated with the author of the book for many years.

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Introduction

Physics is one of the fundamental natural sciences studying the laws of inanimate nature.

The universe consists of material bodies which are in permanent interaction and motion. All the observable natural phenomena are governed by certain laws. The study and the explanation of the laws governing the connections between various processes and phenomena are the fundamental aim of every branch of science. The analysis of the interaction of material bodies and of the laws of electromagnetic phenomena is the aim of physics.

It is rather difficult to give an exhaustive description of the phenomena studied in physics and to outline definitely its boundaries; in this connection we limit ourselves to noting that new scientific discoveries and new fields of practical applications of our knowledge permanently extend these boundaries. For instance, in recent years such new divisions as plasma physics, elementary particle physics, physics of semiconductors, biophysics, solid-state physics, etc. have been intensively developed. These new branches of science are partially dealt with in the general course of physics but the primary aim of the course is *to give the necessary prerequisites for studying the new divisions of modern physics*. The new divisions are basically treated in special courses and studied in laboratories after the study of the general course of physics. It is impossible to pass to the special courses without knowing the basic laws and phenomena studied in the general course of physics.

The general course of physics is usually subdivided into (1) mechanics, (2) molecular physics, (3) electricity and magnetism, (4) optics, and (5) atomic and nuclear physics.

The motion of the substance has various forms: mechanical, electromagnetic, thermal, etc. The mechanical form of motion of the substance is the simplest one: it is connected with displacements of various bodies relative to each other and with changes of the shape of the bodies. The laws of mechanical motion are studied in *mechanics* which is the initial part of the general course of physics. The other parts of physics cannot be studied without mechanics since various displacements are observed almost in all physical phenomena.

Mechanics is usually subdivided into three parts: *kinematics*, *statics*, and *dynamics*. In kinematics the motion of the bodies is considered irrelative to the factors causing the motion or changing its character. Statics deals with the laws of equilibrium of the bodies. Finally, dynamics studies the laws of motion and the causes producing the motion and changing it. On the other hand, if the laws of motion are known it is possible to derive the laws of equilibrium as

a special case of the former. Therefore usually in physics the laws of statics are not considered separately and are studied in connection with the general laws of dynamics.

Before proceeding to the study of mechanics as a part of the general course of physics we shall briefly discuss some questions concerning the subject and the methods of physics and also define some basic notions.

Physical Phenomenon. By a *physical phenomenon* (or a physical process) is meant a sequence of changes of definite bodies developing with time according to certain laws. All the changes of this kind studied in physics are estimated *qualitatively* by means of measurements.

Physical Experiment. The laws defining the connections between various changes of the bodies are studied by means of *observation* of phenomena taking place both in nature and in special *laboratories* with the aid of *experiments* providing the conditions in which these connections can be observed in the clearest and most distinct way. Laboratory and technical experiments together with the observation of natural phenomena are the basis of physics and provide the basic criterion for the correctness of our conclusions concerning natural processes or phenomena. This criterion lies in the coherence between the results of the scientific analysis and those of the experiment.

Physical Measurements and Physical Quantities. Physics belongs to the class of *exact sciences* in which the quantitative analysis of the processes plays the most important role. In physics we deal with various physical quantities, such as force, velocity, length, potential difference, and the like. *Physical quantities* define properties of bodies or characterize processes in which the changes involved should be described quantitatively, that is by means of measurements comparing a given quantity with a definite quantity of the same kind chosen as a *unit.*

A precise and correct measurement of physical quantities in the course of observations and experiments is the basic part of every scientific analysis in physics.

Physical Laws. All the phenomena and processes are in definite causal connections with each other. By means of observations and experiments a scientist discovers these connections and finds a definite causal interrelation between the changes of various quantities.

The general laws governing various processes are established on the basis of the analysis of the results of observations and experiments. It is these general rules that are called *physical laws* and provide the basis for the analysis of every concrete phenomenon.

Abstraction and Simplification. When analysing complex processes in which it is difficult to find and elucidate the basic causal connections and laws because of the presence of a number of additional connections and interrelations we first of all try to separate the basic phenomena and connections from the secondary ones. To decide what plays the most important role in a process in question and what is of secondary importance we rely on the results of check experiments. For instance, laboratory experiments show that the free fall of a steel ball in the air and the free fall in vacuum give the same results whence we conclude that the air friction affects the motion of the ball very slightly and therefore the free

fall of the ball in the air can be considered with a high accuracy as a uniformly accelerated motion under the action of gravity. The analysis of a phenomenon makes it possible to separate the basic factors and to drop the secondary ones; this leads to the construction of a conditional *scheme* (*model*) of the phenomena by means of scientific *abstraction*. By abstraction is meant a notion which describes only some definite basic properties of things or some definite basic characteristics of a process. Examples of this kind are, for instance, the notions of a material point, of a straight line, of a force applied at a point, of a fluid without viscosity etc.

When an abstract model is applied, which describes a real process approximately or represents only a certain characteristic of a phenomenon, one should always take into account the *limitedness* of every abstract scheme.

For instance, in mechanics when studying motions of various bodies we use the notion of a material point (a particle) but it is insufficient to say simply that a given body can be regarded as a particle because it is necessary to indicate the type of motion for which the body in question can be treated as a material point. For example, the annual motion of the Earth round the Sun can be considered as the motion of a particle while there are cases when the motion of a molecule cannot be treated as the motion of a material point. A material point is an abstract notion by means of which only some features of a given motion can be described; for instance, when considering the Earth as a material point we can describe correctly the orbital motion of the Earth but at the same time this abstraction provides no means for studying the rotation of the Earth about its axis.

If there is a discrepancy between the results of a theoretical investigation and those of an experiment it is necessary to subject to a thorough analysis the correctness and the admissibility of the simplifications lying in the foundation of the model we deal with. For instance, in many phenomena water can be treated as an imcompressible fluid because even under considerable variations of the pressure a given volume of water changes very slightly and therefore for small variations of the pressure the changes of the volume can be neglected; at the same time, in the investigation of the phenomenon of the propagation of acoustic waves in the water the compressibility of the latter should be taken into account although the variations of the pressure in the acoustic waves are not very large. Thus, when investigating a phenomenon one should always know exactly which features of the phenomenon are taken into account in the concrete choice of the model and to what extent is it allowable to use this abstraction.

In the choice of a model of a phenomenon it is necessary to estimate accurately which quantities are large and which are small. For instance, when saying that a certain quantity is very small we must indicate clearly in comparison with what quantity the former quantity is considered small; for example, if we deal with a free fall of a ball in a room the force of friction exerted by the air is small compared with the force of gravity.

It may turn out that in experiments and phenomena of one kind small variations of some quantities play an essential role while even considerable variations of the same quantities in experiments and phenomena of another kind do not affect their course. Only a thorough and repeated checking with the aid of various experiments and calculations may convince us that our assumptions are correct.

Experimental and Theoretical Investigations. Physics is an experimental science because the basic facts it deals with and the conclusions it comes to

are obtained as results of certain experiments. However, it would be impossible to subject to a detailed study the laws of physics without theoretical analysis whose fundamental means are *mathematical* methods.

In physics, when investigating a complex phenomenon, some schematic model is constructed which describes the basic facts and the sought-for laws of the phenomenon. In the construction of such a model admitting of the application of mathematical methods the general theoretical assumptions and the known facts concerning analogous phenomena are taken into account.

The inferences obtained on the basis of a theoretical investigation represented by some qualitative relations are then checked by measurements with the aid of observations and laboratory experiments. The results of such a comparison and checking confirm or disprove the adequacy of the model under consideration. Such a comparison can at least lead to a construction of a more perfect model which in its turn should be subjected to a mathematical investigation followed by an experimental check, etc.

Such a successive mutual influence of theoretical and experimental analyses is a continuous process leading to a more thorough knowledge of the laws of the phenomena in question. The history of the discoveries in physics gives many convincing examples of extremely productive collaboration of theoreticians and experimentalists.

In recent years investigation processes of this type have been considerably accelerated. It is characteristic of modern science that the methods of experimental studies are permanently perfected and that the effective means of applied mathematics provided by electronic computers are successfully used.

Length and Time. In physical phenomena we encounter various physical quantities among which *length* and *time* are of special importance and are dealt with in almost all the cases.

Length measures the extension of bodies and time is the measure of duration of processes and phenomena. The definition of these quantities is closely related to the philosophical notions of space and time. Materialism considers space and time to be the forms of the existence of substance without which there cannot be any substance and phenomena.

Every material body has certain sizes which are determined by such quantities as lengths, areas, and volumes known from geometry.

The fundamental quantity by means of which the spatial properties of the bodies are determined is the length of a line segment taken as *unit length*. In physics as unit length is taken the *metre* defined as the distance between two lines on a standard bar; area and volume are measured in square and cubic metres respectively.

Physical bodies move relative to each other or, in other words, their motion is connected with changes of mutual disposition of bodies or of various parts of a body (deformation).

Each motion involves at least two bodies and therefore to describe the motion we can choose one of the moving bodies and connect a *frame* (*system*) *of reference* with it. Any of the bodies can be chosen as such a body.

A frame of reference connected with a chosen body can, for instance, be thought of as a system of rectangular coordinates. The positions of all the points in space are then uniquely determined relative to three (imaginary) hard mutually perpendicular rectilinear bars rigidly connected with the body in question and passing through a known point called the *origin*. The lengths of the segments of the bars should be measured with the aid of a certain unit of length. Then

the position of every point is uniquely determined by *three* numbers called its *coordinates* which indicate the distances reckoned along the axes of the bars from the origin to the foots of the perpendiculars dropped from the given point to the axes of the bars.

In the present course we primarily deal with motions for which it is known from experiments that the properties of space are independent of the choice of the frame of reference and of the direction of motion. In other words, when passing from one reference system to another, we should only take into account the geometrical properties of the frames of reference while the physical properties of the bodies with which the frames of reference are connected can be neglected. However, as is known from more sophisticated investigations, such an approach to the notion of space is not always adequate: there are certain kinds of motion in some phenomena (their thorough analysis falls outside the limits of the present course) whose investigation is connected with the corresponding changes of some properties of space dependent on the properties of the material bodies involved (the theory of relativity).

Mechanical motions reduce to changes of the positions of the moving bodies in space with time. As was mentioned, time is the measure of the duration of processes. Therefore the time elapsed can be measured by means of the duration of a certain process (or phenomenon).

A measurement of time can be performed with the aid of the processes whose course is periodically repeated (watches and clocks). The uniformity and the periodicity of a process is established on the basis of experiments in which the durations of various processes are compared. The time interval corresponding to the duration of a certain process of this kind is chosen as a *unit of time*. In physics as a unit time the *second* is taken which is 1/86,400th of the mean solar day, the latter being the duration of one full revolution of the Earth around its axis.

Experiments show that in the investigation of motions of bodies with velocities very small relative to the speed of light (3×10^{10} cm/s) it can be assumed that time is independent of the properties of the bodies and of their motion (Newtonian *absolute* time). Under this assumption we can regard the course of time for various processes and phenomena as being uniform and independent of the character of the phenomena and of the properties of the bodies taking part in the phenomena. However, some special investigations indicate that the duration of a physical process considered in different moving systems of reference depends in fact on the relative motion of these systems. Consequently, there exists no common time for different systems of reference when the bodies involved move with arbitrary velocities (*Einstein's relativity theory*). But this difference is practically negligibly small for ordinary velocities of motion sufficiently small relative to the velocity of light.

We conclude this introductory discussion with mentioning that the concept of absolute time, which is invariable for different frames of reference, is an abstraction whose admissibility is confirmed by experiments in which the velocities of motion of frames of reference relative to each other are much smaller than the speed of light. As a measure of the error appearing in the application of this abstraction the quantity $\beta = \frac{v^2}{c^2}$ can serve, where v is the velocity of motion of the frame of reference and c is the velocity of light. The smaller the quantity β relative to unity, the more precise are the results of the application of the concept of absolute time whose course is invariable for moving reference systems.

Dimensions of Physical Quantities. Every physical quantity is determined on the basis of certain laws established in experiments. A numerical value of a quantity appears as the result of a measurement, that is of the comparison of the given quantity with a certain quantity of the same kind chosen as unit measure. Generally speaking, the choice of a unit measure can be made quite arbitrarily. It is quite possible to choose conditional unit measures (units) for all physical quantities independently of each other. However, there are some reasons why in physics we do not follow this way and choose arbitrarily only certain units for some basic quantities (*basic units*) while the units for all the other quantities (the *derived units*) are chosen to be dependent on the basic ones. It turns out that using the known physical laws we can find the dependence of derived units on the basic ones. This dependence becomes quite definite if we indicate in each case how the proportionality coefficients are chosen in the formulas expressing the corresponding physical laws. When determining the derived units we try to choose these coefficients in the simplest way.

In mechanics we choose as basic units the units of length and time, for instance, the *metre* and the *second* (1 m and 1 s). Besides, as a third additional basic unit the unit of mass is introduced, for instance, the *kilogram* or the *gram* (1 kg, 1 g).

There exist different *systems of units* each of which is determined by the choice of the basic units. At present in physics is preferably used the *International System of Units* (SI).

In this course we shall use the SI system and the system cgs.

Since there exist extensive technical literature and engineering handbooks in which the *technical system of units* is used we will in some cases (for reference purposes) give the values of physical quantities in the technical system as well.

All the derived physical units depend on the choice of the basic units. The formulas expressing the derived units in terms of the basic ones are called *dimension formulas.* Every quantity has a certain *dimension*, which makes it possible to determine the variation of the corresponding derived unit caused by that of the basic units. For instance, if time is denoted t and length is denoted l the dimension formula for the acceleration a will be

$$[a] = lt^{-2}$$

This formula implies that when the unit of length is increased n times the unit of acceleration is also increased n times and when the unit of time is increased m times the unit of acceleration decreases m^2 times. The dimension formulas for derived units are determined by the laws connecting the corresponding physical quantities.

Physics and Engineering. Physics is closely related with all natural sciences and, especially, with technical sciences. The laws of physics create the foundation for a number of technical sciences.

Discoveries and investigations in new divisions of physics lead to the appearance of new branches of engineering. For example, machine-building is based on the laws of mechanics, electrotechnics and radio engineering are based on the laws of electromagnetic phenomena. The achievements of physics are applied in technology to solving practical problems. For instance, at present there are industrial atomic power stations, that is there has appeared a new field of technology, atomic power energetics, the idea of which seemed fantastic even 30 years ago although the existence of enormous reserves of atomic energy was discovered

by physicists about 70 years ago. Investigations in the field of solid-state physics have led to the creation of rapidly advancing semiconductor technology which has raised to a much higher level radio engineering, communications engineering, high-speed electronic computer technology, etc. In its turn, the development of technical sciences contributes to the perfection of exploratory methods of physics; for example, observation radio aids in radio astronomy have provided new effective means for investigating astrophysical phenomena, the construction of powerful charged-particle accelerators has become possible only because of the high level of modern technology, etc. Space exploration is setting new physical problems and expanding our knowledge of the laws of nature.

Mathematics and Physics. Mathematics and physics (and other natural sciences) are closely interrelated in their development. Physics cannot be studied without the knowledge of mathematics because all the laws of physics are expressed by means of relations between numbers. Only the application of mathematical methods makes it possible to analyse complicated laws governing physical phenomena.

The development of mathematical methods is to a certain extent always directed to practical aims providing means for studying laws of nature. That is why even those parts of physics which are studied in the general course or belong to experimental physics are closely connected with the study of mathematics.

Historical Notes. Before proceeding to the study of mechanics it is advisable to get familiar with the basic facts of the history of that science. The development of mechanics is inseparably linked with the history of the human culture.

The Pyramids and other remnants of ancient structures show that the ancients were to a certain extent familiar with the basic laws of mechanical equilibrium without which such grand structures could not have been built. The Greek philosopher Aristotle (384-322 B.C.) summed up in his *Physics* the facts of mechanics known to the ancients; however the basic law he stated connecting force and motion was incorrect, which was found 19 centuries later. The law of the lever on which the construction of all machines is based and the law of equilibrium for floating bodies were stated quite clearly and correctly by the great Greek scientist Archimedes (the 3rd century B.C.). Since then mechanics has been developing as an independent science. The medieval scientists obtained some new results concerning the equilibrium of bodies and their properties but they stuck to the incorrect idea of Aristotle concerning the basic law of motion of physical bodies.

Only in the seventeenth century did G. Galileo (1564-1642) discover and state correctly the basic law of motion of the bodies. Proceeding from that law and from the achievements of his contemporaries the great English scientist I. Newton (1643-1727) discovered the fundamental laws of mechanical motion and stated them so clearly and concisely that they are still used in that very form both for solving practical and technical problems and in research work.

The successors of G. Galileo and I. Newton elaborated a more general form of the basic laws of mechanics and perfected the methods of investigation of complex mechanical phenomena. The outstanding results of the studies of these scientists among which we should first of all mention L. Euler, D. Bernoulli, and J. D'Alembert are treated in courses in theoretical mechanics.

The new stage in the development of mechanics is connected with the fundamental works of A. Einstein (1879-1956) and some of his predecessors. Their

results are an important generalization of the laws of mechanics; they include, as a special case, the laws of motion of bodies with velocities smaller than the speed of light. Newtonian mechanics can now be regarded as a part of Einstein's relativistic mechanics.

The laws of motion and interaction of particles forming atoms and molecules and also those motions which take place in very small regions of space (of dimensions of the order of 10^{-10} m) differ in principle from the laws of classical mechanics, that is from the laws of motion of macroscopic bodies containing a great number of molecules. The laws of intraatomic and intramolecular phenomena are studied in *quantum* (or *wave*) *mechanics*. It should be noted that, under certain conditions, quantum mechanics and Einstein's mechanics include the classical Newtonian mechanics as a special case.

Many generations of engineers and scientists have made valuable contributions to the development of mechanics; the high level of modern technology is to a large extent based on their discoveries, and their practical results proved the correctness of our knowledge in the field of mechanics. The investigations of the scientists of the nineteenth and twentieth centuries have considerably expanded our understanding of mechanical phenomena.

PART ONE

Mechanics of Rigid Bodies

CHAPTER 1

Kinematics of a Particle

1. Motion of Bodies

If we observe a moving body, for instance, a rolling wheel, a moving car, a falling pellet or a skater sliding on the ice, we are struck by the variety and complexity of the motion. Any body such as a wheel, an automobile, an engine can be mentally divided into parts, and the motion of any part can differ from the motions of the other parts.

To investigate the motion of the parts of a moving body we can attach small electric bulbs to the parts and then photograph the moving body with the aid of a fixed camera, the exposure being chosen so that the picture only shows the

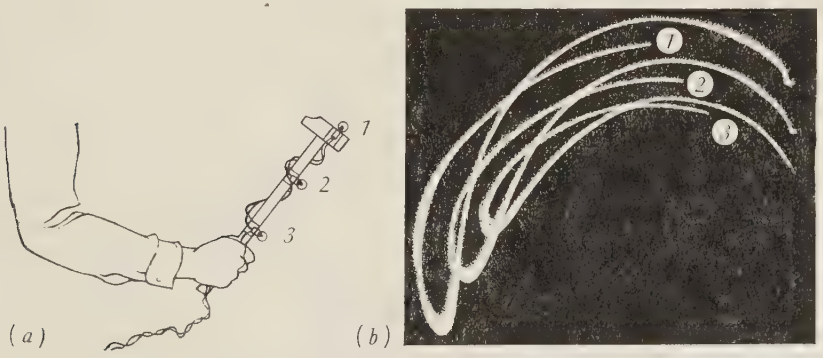

FIG. 1

traces of the moving shining bulbs. Knowing the time and the scale of the photography we can thus analyse complex motions. As an instance, see Fig. 1*b* showing the picture of the traces of three electric bulbs placed along the handle of a hammer with the aid of which a man drives in nails (Fig. 1*a*). Changing the positions of the bulbs, photographing them from different sides, and using the blinker light we can study in this way the motions of various bodies whose form may change in the process of motion (deformable bodies).

Each shining bulb (together with the part of the moving body it is attached to) moves along a definite curve; if the dimensions of the bulbs are sufficiently small the picture only shows the traces of the bulbs. Such a curve represents

the projection of the trajectory of motion of the corresponding bulb onto the plane of the photographic plate or film. Hence, to study the laws of motion of a complex body whose different parts are in different motions one should begin with deriving the laws of motion of the particles that form the body.

When the position of a part of a body (or of the whole body) is uniquely determined by the position of a point on a line we use the notion of a *material point*. By a material point is meant a small part of a moving body whose dimensions are sufficiently small relative to the dimensions of the whole body or even the whole body itself provided that its dimensions are small compared with the distance travelled by the body in the phenomenon in question. For instance, the annual motion of the Earth round the Sun can be regarded as the motion of a material point. A translatory motion of a rigid body can always be represented as a motion of a material point because in this case the position of the whole body is completely determined by the motion of any of its parts. After the laws of motion of a material point have been studied we can proceed to the study of the laws of motion of rigid and deformable bodies. Such bodies can always be thought of as collections of separate material points connected with each other.

In what follows, when studying a motion of a material point, we may simply speak of a motion of a *point* (of a *particle*).

2. Rectilinear Motion of a Particle

The simplest motion of a point is its rectilinear motion. When a particl is in a rectilinear motion it moves along a straight line, its distance from a chosen point on the line increasing or decreasing with time. In such a case we can connect a frame of reference with that straight line and consider the motion in question relative to the chosen fixed point serving as the origin.

If the coordinate x of the point (that is its distance from the chosen origin O) is known as function of time t the law of motion of the point (the particle) is completely determined. To analyse the motion it is convenient to plot the graph of the dependence of the coordinate x on time t (see Fig. 2) by choosing a certain length scale and setting off the values of the coordinate x along the axis of ordinates while the values of time t are set off along the axis of abscissas, a certain segment of the latter representing unit time. The graph shown in Fig. 2 makes it possible to investigate the motion of the point. If the variable point of the curve x (t) moves upwards as t increases, the distance of the particle from the origin O increases; the steeper the curve ascends, the faster does the distance between the particle and the point O increase. The parts of the curve that are line segments parallel to the axis of abscissas correspond to the states of rest of the particle; if the variable point of the curve goes downwards the distance between the particle and the origin decreases.

To construct the graph representing the motion of the particle in the straight line it is necessary to measure the distances x at the corresponding moments t. The quantity x can be measured in different ways; for instance, we can make snap shots of the moving particle at definite moments with a fixed camera. We can also attach to the moving body (the particle) a device which makes marks, in certain time intervals, on a fixed body along which the particle moves, and the like.

Generally speaking, in this way we find the coordinates x of the point at the given moments t but not the distances travelled by the point. The distance travelled can be found from the coordinate x only in the case when the particle moves in one direction. For instance, in the motion of the point represented by the diagram in Fig. 2 the coordinate of the point cannot be greater than x_0 although after time t_0 the distance S (t) travelled by the point exceeds x_0 (while the coordinate x becomes less than x_0). To elucidate what has been said let us consider an example of a moving car. If we read the data from a tachometer indicating the number of revolutions of the wheels of the car at certain fixed moments we find, with a certain accuracy, the dependence of the distance travelled by the car on time. If the car moves constantly in one direction the distance travelled is equal to the coordinate describing the motion but when the direction of motion changes to the opposite the distance still increases while the coordinate decreases.

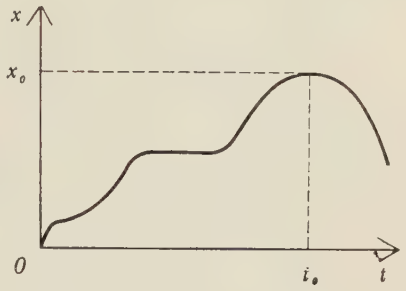

FIG. 2

The dependence of the coordinate on time completely determines the motion of the particle in the line but in mechanics it is important to know two more quantities, namely *velocity* and *acceleration* of the moving point.

3. Velocity of a Particle in Rectilinear Motion

The velocity of a point is a physical quantity determining the rate of change of the coordinate with time. The numerical value of the *average velocity* is equal to the ratio of the distance travelled by the point to the time taken. Let the particle be at a point x_1 at time t_1 and at a point x_2 at time t_2; then the displacement of the point is equal to $x_2 - x_1$ whence its average velocity is

$$v_{av} = \frac{x_2 - x_1}{t_2 - t_1} \tag{3.1}$$

Displacements are measured in units of length (that is in centimetres, metres, kilometres, and the like) while time is measured in time units such as seconds, minutes, hours, etc. Velocity is a derived physical quantity (distinct from length and time) which is measured in *derived units* dependent on the choice of unit length and unit time.

The dimension of velocity is equal to the ratio of the dimensions of length and time:

$$[v] = lt^{-1} \tag{3.2}$$

The velocity is measured in cm/s, m/s, km/h, etc.

It is obvious that the average velocity depends on the time interval for which it is computed. If, in a given motion, the average velocity is the same for *any* time interval the motion has a *constant* velocity and is said to be *uniform*.

For a uniform rectilinear motion the graph representing the dependence of the coordinate on time is a straight line. In the case of a uniform motion which starts from the origin there is no difference between the value of the coordinate and that of the path travelled.

In a non-uniform motion the average velocity is no longer a constant quantity and it depends on the time interval for which it is computed. For a non-uniform motion the average velocity no longer gives an adequate description of the variations of the motion of the body for different parts of its path; in

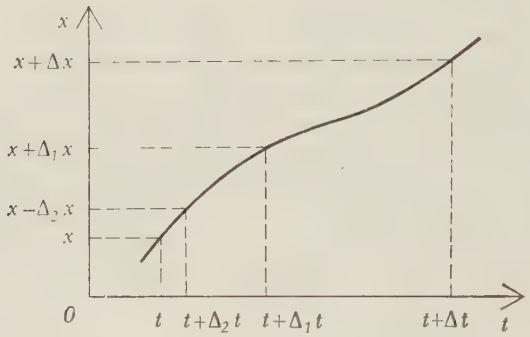

FIG. 3

order to characterize such a motion more completely it becomes necessary to introduce the notion of the *instantaneous velocity* (or, simply, the *velocity*) of the point at a given time t.

Suppose that we know the value of the displacement Δx of the point corresponding to a small time interval Δt; then, for this time interval, the average velocity is

$$v_{av} = \frac{\Delta x}{\Delta t}$$

Let the point have a coordinate x at time t (Fig. 3); then at time $t + \Delta t$ the value of its coordinate is $x + \Delta x$. Suppose that we have performed a number of measurements of Δx for various values of Δt which consecutively decrease, that is we have measured the coordinate $x + \Delta x$ for the moments $t + \Delta_1 t$, $t + \Delta_2 t$, etc., where $\Delta_1 t > \Delta_2 t > \ldots$. It is obvious that if the value of $\Delta_i t$ and the corresponding value of $\Delta_i x$ are so small that the motion over the path $\Delta_i x$ can be approximately regarded as uniform then for all sufficiently small values of Δt the values

$$v = \frac{\Delta x}{\Delta t}$$

of the average] velocity are almost the same. It is this (limiting) value of the average velocity that is meant by the (instantaneous) velocity of the point at time moment t.

A more rigorous mathematical definition of the velocity at time t is expressed in the form

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \tag{3.3}$$

which implies that the velocity is the limit to which tends the ratio of the increment Δx of the coordinate to the corresponding time increment Δt as the latter tends to zero. In mathematics this limit is called the derivative of the coordinate x with respect to time t and is denoted

$$v = \frac{dx}{dt} \tag{3.4}$$

In the last expression the quantity dt is by definition equal to Δt while dx differs from the corresponding value of Δx by quantities proportional to dt^2, dt^3, etc. (see Fig. 4). Formulas (3.3) and (3.4) read: *the value of the velocity is equal to the derivative of the coordinate with respect to time.* Relation (3.4) can also be rewritten in the form

$$dx = v\,dt \tag{3.5}$$

Relation (3.5) indicates that a sufficiently small increment of the coordinate is equal to the product of the velocity by the increment of time.

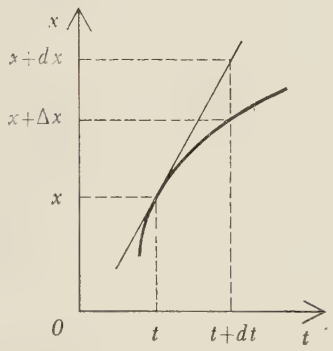

FIG. 4

The value of the velocity can be found and the velocity-time diagram can be constructed from the graph representing the dependence of the coordinate of the moving point on time (i.e. from the displacement-time graph). Suppose that the graph of $x\,(t)$ is as shown in Fig. 5*a*. Let the axis of abscissas t be divided into subintervals dt so small that the parts of the curve $x\,(t)$ corresponding to them can be regarded, with high accuracy, as segments of the tangent line*; then the slope of the tangent line which is equal to $\tan \alpha = \frac{dx}{dt}$ coincides with the numerical value of the velocity measured in derived units corresponding to the chosen basic units of length and time.

It is obvious that for the values of time for which the curve $x\,(t)$ is close to a straight line in its shape (or coincides with a straight line) the time intervals dt must not necessarily be very small; however for the parts of the line $x\,(t)$ which are sharply curved we should take very small intervals dt to obtain $\frac{dx}{dt} = v$ with high accuracy. Practically this can be very difficult to perform and therefore the accuracy of the determination of the velocity from the displacement-time diagram may not be sufficiently accurate for such time moments. Figure 5*b* represents the values of the velocity $v\,(t)$ corresponding to the graph of $x\,(t)$ shown in Fig. 5*a*, the time scales in both figures being the same. The comparison of these graphs shows that to the moments at which the greatest values of the velocity $v\,(t)$ are achieved there corresponds the greatest rate of increase of the coordinate $x\,(t)$, that is the steepest ascent of the graph of $x\,(t)$.

*The tangent line to a curve at its point is defined as the limiting position of the secant passing through that point and some other point of the curve as the latter point approaches indefinitely the former.

We see that, in principle, the values of the velocity of a moving point at given moments can be determined, as described above, by measuring the values of the coordinate in small time intervals, but practically this method is inefficient. Therefore there were constructed a number of special devices measuring directly the values of the velocity of a moving body. An example of an apparatus of this kind is the speedometer measuring the velocity of an automobile.

The (instantaneous) velocity of a point at a given time is a physical quantity characterizing the rate of change of the coordinate with time. This physical

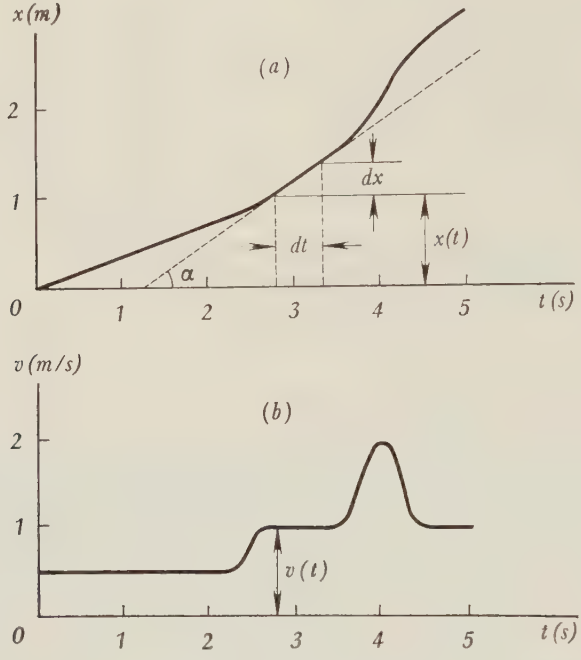

FIG. 5

quantity exists because of the continuity of the change of the coordinate with time and because there is a certain law governing the variations of these quantities. The value of the velocity is determined by the difference in the position of the moving point in space corresponding to two moments arbitrarily close to each other.

The strict definition of the value of the instantaneous velocity was given by I. Newton; proceeding from this definition he developed the foundation of the analysis of infinitesimals.

4. Relationship Between the Velocity and the Distance Travelled

For a point moving with a constant velocity v_0, the distance travelled from time t_1 to time t_2 is obviously equal to the product of the velocity v_0 by the time difference $t_2 - t_1$:

$$x_2 - x_1 = v_0 (t_2 - t_1) \tag{4.1}$$

This formula makes no sense when the velocity of motion is not constant. If the average velocity v_{av} corresponding to the time interval $t_2 - t_1$ is known then the distance travelled can be expressed by a formula analogous to (4.1) but v_0 should be replaced by v_{av}.

In the case when the average velocity is unknown the computation of the path length travelled by a body is performed with the aid of a special method based on the fact that every motion can be regarded, with a sufficient accuracy, as being uniform for any sufficiently small time interval. Based on this principle, we can determine the distance dx travelled by the body during a sufficiently small time interval dt by multiplying the velocity v at the given time moment t by the corresponding increment of time dt:

$$dx = v\,dt \tag{4.2}$$

Now let us suppose that the whole time interval $t_2 - t_1$ is divided into infinitely many infinitesimal subintervals dt. To each (infinitely) small interval dt there corresponds a certain small increment dx. The distance $x_2 - x_1$ travelled during the time $t_2 - t_1$ can be written as the sum of all dx. A sum of this kind is called an *integral* and is written in the form

$$x_2 - x_1 = \int_{x_1}^{x_2} dx$$

Next we replace dx under the integral sign by the equal quantity $v\,dt$ and perform the summation with respect to time t ranging from t_1 to t_2. It follows that the distance covered by the body during the time interval $t_2 - t_1$ can be written as

$$x_2 - x_1 = \int_{t_1}^{t_2} v\,dt$$

The computation of the quantity $x_2 - x_1$ from a known value of the velocity $v\,(t)$ is a problem of the integral calculus. Integration is an operation inverse to differentiation, that is inverse to the computation of the derivative. The value of the distance travelled is equal to the integral of the velocity v with respect to time t. Thus, the laws connecting the three physical quantities—the coordinate, the velocity, and the time—are expressed mathematically with the aid of the derivative and the integral.

In Fig. 6 we see the graphs of the coordinate and of the velocity for two motions one of which is uniform and the other non-uniform. When the graph of the velocity of motion $v\,(t)$ is known it is possible to find the graph of $x\,(t)$* with a sufficiently high accuracy. In Fig. 7*a* we see the graph of the velocity v as function of t. Let us consider the interval dt shown in the figure; the numerical value of the area $v\,dt$ (this area is densely dashed in the figure) measured in the units corresponding to the scales chosen for the coordinate axes v and t is equal to the increment of $x\,(t)$ corresponding to dt (see Fig. 7*b*). The coordinate of the moving point at a moment t_1, that is $x\,(t_1)$, is obviously equal to the whole area bounded by the curve $v\,(t)$ on the interval ranging from $t = 0$ to $t = t_1$ because

$$x\,(t_1) = \int_0^{t_1} dx = \int_0^{t_1} v\,dt$$

* If, in addition, the position of the point at a certain time is given.

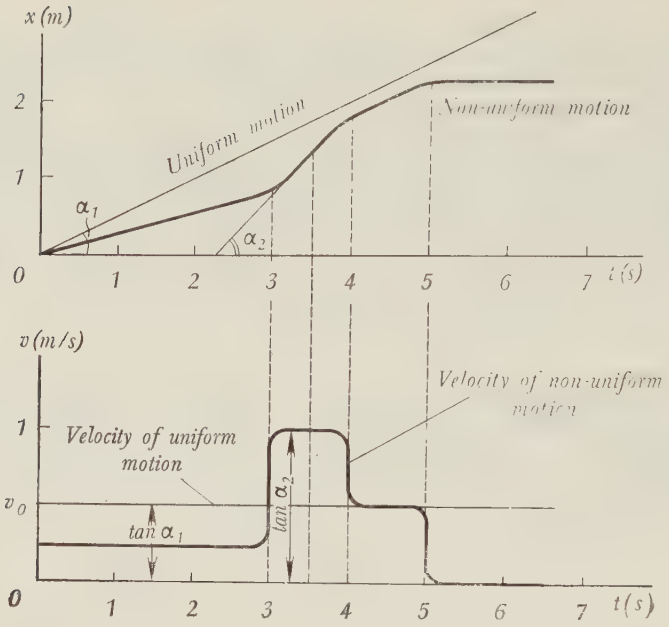
x (m)
Uniform motion
Non-uniform motion
α1
α2
t (s)
v (m/s)
Velocity of non-uniform motion
Velocity of uniform motion
v0
tan α1
tan α2
t (s)

FIG. 6

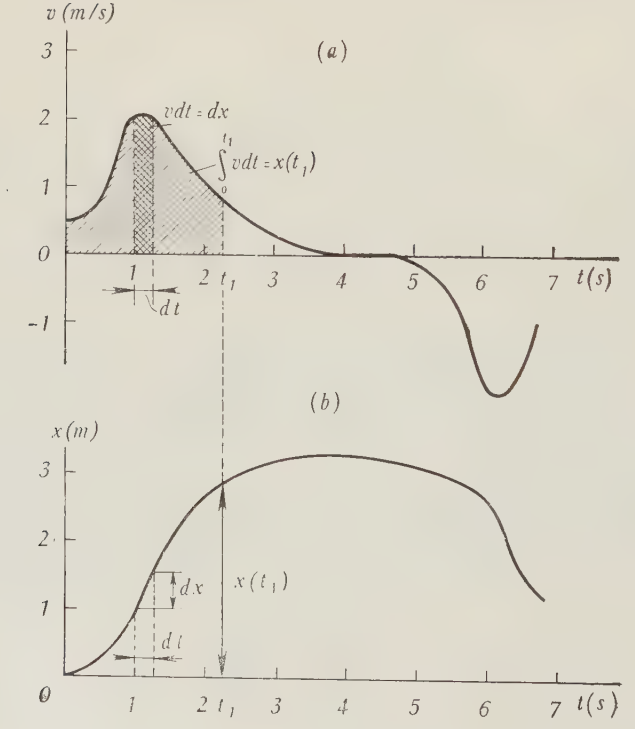
v (m/s)
(a)
vdt = dx
∫ vdt = x(t1)
dt
t (s)
(b)
x (m)
dx
x (t1)
dt
t (s)

FIG. 7

Thus, the computation of the area lying under the curve $v(t)$ on the interval from $t = 0$ to $t = t_1$ gives us the value of the coordinate x for any time t_1. The result of the numerical integration is shown in Fig. 7*b*. The computation of the area can be performed sufficiently accurately and therefore the determination of the coordinate from the given velocity-time diagram is more precise than the determination of the velocity from the graph representing the time variation of the coordinate.

5. Acceleration of a Particle in Rectilinear Motion

The velocity of a point moving non-uniformly is a variable quantity and, like the coordinate, can be regarded as a function of time. As we know, the value of the velocity determines the time rate of change of the displacement of the body. As to the rate of change of the velocity, it is characterized by another quantity called *acceleration*.

The acceleration of a body at a given moment of time is defined as follows. Let the velocity have a value v at time t and a value $v + dv$ at another moment $t + dt$ where dv and dt are infinitesimals; then the ratio

$$w = \frac{dv}{dt} \tag{5.1}$$

is called the (*instantaneous*) *acceleration*. Thus, *the value of the acceleration i equal to the derivative of the velocity with respect to time.*

The dimension of the acceleration is equal to the ratio of the velocity and time dimensions:

$$[w] = \frac{[v]}{[t]} = lt^{-2} \tag{5.2}$$

If the velocity is measured in m/s and the time of motion is measured in seconds the acceleration is measured in m/s^2. As a rule, the units of velocity and acceleration have no special names. However, in navigation a special unit of velocity is used which is called the *knot* and is equal to one nautical mile (that is 1.853 km) per hour. In aviation and astronautics the acceleration is sometimes expressed with the aid of the normal acceleration due to gravity (this unit is equal to 9.81 m/s^2 and is denoted g). Usually in calculations in physics and engineering the derived units of velocity and acceleration are used; they are specified by the choice of the basic units of length and time.

In a motion with a constant acceleration the velocity increases or decreases uniformly with time while the coordinate increases or decreases as a quadratic function of time. A free fall of a body on the Earth is an example of a rectilinear motion with constant acceleration.

Analysing the results of experiments with bodies falling in the air G. Galileo found that the acceleration of free fall (near the Earth's surface) is the same for all bodies (provided that the effect of the air friction is negligibly small). This acceleration depends on the latitude; for instance, at the latitude of Moscow it is approximately equal to 9.81503 m/s^2. The time diagrams of the acceleration, of the velocity and of the coordinate of a body thrown vertically upwards in vacuum are shown in Fig. 8.

Free fall of bodies in vacuum is an example of a motion with constant acceleration. When the acceleration of motion varies with time the law of the variation of the velocity differs from the linear law characteristic of the motion with

constant acceleration. Arguing as in the case of a variable velocity we readily conclude that *the increment of the velocity corresponding to a given time interval is equal to the integral of the acceleration with respect to time over that interval.*

Thus, the acceleration is the rate of change of the velocity. Mathematically, this means that the acceleration is equal to the time derivative of the velocity.

The mathematical notions of the derivative and of the integral were introduced by I. Newton in order to describe the laws of the non-uniform motion of a material point in the line. It is impossible to analyse the laws of complex

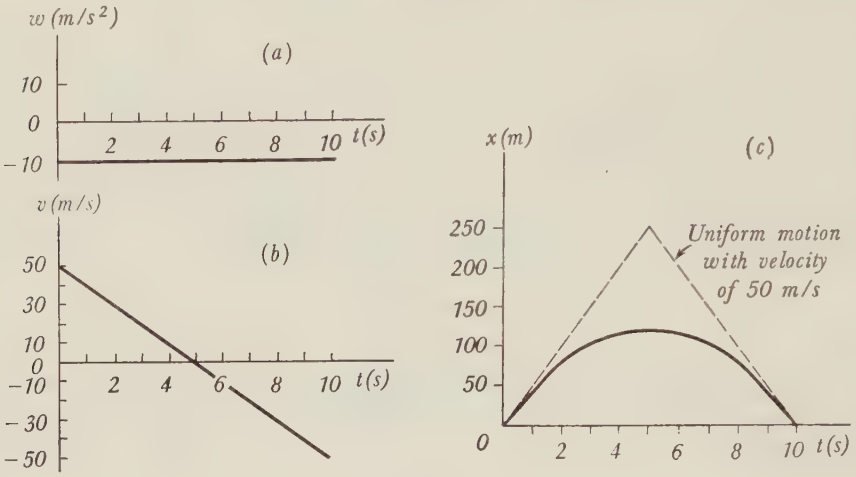

FIG. 8

mechanical motions without using these important notions. That is why the study ofi complex motions is related to the development of mathematical hods.

6. Three-dimensional Motion of a Particle

When a point moves in space along a straight line the possible directions of the motion are determined by the position of that line in space. If the direction of motion in the line changes to the opposite the velocity of the motion changes sign. The velocity is considered positive when the coordinate of the point reckoned along the line in the (chosen) positive direction increases.

Any motion of a body relative to other bodies can be thought of as its motion relative to *a frame of reference* consisting of three mutually perpendicular coordinate axes. For instance, to determine the motion of a falling pellet relative to the walls and the floor of the room we can take as coordinate axes the lines of intersection of the walls and the floor passing through one of the lower corners of the room. The axes of coordinates thus chosen form a frame of reference connected with the room, the origin being at the chosen lower corner. The distances between the origin and the feet of the perpendiculars dropped from the point under consideration on the coordinate axes are certain numbers which we can denote x, y, and z, the coordinate x being reckoned along one of the horizontal axes, the coordinate y along the other horizontal axis, and the coordinate z along the vertical axis (see Fig. 9).

If x_1 and y_1 are the coordinates of a point in the plane of the floor then the position of the pellet placed at that point is uniquely determined by the two coordinates x_1 and y_1 and any motion of the pellet in the plane of the floor can be described with the aid of two functions

$$x_1(t) \quad \text{and} \quad y_1(t)$$

The position of any point in the room relative to the frame of reference formed by the three mutually perpendicular coordinate axes we have chosen is precisely determined by the three numbers x_1, y_1, and z_1 where z_1 is the distance between that point and the floor and x_1 and y_1 are the coordinates of the foot of the perpendicular dropped from the point on the horizontal plane of the floor. The numbers x_1, y_1, and z_1 are thus the coordinates of the point in space.

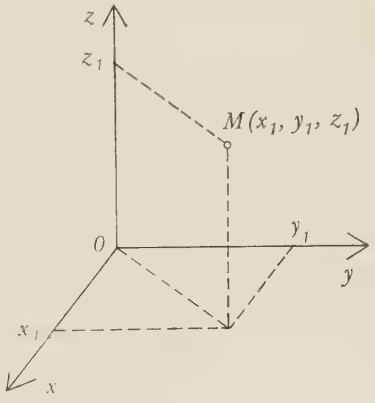

FIG. 9

Generally speaking, in a motion of a point relative to a system of reference (for example, in a motion of the pellet relative to the room) all the three coordinates vary simultaneously and therefore, in the general case, a motion of a point is described mathematically by three functions of time

$$x_1(t), \quad y_1(t), \quad \text{and} \quad z_1(t)$$

All the points in space through which the moving point passes form a curve which is called the *trajectory of motion.* The distance travelled by the point along the trajectory is spoken of as the path length covered by the point. For example, we can speak of the trajectory of a car moving about the town which can be a rather complicated curve. Thus, the trajectory is a curve which the moving point "traces" in space (for instance, the curve along which the car moves, the distance travelled by the car being equal to the number shown by the speedometer).

In the course of time a moving point passes from some points in space to other points lying on the trajectory. The displacement ds of the point corresponding to a small time interval dt is determined not only by the numerical value of its velocity but also by the direction of the velocity in space. By the direction of the velocity of the body at a given time moment is meant the one in which the body moves at that moment.

Thus, the velocity of a spatial motion of a point is characterized not only by the ratio $\frac{ds}{dt}$ where ds is the length of the segment of the trajectory travelled during the time interval dt but also by the direction of that infinitesimal segment ds. For operations on physical quantities of this kind characterized by some numbers and definite directions in space and called *vector quantities* (or, simply, *vectors*) there are special rules elaborated in mathematics; the most important of these rules are discussed in the next section.

7. Basic Properties of Vectors

A *vector* is represented geometrically as a directed line segment with a definite orientation in space. For a point moving in space its displacement during any time interval can be represented by a vector. For instance, as is shown in Fig. 10*a* the moving particle is at the point A at the moment t_1 and at the point B at moment t_2; the displacement corresponding to the time interval from $t = t_1$ to $t = t_2$ is represented by the vector $\overrightarrow{AB}$*.

It should be noted that the line segment representing the displacement vector of the moving point coincides with the trajectory of motion only in the case of a rectilinear motion. When the motion is curvilinear the trajectory of the point is a curve passing through the points A and B (see the dotted line in Fig. 10*a*).

Two vectors are considered equal when they are represented by two equal parallel line segments and have the same direction. In Fig. 10*b* we see several vectors lying in the plane of the figure. The length of the line segment representing a vector measured in definite units of length is called the *absolute value* or the *modulus* (or the magnitude) of the vector** while the position of that segment in space and the arrow indicate its direction. The vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ in Fig. 10*b* have the same absolute values but their directions are different and therefore the vector $\overrightarrow{AB}$ is not equal to the vector $\overrightarrow{CD}$. The vectors $\overrightarrow{AB}$ and $\overrightarrow{MN}$ differ both in the directions and in the absolute values. The vectors $\overrightarrow{AB}$ and $\overrightarrow{B'A'}$ are equal because they have the same absolute values and coincide in their directions.

Let the moving point, as shown in Fig. 10*a*, pass from the point A to the point B, the displacement vector being $\overrightarrow{AB}$. Such a displacement of the point can be regarded as a sum of two displacements; for instance, we can imagine that the moving point first passes from the point A to an arbitrary point C (the corresponding displacement is represented by the vector $\overrightarrow{AC}$) and then goes from the point C to the point B (the latter displacement is represented by the vector $\overrightarrow{CB}$). Consequently, the displacement $\overrightarrow{AB}$ is equal to the sum of the displacements $\overrightarrow{AC}$ and $\overrightarrow{CB}$ or, which is the same, the sum of the vectors $\overrightarrow{AC}$ and $\overrightarrow{CB}$ is equal to the vector $\overrightarrow{AB}$. Thus, the addition of vectors is performed geometrically: the sum of two vectors is equal to the diagonal of the parallelogram whose sides are the added vectors. The subtraction rule for vectors follows from the addition rule: the vector $\overrightarrow{CB}$ is equal to the difference of the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, that is the displacement $\overrightarrow{CB}$ is obtained by the subtraction of the displacement $\overrightarrow{AC}$ from the displacement $\overrightarrow{AB}$.

The vectors $\overrightarrow{AC}$ and $\overrightarrow{CB}$ are called *components* of the vector $\overrightarrow{AB}$. The sum of several vectors can be found by constructing the "chain" of vectors as shown

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* The arrow in the symbol $\overrightarrow{AB}$ indicates that the vector is directed from A to B.

** The absolute value (the modulus) of a vector $\overrightarrow{AB}$ is denoted $|\overrightarrow{AB}|$ or simply AB (without an arrow).

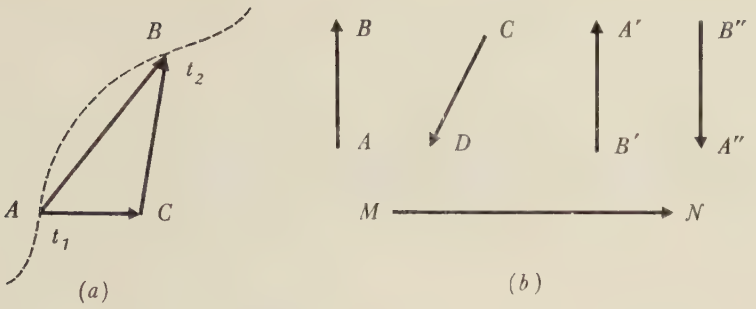

FIG. 10

in Fig. 11: the vector summands are arranged so that the origin of the third vector is placed at the terminus of the second and so on, after which the origin of the first vector is connected with the terminus of the last vector. The vector connecting the origin of the first vector with the terminus of the last vector of the chain represents the sum of all these vectors.

It is obvious that, given constituent vectors, we can always find their sum with the aid of the above addition rule. When operating on vector quantities, it is sometimes convenient to represent the vectors as sums of their components

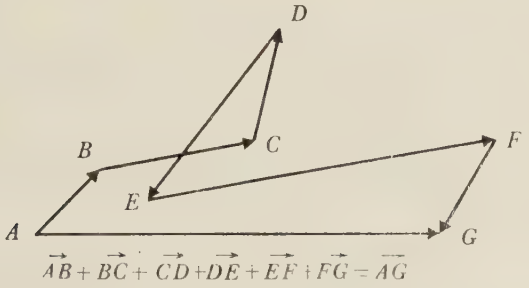

FIG. 11

along certain definite directions. Let us first demonstrate what has been said for vectors lying in a plane (see Fig. 12). Suppose we have chosen in the plane a rectangular coordinate system x, y; then any vector $\boldsymbol{a}$* is representable as the sum

$$\boldsymbol{a} = \boldsymbol{a}_x + \boldsymbol{a}_y \tag{7.1}$$

where the vector $\boldsymbol{a}_x$ is directed along the x-axis and is called the *component of the vector* $\boldsymbol{a}$ *along the x-axis* and the vector $\boldsymbol{a}_y$ is the *component of* $\boldsymbol{a}$ *along the y-axis*. It is evident that the two components of the vector $\boldsymbol{a}$ uniquely determine the modulus and the direction of that vector.

Let us denote by $\boldsymbol{i}$ the unit vector along the x-axis and by $\boldsymbol{j}$ the unit vector along the y-axis. By definition, a *unit vector* is a vector whose modulus is equal

* In what follows we shall denote a vector by a boldface-type letter and its modulus either by the same standard-type letter or by the boldface-type letter with two vertical lines: $|\boldsymbol{a}|$.

to unity. A unit vector indicates a certain direction in space. Then the components $\boldsymbol{a}_x$ and $\boldsymbol{a}_y$ can be written as

$$\boldsymbol{a}_x = a_x \boldsymbol{i} \quad \text{and} \quad \boldsymbol{a}_y = a_y \boldsymbol{j} \tag{7.2}$$

where a_x and a_y are ordinary numbers (not vectors), that is they are *scalars* with the aid of which are represented quantities having no direction in space, for instance, time, distance, etc. The numbers a_x and a_y are called the *projections* of the vector on the directions specified, respectively, by the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$, or, simply, the *projections of $\boldsymbol{a}$ on the given coordinate axes x and y*. In practical problems we shall always consider vector quantities relative to some definite rectangular coordinates, that is relative to a chosen frame of reference specified by the directions of unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$.

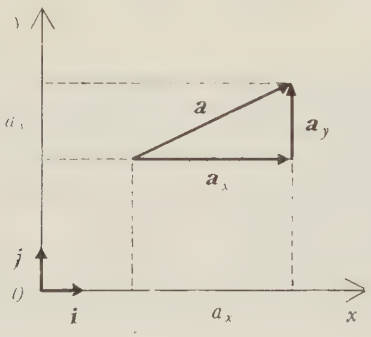

FIG. 12

The modulus (or absolute value or length) of a vector $\boldsymbol{a}$ lying in the xy-plane is expressed by the formula

$$a = |\boldsymbol{a}| = \sqrt{a_x^2 + a_y^2} \tag{7.3}$$

Taking into account formulas (7.1) and (7.2) we can write

$$\boldsymbol{a} = a_x \boldsymbol{i} + a_y \boldsymbol{j} \tag{7.4}$$

All these formulas are valid for any vector $\boldsymbol{a}$ lying in the xy-plane.

When we deal with a vector $\boldsymbol{a}$ lying in space it is convenient to consider it relative to some rectangular coordinates x, y, z. The unit vector along the z-axis, which is perpendicular to the vectors $\boldsymbol{i}$ and $\boldsymbol{j}$, is usually denoted as $\boldsymbol{k}$. Arguing in an analogous manner we can express the vector $\boldsymbol{a}$ in terms of its components in the form

$$\boldsymbol{a} = \boldsymbol{a}_x + \boldsymbol{a}_y + \boldsymbol{a}_z \tag{7.5}$$

where $\boldsymbol{a}_z$ is the component of $\boldsymbol{a}$ along the z-axis; on expressing the components in terms of the projections a_x, a_y, and a_z we can rewrite (7.5) as

$$\boldsymbol{a} = a_x \boldsymbol{i} + a_y \boldsymbol{j} + a_z \boldsymbol{k} \tag{7.6}$$

The modulus of the vector $\boldsymbol{a}$ is given by the formula

$$a = |\boldsymbol{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} \tag{7.7}$$

We now proceed to the basic rules for the multiplication of vector quantities. The result of the multiplication of a vector $\boldsymbol{a}$ by a number β is written as $\beta\boldsymbol{a}$, the meaning of the operation being that the modulus of the vector is increased $|\beta|$ times while its direction remains the same when $\beta > 0$ and is changed to the opposite when $\beta < 0$. In the multiplication of two vectors by each other we distinguish between the *scalar* and the *vector* products; in the former case the product of the vectors is a scalar and in the latter case a vector.

The *scalar product* of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as a number equal to the product of the moduli of these vectors by the cosine of the angle α between the directions of the vectors. The scalar product is denoted $\boldsymbol{a}\boldsymbol{b}$ and, according

to the definition, we have

$$\boldsymbol{ab} = ab \cos \alpha \tag{7.8}$$

It is readily seen that the scalar product $\boldsymbol{ab}$ is equal to the modulus of one of the vectors times the projection of the other vector on the former. For example, if one of these vectors represents a force while the other represents a displacement then the work performed by the force is equal to the scalar product of the vector of the force by the displacement vector.

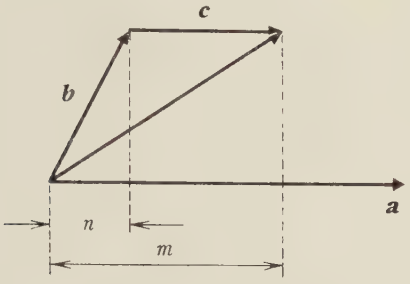

FIG. 13

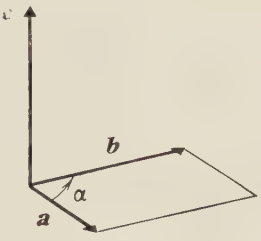

FIG. 14

The scalar product of two mutually perpendicular vectors is equal to zero. It is also obvious that $\boldsymbol{ab} = \boldsymbol{ba}$.

The scalar product possesses the *distributive property*:

$$\boldsymbol{a}(\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{ab} + \boldsymbol{ac}$$

It is proved quite easily (see Fig. 13) because we have

$$\boldsymbol{a}(\boldsymbol{b} + \boldsymbol{c}) = am, \quad \boldsymbol{ab} = an \quad \text{and} \quad \boldsymbol{ac} = a(m - n)$$

Proceeding from this property we can readily prove that

$$\boldsymbol{ab} = (a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k})(b_x\boldsymbol{i} + b_y\boldsymbol{j} + b_z\boldsymbol{k}) = a_xb_x + a_yb_y + a_zb_z \tag{7.9}$$

taking into account that $\boldsymbol{ij} = \boldsymbol{ik} = \boldsymbol{jk} = 0$ and $\boldsymbol{ii} = \boldsymbol{jj} = \boldsymbol{kk} = 1$. Hence, the scalar product of two vectors is equal to the sum of the products of its corresponding projections on the coordinate axes.

The *vector product* of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as the vector $\boldsymbol{c}$ which is perpendicular (normal) to the plane of the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and has the modulus equal to $ab \sin \alpha$ where α is the angle between the vectorial factors. The vector product is denoted $[\boldsymbol{ab}]$:

$$\boldsymbol{c} = [\boldsymbol{ab}]$$

The direction of the vector $\boldsymbol{c}$ is determined by the *right-hand screw rule*: if a screw (with a standard right-hand thread) placed perpendicularly to the plane containing the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is rotated by the smallest angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ in the direction from $\boldsymbol{a}$ to $\boldsymbol{b}$ then the direction in which this screw is driven is that of the vector $\boldsymbol{c}$ (see Fig. 14). It follows that, in contrast to the scalar product, the vector product $\boldsymbol{c}$ depends on the order of the factors, that is *the vector multiplication* is *non-commutative*:

$$[\boldsymbol{ab}] = -[\boldsymbol{ba}]$$

From the definition of the vector product it follows that the modulus of the vector $\boldsymbol{c}$ is equal to the area of the parallelogram constructed of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as sides.

In vector algebra it is proved that the vector product possesses the *distributive property*, that is

$$[\boldsymbol{a}\,(\boldsymbol{b}+\boldsymbol{c})] = [\boldsymbol{ab}] + [\boldsymbol{ac}]$$

It should be noted that in the last formula it is important to write the factors in the correct order because the vector product is not commutative.

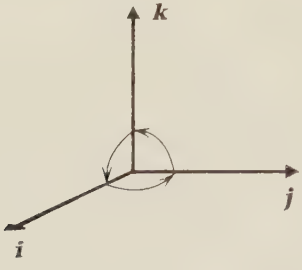

FIG. 15

Using the indicated property we can derive the following expression for the vector product:

$$\boldsymbol{c} = [\boldsymbol{ab}] = [(a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}) \times (b_x\boldsymbol{i} + b_y\boldsymbol{j} + b_z\boldsymbol{k})] = (a_yb_z - a_zb_y)\boldsymbol{i} + (a_zb_x - a_xb_z)\,\boldsymbol{j} + (a_xb_y - a_yb_x)\,\boldsymbol{k} \qquad (7.10)$$

In the derivation of this formula one should take into account (see Fig. 15) that

$$[\boldsymbol{ij}] = -[\boldsymbol{ji}] = \boldsymbol{k}, \quad [\boldsymbol{jk}] = -[\boldsymbol{kj}] = \boldsymbol{i}, \quad [\boldsymbol{ki}] = -[\boldsymbol{ik}] = \boldsymbol{j},$$
$$\text{and} \quad [\boldsymbol{ii}] = [\boldsymbol{jj}] = [\boldsymbol{kk}] = 0$$

It is also expedient to write down the expressions of the projections of the vector $\boldsymbol{c}$ which follow from formula (7.10):

$$\left.\begin{aligned} c_x &= a_yb_z - a_zb_y \\ c_y &= a_zb_x - a_xb_z \\ c_z &= a_xb_y - a_yb_x \end{aligned}\right\} \qquad (7.11)$$

Formulas (7.11) can easily be memorized if we note that the indices x, y and z in these formulas follow each other in a circular order in all the columns (see Fig. 15).

To stress once again the distinction between a scalar and a vector we note that the former is determined by *one* number while the latter is determined by *three* numbers (its projections).

A more general definition of a vector (physical) quantity reads: a *vector is an ordered triple of numbers* (*representing some scalar physical quantities*) *which depend on the choice of the coordinate system and change like the coordinates of a point when the coordinate system is rotated.* A parallel translation of the coordinate system does not affect the projections of a vector; they only vary when the system is rotated. In mechanics we deal with various vector quantities such as displacements, velocities, accelerations, forces, and the like.

8. Velocity of a Particle

On a sufficiently small part of the trajectory any motion can be considered rectilinear. A point moving along a curve (see Fig. 16) travels a very small distance during a sufficiently small time interval dt and receives a small displace-

ment dS. The displacement dS should be regarded as a vector going along the tangent to the trajectory in the direction of motion.

In accordance with (3.5), the velocity vector $\boldsymbol{v}$ can be found from the equality

$$d\boldsymbol{S} = \boldsymbol{v}\, dt \tag{8.1}$$

Since dt is a scalar the direction of the vector $\boldsymbol{v}$ coincides with that of $d\boldsymbol{S}$. Hence we can say that the velocity is equal to the ratio of $d\boldsymbol{S}$ to dt:

$$\frac{d\boldsymbol{S}}{dt} = \boldsymbol{v}$$

The velocity of a point is a vector quantity whose direction coincides with that of the tangent line to the trajectory. Therefore velocities can be represented by directed line segments (vectors) provided that the length of a certain line segment is chosen to represent the unit of velocity (which is a derived unit). The addition and subtraction of velocities should be performed in accordance with the addition and subtraction rules for vectors.

The velocity vector $\boldsymbol{v}$ and a very small displacement vector $d\boldsymbol{S}$ coincide in their directions when the corresponding time interval dt is sufficiently small; however, as physical quantities, the displacement $d\boldsymbol{S}$ and the velocity $\boldsymbol{v}$ are quite different, only their directions in space coinciding. The line segment $d\boldsymbol{S}$ in Fig. 16 represents, for the chosen length scale, the real displacement of the moving point corresponding to the time interval dt; the line segment denoted by $\boldsymbol{v}$ in the figure represents the velocity only conditionally: the direction of the segment coincides with that of $\boldsymbol{v}$ while its length symbolizes the (absolute) value of $\boldsymbol{v}$ in the sense that this length measured in conditional units of length representing the chosen unit of velocity is equal to $|\,\boldsymbol{v}\,|$.

In Fig. 16 are also shown the unit lengths chosen for measuring displacements and velocities (in the above sense). In what follows we do not indicate unit measures in the figures but the reader should bear in mind that it is always meant that for every quantity we deal with a certain unit has been chosen.

In an arbitrary motion of a point along a curve both the direction and the absolute value of the velocity vary. The velocity remains constant only in the case when its absolute value and its direction are invariable. *A motion with a constant, invariable velocity is a uniform rectilinear motion. Any motion along a trajectory distinct from a straight line is a motion with a variable velocity.*

When considering the velocity of a simple motion, it is sometimes convenient to represent it as a vector sum of two or more components. For instance, let us consider a ball moving along a horizontal plane. At any moment its velocity vector $\boldsymbol{v}$ has a definite orientation relative to the chosen coordinate axes x and y which is determined by the angle α between the direction of the vector $\boldsymbol{v}$ and that of the x-axis. The modulus and the direction of the velocity vector can be found if we are given the projections of the vector on the coordinate axes x and y. We can consider the vector $\boldsymbol{v}$ as the sum of its two components (see Fig. 17) one of which goes in the direction of the x-axis and the other in the direction of the y-axis:

$$\boldsymbol{v} = v_x\boldsymbol{i} + v_y\boldsymbol{j} \tag{8.2}$$

where v_x and v_y are the projections of the velocity on the coordinate axes x and y, $\boldsymbol{i}$ and $\boldsymbol{j}$ being the unit vectors of the same axes. The vector $v_x\boldsymbol{i}$ is the component of the velocity along the x-axis and the vector $v_y\boldsymbol{j}$ its component along the y-axis. If, in a special case, it turns out that one of the components

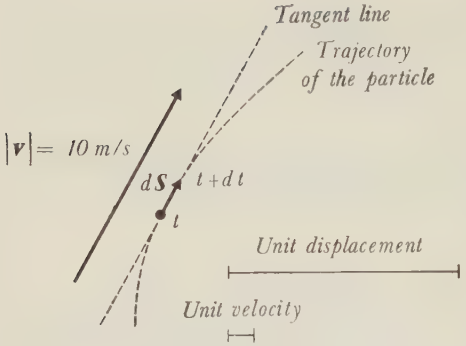

FIG. 16

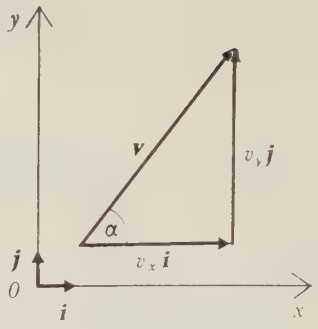

FIG. 17

is equal to zero, the velocity v is parallel to the other coordinate axis. The modulus of the velocity is

$$v = |v| = \sqrt{v_x^2 + v_y^2} \tag{8.3}$$

and the angle α between the vector v and the x-axis is obviously given by the formula

$$\alpha = \arctan \frac{v_y}{v_x} \tag{8.4}$$

A component of the velocity vector is the velocity with which moves the projection of the moving point on the corresponding coordinate axis and it can

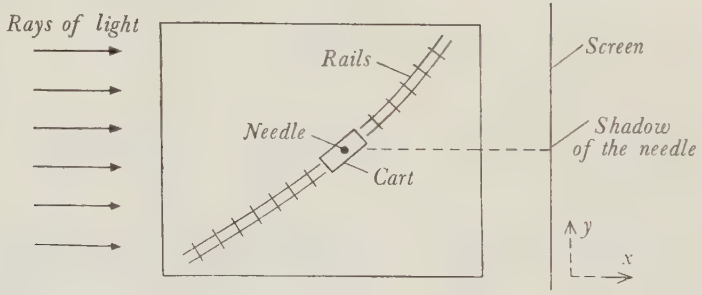

FIG. 18

be simply called the velocity of the point along that axis. Knowing the velocities along two coordinate axes we can readily find the absolute value of the velocity and determine its direction in the plane of motion.

As has been said, the component of the velocity along a definite axis is the velocity of the projection of the moving point on that axis, and it can be thought of as the velocity of motion of the shadow of the moving body. For instance, consider a small cart running with velocity v along the rails placed on the table in Fig. 18 which shows the top view. Let there be a light source placed far from the table so that the rays of light can be regarded as being parallel to the x-axis. Then the shadow of the vertical needle attached to the cart moves in the

screen perpendicular to the x-axis with velocity v_y. It is obvious that the velocity of the motion of the shadow is nothing but the component of the velocity of the needle along the y-axis. Given the velocities of the shadow on two mutually perpendicular screens, we can determine the absolute value and the direction of the velocity of motion of the needle.

When a point moves in space arbitrarily the infinitesimal displacement vector $d\boldsymbol{S}$ is uniquely determined by its three projections on three mutually perpendicular coordinate axes x, y, and z (Fig. 19*a*). The vector $d\boldsymbol{S}$ is representable in the form

$$d\boldsymbol{S} = dx\,\boldsymbol{i} + dy\,\boldsymbol{j} + dz\,\boldsymbol{k} \tag{8.5}$$

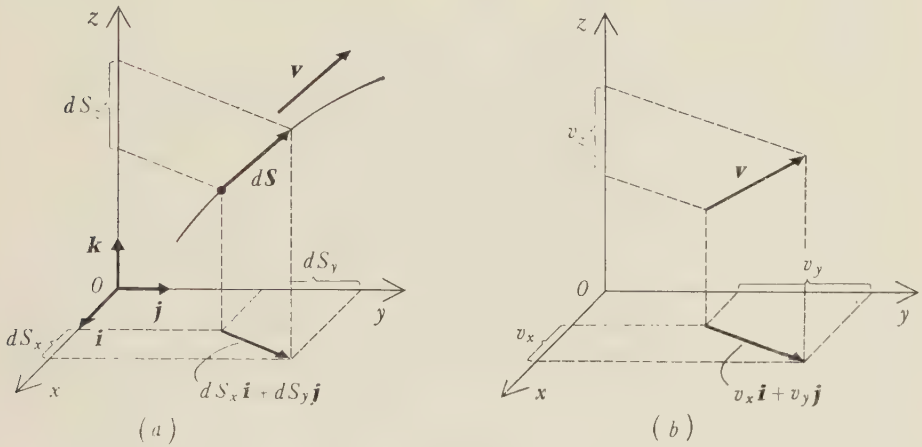

FIG. 19

where dx, dy, and dz are the projections of the vector $d\boldsymbol{S}$ on the coordinate axes, whence follows that the velocity can be written as

$$\boldsymbol{v} = \frac{d\boldsymbol{S}}{dt} = \frac{dx}{dt}\,\boldsymbol{i} + \frac{dy}{dt}\,\boldsymbol{j} + \frac{dz}{dt}\,\boldsymbol{k}$$

or, which is the same,

$$\boldsymbol{v} = v_x\boldsymbol{i} + v_y\boldsymbol{j} + v_z\boldsymbol{k} \tag{8.6}$$

where v_x, v_y, and v_z are the projections of the velocity vector on the corresponding coordinate axes. Thus, the projections of the velocity on the coordinate axes are equal to the derivatives of the coordinates x, y, and z of the moving point with respect to time. The velocity of motion of a point in space can also be thought of as the sum of the two velocities $v_x\boldsymbol{i} + v_y\boldsymbol{j}$ and $v_z\boldsymbol{k}$, the former being the velocity of motion of the projection of the point onto the (horizontal) coordinate plane xOy and the latter the velocity of the projection of the point on the (vertical) z-axis (see Fig. 19*b*). For instance, we can say that the velocity of an airplane has a horizontal component and a vertical one.

The absolute value v of the velocity is expressed in terms of its projections by the formula

$$v = |\boldsymbol{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2} \tag{8.7}$$

whose validity is readily seen from Fig. 19. In the general case when a point moves arbitrarily in space all the three projections of its displacement vector vary and its velocity changes both in the absolute value and in the direction.

9. Acceleration of a Particle in Plane Motion. Centripetal Acceleration

The acceleration of a moving body is a physical quantity characterizing the rate of change of the velocity at a given moment. As an example, let us consider a car going through the gate (Fig. 20) with velocity v at time t. For the sake

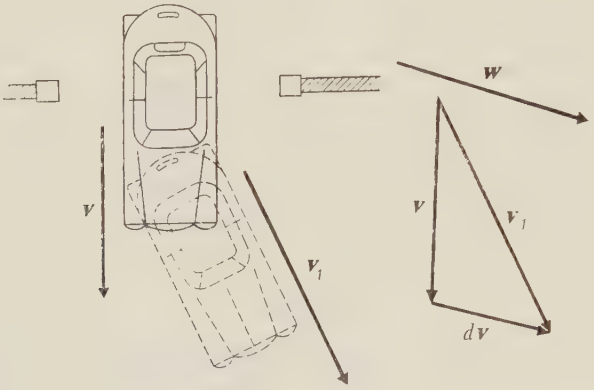

FIG. 20

of visuality, the velocity vector is placed in the figure near the car and is also depicted separately in the right-hand art of the figure (it is of course meant that the length of the line segment representing the velocity vector is measured in certain units corresponding to the chosen unit of velocity). At a new moment of time $t + dt$ the value of the velocity is $\boldsymbol{v}_1 = \boldsymbol{v} + d\boldsymbol{v}$ where $d\boldsymbol{v}$ is the (vectorial) increment of the velocity corresponding to the time interval dt.

We can also say that during the time interval dt the vector $\boldsymbol{v}$ gains an increment equal to the vector $d\boldsymbol{v}$, and this results in the new velocity $\boldsymbol{v}_1$ of the car. In the general case the direction of the vector $d\boldsymbol{v}$ does not coincide with that of the velocity $\boldsymbol{v}$.

When dt is sufficiently small the increment $d\boldsymbol{v}$ is also a very small (vector) quantity, and the ratio

$$\boldsymbol{w} = \frac{d\boldsymbol{v}}{dt} \tag{9.1}$$

is a new *vector* physical quantity which is called the *acceleration* of the moving body, its direction coinciding with that of $d\boldsymbol{v}$. The acceleration characterizes the rate and the direction of change of the velocity vector. To find the (vectorial) value $\boldsymbol{w}$ of the acceleration we must determine the increment of the vector $\boldsymbol{v}$ for an infinitesimal time interval dt and then divide that increment by dt.

In a uniform motion of a point in a circle the absolute value of the velocity is invariadle while its direction continuously changes. Consequently in this

case the velocity is not constant and gains an increment. Let us consider two values of the velocity corresponding to two moments of time differing by a small quantity Δt (Fig. 21). On subtracting the velocity at time $t + \Delta t$ from the velocity at time t we obtain the increment $\Delta \boldsymbol{v}$. To find the increment of the velocity geometrically we construct the directed line segments representing (in their directions and in their magnitudes measured in the corresponding units) the vector $\boldsymbol{v}$ (for time t) and the vector $\boldsymbol{v}_1$ (for time $t + \Delta t$). The directions of these vectors coincide with the directions of the tangent lines to the circle at the point through which the moving point passes at the corresponding moments of time. Then we subtract the vector $\boldsymbol{v}$ from the vector $\boldsymbol{v}_1$

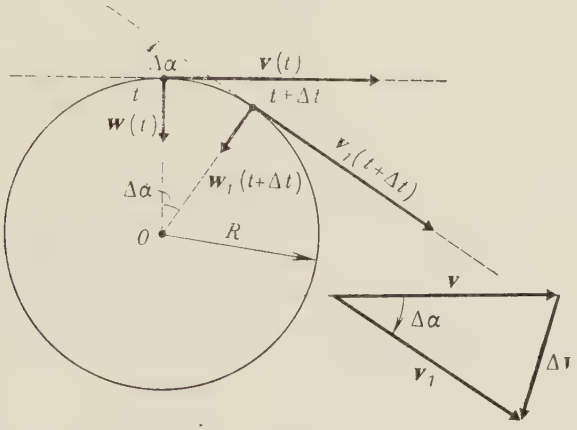

FIG. 21

to obtain the vector $\Delta \boldsymbol{v}$. It is apparent that the vector $\Delta \boldsymbol{v}$ is not perpendicular either to the initial value $\boldsymbol{v}$ of the velocity or to the terminal value $\boldsymbol{v}_1$. However, when $\Delta t \to 0$ we also have $\Delta \boldsymbol{v} \to 0$, and, in the limit, the direction of the vector $\Delta \boldsymbol{v}$ tends to become perpendicular to the velocity vector $\boldsymbol{v}$. Hence, a sufficiently small increment $d\boldsymbol{v}$ of the vector $\boldsymbol{v}$ is perpendicular to the latter, and therefore the acceleration $\boldsymbol{w} = \frac{d\boldsymbol{v}}{dt}$ is perpendicular to the velocity and is directed towards the centre of the circle.

The value of the acceleration can be expressed in terms of the absolute value v of the velocity $\boldsymbol{v}$ of the circular motion and the radius R of the circle. From Fig. 21 we see that for sufficiently small $\Delta\alpha$ the approximate relation

$$| \Delta \boldsymbol{v} | = \Delta v \approx v \Delta \alpha \tag{9.2}$$

holds, and the distance travelled by the moving point in time Δt is equal to

$$v \, \Delta t \approx R \, \Delta \alpha \tag{9.3}$$

On eliminating $\Delta\alpha$ from two equations (9.2) and (9.3) we obtain

$$\Delta v \approx \frac{v^2}{R} \Delta t \tag{9.4}$$

whence

$$\frac{\Delta v}{\Delta t} \approx \frac{v^2}{R}$$

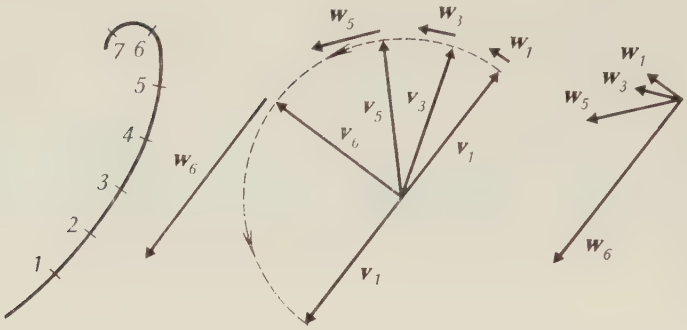

FIG. 22

and thus $$\frac{dv}{dt} = \lim_{\Delta t \to 0} \left(\frac{\Delta v}{\Delta t} \right) = \frac{v^2}{R} \tag{9.5}$$

We see that the absolute value of the acceleration of a body moving uniformly in a circle is equal to the square of the absolute value of the velocity divided by the radius of the circle. This acceleration is directed towards the centre of the circle and is called the *centripetal* acceleration.

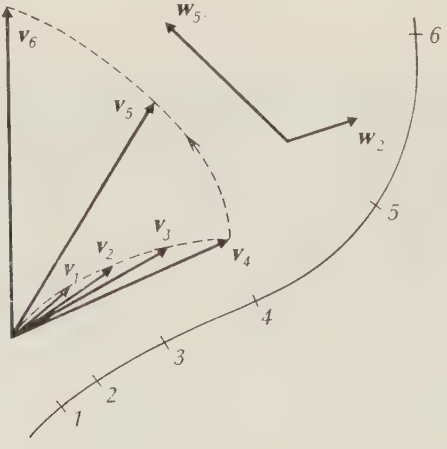

FIG. 23

We now investigate an arbitrary motion of a point along a plane with a velocity having a constant modulus. As an example, let us consider the motion of the point shown in Fig. 22; the absolute value of the velocity is constant and the point moves from position *1* to position *7*. Near the trajectory in the figure are depicted the velocity vectors v_i for different moments t_i, and in the right-

most part of the figure are shown the acceleration vectors $\boldsymbol{w}_i$ for the same moments of time. As is seen, the acceleration increases together with the curvature of the trajectory. The acceleration is at all the points of the trajectory perpendicular to the velocity since the motion along the trajectory is uniform.

Compare this case with the one shown in Fig. 23 where the velocity of a point moving along a plane curve increases in its absolute value. Near the trajectory of motion in Fig. 23 are shown the velocity and the acceleration vectors for the moments of time at which the moving point passes through the positions marked on the trajectory. As in Fig. 22, the positions *1*, *2*, *3*, . . . in Fig. 23 correspond to equally spaced moments of time.

In the latter case the acceleration is no longer perpendicular to the velocity but forms an angle different from 90° with the velocity vector. That is why the acceleration vector is usually considered as being resolved into two components one of which goes along the tangent to the trajectory and is called the tangential acceleration while the other is perpendicular to the former and is called the *normal acceleration.*

If the laws according to which the absolute value and the direction of the velocity vary as functions of time are known it is possible to determine the acceleration for any moment of time.

Like any vector, the acceleration can be represented in the form of the sum of its components along the coordinate axes. For example, if the acceleration vector lies in the xy-plane it can be written in the form

$$\boldsymbol{w} = w_x\boldsymbol{i} + w_y\boldsymbol{j} \tag{9.6}$$

where w_x and w_y are the projections of the vector $\boldsymbol{w}$ on the coordinate axes and, as before, the vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ are the unit vectors of the corresponding coordinate axes x and y.

The projections w_x and w_y of the acceleration are equal to the derivatives of the corresponding projections of the velocity with respect to time:

$$w_x = \frac{dv_x}{dt} \text{ and } w_y = \frac{dv_y}{dt} \tag{9.7}$$

Indeed, let the velocity vector be

$$\boldsymbol{v} = v_x\boldsymbol{i} + v_y\boldsymbol{j}$$

Since $\boldsymbol{i}$ and $\boldsymbol{j}$ are constant the differentiation of both members of this relation results in

$$\frac{d\boldsymbol{v}}{dt} = \frac{dv_x}{dt}\boldsymbol{i} + \frac{dv_y}{dt}\boldsymbol{j} \tag{9.8}$$

On comparing (9.8) with (9.6) we obtain (9.7).

Example 1. Let the reader investigate the time dependence of the projections of the velocity and of the acceleration on rectangular coordinate axes when the point under consideration is in a uniform motion with velocity v_0 along a circle of radius R. In Fig. 24 we see the graphs of the projections of the velocity and of the acceleration for this motion.

Example 2. Let us determine the acceleration of a point moving in a circle of radius R when the absolute value of the velocity varies according to the law $v = bt$ (b = const).

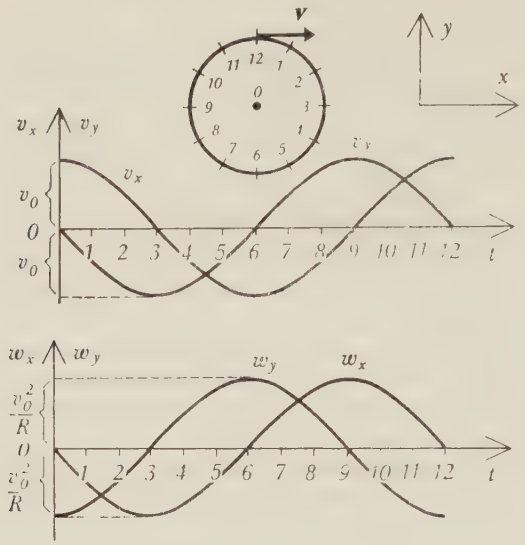

FIG. 24

Let the velocity have a value v_1 at time t and a value v_2 at time $+ \Delta t$ (see Fig. 25*a*). The vector v_2 is turned through a very small angle $\Delta\alpha$ relative to v_1 and has a greater absolute value. Let us mark a point A on the line segment representing the vector v_2 so that $OC = OA$. If the velocity were constant in its absolute value it would be represented by the vector $\overrightarrow{OA}$ at time $t + \Delta t$; consequently the increase of the absolute value of the velocity during time Δt

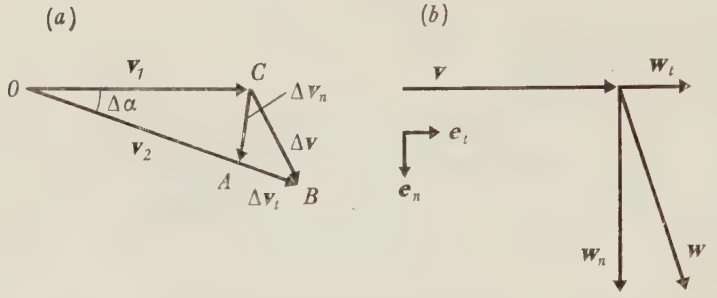

FIG. 25

is equal to the absolute value of the vector $\overrightarrow{AB}$: $|\overrightarrow{AB}| = \Delta v_t$. This increase is obviously equal to

$$v_2 - v_1 = b\,(t + \Delta t) - bt = b\,\Delta t$$

whence

$$\Delta v_t = b\,\Delta t \tag{9.9}$$

The increment of the velocity during the time interval Δt, which is equal to $\Delta \boldsymbol{v} = \boldsymbol{v}_2 - \boldsymbol{v}_1$, can be represented as the sum of the two vectors $\Delta \boldsymbol{v}_n$ and $\Delta \boldsymbol{v}_t$ (Fig. 25*a*), that is

$$\Delta \boldsymbol{v} = \Delta \boldsymbol{v}_n + \Delta \boldsymbol{v}_t$$

The absolute value of the vector $\Delta \boldsymbol{v}_n$ is equal to $\Delta v_n \approx \Delta\alpha \cdot v$ and, as in the case of a uniform motion (see formula (9.4)), it can be written in the form

$$\Delta v_n \approx \frac{v^2}{R} \Delta t \tag{9.10}$$

The average acceleration during the time interval Δt is

$$\frac{\Delta \boldsymbol{v}}{\Delta t} = \frac{\Delta \boldsymbol{v}_n}{\Delta t} + \frac{\Delta \boldsymbol{v}_t}{\Delta t}$$

and the (instantaneous) acceleration at time t is equal to

$$\frac{d\boldsymbol{v}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{v}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{v}_n}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{v}_t}{\Delta t} \tag{9.11}$$

Let us consider the limits of the absolute values of the vectors on the right-hand side of (9.11) as $\Delta t \to 0$. From formulas (9.9) and (9.10) we find

$$\frac{\Delta v_t}{\Delta t} = b \text{ and } \frac{\Delta v_n}{\Delta t} \approx \frac{v^2}{R}$$

whence it is seen that in the limit, as $\Delta t \to 0$, we have

$$\frac{dv_t}{dt} = b \text{ and } \frac{dv_n}{dt} = \frac{v^2}{R} \tag{9.12}$$

Now let us consider the limiting behaviour of the directions of the vectors $\Delta \boldsymbol{v}_n$ and $\Delta \boldsymbol{v}_t$; in the limit, when $\Delta t \to 0$ (that is, when $\Delta\alpha \to 0$) the direction of the vector $\Delta \boldsymbol{v}_n$ tends to that of the perpendicular to $\boldsymbol{v}$ while the direction of $\Delta \boldsymbol{v}_t$ tends to that of the vector $\boldsymbol{v}$ (see Fig. 25*b*). If we denote by $\boldsymbol{e}_t$ the unit vector in the direction of $\boldsymbol{v}$ (along the tangent to the trajectory) and by $\boldsymbol{e}_n$ the unit vector perpendicular to the former and directed towards the centre of the circle along its radius, then, by virtue of (9.12), formula (9.11) expressing the acceleration takes the form

$$\boldsymbol{w} = \frac{d\boldsymbol{v}}{dt} = \frac{v^2}{R} \boldsymbol{e}_n + b\boldsymbol{e}_t \tag{9.13}$$

The normal acceleration is $\boldsymbol{w}_n = \frac{v^2}{R} \boldsymbol{e}_n$ and the tangential acceleration is $\boldsymbol{w}_t = \frac{dv_t}{dt} \boldsymbol{e}_t = b\boldsymbol{e}_t$.

It should be noted that the normal acceleration $\boldsymbol{w}_n$ always has the same value as in the case of a uniform motion. The vector $\boldsymbol{w}$ is not directed towards the centre of the circle and forms an acute angle with the vector $\boldsymbol{v}$ for $b > 0$ (an accelerated motion) and an obtuse angle for $b < 0$ (a decelerated motion).

The tangential acceleration $\boldsymbol{w}_t$ determines the rate of change of the absolute value of the velocity of motion and the normal acceleration $\boldsymbol{w}_n$ characterizes the rate of change of the direction of the velocity $\boldsymbol{v}$ of the moving point.

10. Acceleration of a Particle in Three-dimensional Motion

In the foregoing section we considered the basic relations connecting the velocity and acceleration vectors with their projections only for the case when both vectors are all the time in one plane (a plane motion).

The case when a point moves in space arbitrarily and its velocity has an arbitrary orientation can be treated in a completely analogous way. In this case the motion of the particle is regarded relative to a frame of reference specified by orthogonal coordinates x, y, and z, and therefore the acceleration, like the velocity, can be resolved into components along the three coordinate axes:

$$\boldsymbol{w} = w_x\boldsymbol{i} + w_y\boldsymbol{j} + w_z\boldsymbol{k} \tag{10.1}$$

where $\boldsymbol{k}$ is the unit vector of the z-axis.

The relationship between the projections of the velocity and of the acceleration is given by the formulas

$$w_x = \frac{dv_x}{dt}, \quad w_y = \frac{dv_y}{dt} \text{ and } w_z = \frac{dv_z}{dt} \tag{10.2}$$

A spatial motion can be considered in any other coordinate systems (including curvilinear ones); such analysis is connected with the investigation of the projections of the velocity and of the acceleration on the corresponding coordinate axes. In our course we shall most often deal with rectangular coordinates.

If the three functions $x(t)$, $y(t)$, and $z(t)$ describing the variation of the coordinates of the moving point are known the differentiation of each coordinate gives us the corresponding projection of the velocity and thus the velocity vector can be completely determined. Further, on differentiating the projections of the velocity on the coordinate axes we obtain the projections of the acceleration vector and thus determine the acceleration completely.

The inverse operations, for instance, the operation of finding the coordinates of the moving point from a given expression of the velocity, are more intricate; the latter operation reduces to the integration of the projections of the velocity with respect to time. If it is required to determine the coordinates from a given expression of the acceleration we should perform the integration twice: the integration of the projections of the acceleration results in the projections of the velocity and the integration of the projections of the velocity gives us the coordinates of the moving point as functions of time.

Example. Let a point move in space with a constant acceleration $\boldsymbol{w}$ (see Fig. 26). Let the initial position of the point at the moment $t = 0$ be at a point lying on the z-axis and having the coordinates $x_0 = 0$, $y_0 = 0$, and $z_0 \neq 0$, the initial velocity $\boldsymbol{v}_0$ being parallel to the xy-plane and having the projections $v_{0z} = 0$, $v_{0y} \neq 0$, and $v_{0x} \neq 0$. Suppose that the acceleration is directed opposite to the z-axis, that is

$$w_x = 0, \quad w_y = 0, \quad \text{and} \quad w_z = -a \tag{10.3}$$

where a is a positive number. The projections of the acceleration on the axes x and y are identically equal to zero and consequently the projections of the velocity on these axes are constant numbers, that is

$$v_x = v_{0x} \text{ and } v_y = v_{0y} \tag{10.4}$$

for any time t. It is only the projection of the velocity on the z-axis that varies. Using (10.2) we can write

$$\frac{dv_z}{dt} = w_z \text{ and } v_z = \int_0^t w_z\, dt \tag{10.5}$$

The projection $w_z = -a$ of the acceleration remains constant all the time and therefore the projection of the moving point on the z-axis is in a uniformly

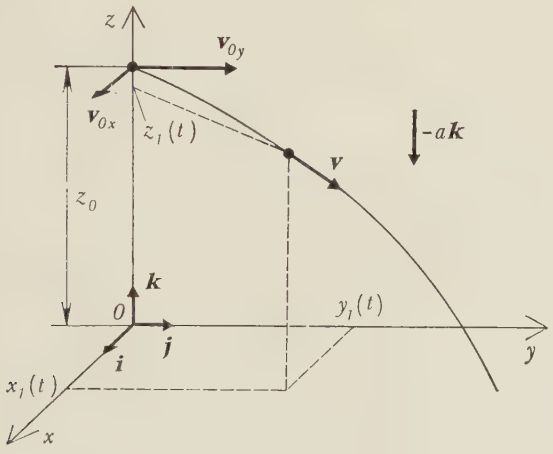

FIG. 26

accelerated motion. The coordinate z is determined as follows:

$$\frac{dz_1}{dt} = v_z = -\int_0^t a\, dt = -at \text{ and } z_1 - z_0 = \int_0^t v_z\, dt = -\int_0^t at\, dt = -\frac{at^2}{2} \tag{10.6}$$

The other two coordinates x and y of the moving point are found from the condition that the projections of the velocity on the x- and y-axes remain constant:

$$\frac{dx_1}{dt} = v_x = v_{0x} \quad \text{and} \quad \frac{dy_1}{dt} = v_y = v_{0y}$$

whence

$$x_1 = \int_0^t v_{0x}\, dt = v_{0x}t \text{ and } y_1 = \int_0^t v_{0y}\, dt = v_{0y}t \tag{10.7}$$

Thus, expressions (10.6) and (10.7) describe completely the variation of all the three coordinates of the point moving with a constant acceleration $w_z = -a$.

A motion of a point with a constant acceleration $\boldsymbol{w}$ directed arbitrarily can be described in vector form. Let us suppose that at any time t the moving point is at the terminus of a vector $\boldsymbol{r}(t)$ dependent on time whose origin is at a fixed point O.

The velocity of the point is $v = \frac{dr}{dt}$ and the acceleration is $w = \frac{dv}{dt}$ whence $dv = w\,dt$. Since $w = \text{const}$ we have

$$v - v_0 = \int_0^t w\,dt = wt \tag{10.8}$$

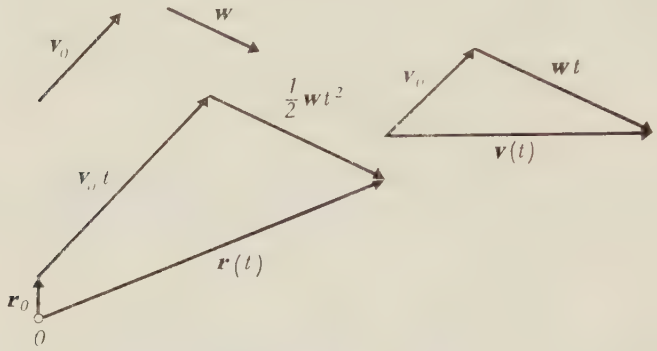

FIG. 27

where v_0 is the initial velocity (at time $t = 0$), that is

$$v = v_0 + wt$$

Using the equality $dr = v\,dt$ we can write the relation

$$r - r_0 = \int_0^t v\,dt$$

where r is the position vector of the point at time t and r_0 specifies the initial position of the point for $t = 0$. Taking into account the expression obtained for v we obtain

$$r = w\int_0^t \xi\,d\xi + v_0\int_0^t d\xi + r_0$$

whence

$$r = \frac{1}{2}wt^2 + v_0 t + r_0 \tag{10.9}$$

Equality (10.9) is the general vector formula describing the motion of a point (particle) with a constant acceleration w starting from a point r_0 with an initial velocity v_0. This relation describes completely kinematics of the motion: the velocity increases linearly as function of time in the direction of the vector w, the vector r increases linearly in the direction of v_0 and quadratically in the direction of w. In Fig. 27 all these vectors are shown for the case when the given vectors r_0, v_0, and w lie in one plane.

This discussion concludes the general consideration of the laws of kinematics of a particle moving in space. The position vector (the radius vector r), the

velocity and the acceleration are the quantities whose variation determines the motion completely. If the dependence on time of one of these vector quantities is known we can find the other two vectors and also the coordinates of the moving point.

As will be seen, the knowledge of the laws governing the motions of a particle is extremely important for the investigation of an arbitrary motion of any body.

11. Space, Time, and Frames of Reference

So far we have not discussed the notions of space and time and simply relied on the reader's intuitive understanding and on his knowledge of the elementary course of physics.

We sometimes simply say that a mechanical motion reduces to a displacement of a body in space with time but it would be more correct to say that by a mechanical motion we mean the displacements of bodies relative to some other bodies. It is senseless to speak of a motion of a single isolated body because in reality we always observe a motion of one body relative to some other bodies. That is why when considering a motion we always choose a *frame of reference* connected in a certain manner with some body or system of bodies. Space cannot be thought of without physical bodies.

The concept of an *absolute* space, motionless and irrelevant to the existence of substance, which is used in classical Newtonian mechanics has no sense. The notion of such a space and of its geometrical elements (such as points, lines, surfaces and solids) appeared as an abstraction applicable to material bodies whose properties are almost invariable. In the Newtonian mechanics space is supposed to be *homogeneous* in all its parts and *isotropic* (the latter means that the properties of the space are independent of the directions, that is they are invariable for all directions). In other words, it is supposed that the physical space is the one that is dealt with in Euclidean geometry. It turns out that for most of the physical phenomena treated in the present course the physical space can be regarded with high accuracy as a Euclidean space, and therefore the concept of a homogeneous and isotropic physical space is an adequate approximation to the reality for such phenomena. However, the idea that there exists an absolutely motionless space irrelevant to the existence of any bodies is incorrect; we always think of space as being connected with certain bodies, that is with some frames of reference.

In Newtonian mechanics time is understood as a measure of *absolute duration* which exists irrespective of physical bodies. This is also only an approximation to reality since duration cannot be separated from substance because time is a form of the existence of substance.

If we limit ourselves to one definite system of reference then it is possible to consider a common measure of duration for all phenomena and processes, that is to speak of a common time. However, as was shown in Einstein's relativity theory, events that are simultaneous and take place in different regions relative to one frame of reference are not simultaneous if we consider them relative to another frame of reference moving relative to the former. Consequently, the course of time is connected with relative motion of frames of reference and there is no common, absolute time for all the frames of reference. All that has been said is a consequence of the fundamental fact that the velocity of light is constant, invariable for all the systems of reference. Thus, the duration of

processes is connected with motions of the bodies involved, and the notion of time cannot be separated from relative motions of physical bodies.

But it turns out that for those motions which are slow (in the sense that their velocities are very small compared with the speed of light) the dependence of time on relative motions of frames of reference is practically insignificant and can be neglected. Therefore, for almost all phenomena and problems considered in the present course we can use the Newtonian notion of absolute time which is common for all systems of reference. The cases when such an approximation is inapplicable will be carefully stipulated.

CHAPTER 2

Fundamental Laws of Dynamics

12. Motion and Interaction of Bodies

Wh n observing and investigating motions of various bodies we see that the motion of a given body is caused or changed by interaction between that body and other bodies. For instance, let a worker start moving a cart, roll it and then stop it. In this example the worker produces the motion of the cart relative to the ground and it is he who stops the motion of the cart. As other examples of this kind we can consider an electric locomotive moving a train or a man throwing a stone. In the latter example the man produces the motion of the stone which moves relative to the ground and then stops because of its interaction with the air and the ground. We can give a lot of similar examples, and in all cases we observe an interaction of at least two bodies. For instance, in the example of the worker moving the cart there are interactions of three bodies: the worker, the cart and the ground (see Fig. 28). When the worker rolls the cart or stops it his feet and the ground interact in a certain way.

The laws of dynamics establish the relationship between the motion of a body and the causes producing or changing the motion.

The study of the laws of mechanics should be started with the investigation of simple motions which can be observed directly. Therefore, we shall begin with considering motions of bodies relative to the Earth's surface and assuming (at the moment, without an adequate justification) that the Earth is a fixed body. We shall first establish the general laws of these motions and then proceed to conclusions concerning the influence of the relative motion of the Earth round the Sun on the motions under consideration and to the experimental verification of these conclusions. A number of experiments which will be discussed later confirm the applicability of the laws we have found to arbitrary motions of bodies. That is why the dynamical laws established for motions of bodies relative to the Earth's surface can be extended to motions of celestial bodies. Based on this generalization, we can perform theoretical calculations and then again verify their consequences by observation. Getting ahead of what follows, we mention here that Newton himself performed such a verification, and since then physical investigations and astronomic observations have perfectly confirmed the validity of the laws of dynamics thus derived.

Motions of bodies relative to each other can be of different character which may vary in the general case. Such variations may develop in different ways; for instance, a body can first be at rest, then begin moving with a certain velocity and finally stop, that is arrive at another state of rest. As another example of this kind we can consider a body moving with a constant velocity which then increases because of some action, that is the body begins to move faster.

A body may first move eastward and then change its motion and move southward and so on.

In all the cases a change of a motion of a body is first of all accompanied by some definite variations of the velocity of the body; the velocity changes its absolute value or direction or both. We can therefore conclude that interactions causing changes of the velocity of a body are in a certain relationship with the velocity.

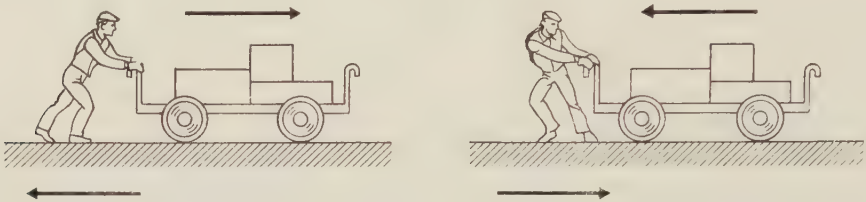

FIG. 28

This is the first conclusion we come to when investigating motions of bodies. This conclusion can also be stated as follows: *it is the actions of other bodies on the given body that determine the velocity of the latter.*

From Aristotle's *Physics* we know that such a formulation of the basic law of mechanical motion was familiar to the ancients. Aristotle classified all the motions into "natural" and "compulsory" ones; into the first group he included the upward and downward motions and into the second group all the rest.

The idea of a relationship between external actions and the velocity of a body is quite correct if it is understood in the following sense: there is a law according to which the external actions specify the velocity of the body. However, the conclusion that an action exerted upon a body at a given moment determines the velocity of the body at that very moment (which can be drawn from Aristotle's ideas) is absolutely wrong. G. Galileo disproved this conclusion on the basis of a thorough analysis of the results of his experiments and stated the following assertion: *if there are no external actions on a body it either moves with a constant velocity or is at rest.*

This assertion is the basic idea of the first law of dynamics discovered by G. Galileo; it will be treated in detail in Sec. 16.

Before proceeding to the study of the laws of dynamics we must consider the general characteristics of an action of a body upon another body; this leads to the important physical notion of a *force.*

13. Force

To lift a pail of water and to hold it in his hand a man must apply to the pail the "force" of the hand; he "feels" the force with which he pulls the pail upwards; this force is equal to the one with which the pail acts on his hand in the downward direction. When a worker pushes a loaded cart he applies the force of his hands to impart the motion to the cart, to roll it and to give the cart its velocity (see Fig. 28). In these actions the man has a feeling of a certain strain in his body; the force we have spoken of in these examples is connected with that feeling.

However, in mechanics *by force is not meant* a physiological feeling. What is understood by force in mechanics is a *physical cause changing the state of motion of bodies and resulting from an interaction of two bodies.* Thus, a physical force in mechanics should by no means be confused with the feeling of strain. For instance, a worker pushing a cart acts upon the cart with a certain force and this action is accompanied with a feeling of strain in his muscles; however the motion of the cart is governed by a certain law connected with the magnitude of the force applied by the worker to the cart and not with his feeling. If the same force is applied to the cart by another body (for example, by a tractor) the motion of the cart will be the same.

The force of gas pressure produced by the explosion of powder pushes the shell out of the barrel of the gun, the force exerted by the locomotive moves the cars of the train, the force produced by the flow of the river rotates the water wheel, etc. In all such examples we speak of a force as a cause producing or changing the motion of a body.

So far we have stressed that a physical force is connected with a change of the motion of a body; but there is a special case frequently encountered when the action of a force does not change the motion of a body, or, more precisely, does not cause a change of the motion observable in ordinary circumstances. For instance, if to a cart is applied a force whose magnitude is not sufficiently large to produce motion, the force is nevertheless acting on the cart. When a man holds a pail of water he acts with a definite force on it but the pail remains at rest.

In all such cases there are some other forces whose action "eliminates" that of the force under consideration.

If for some time period there is only one force acting on a body the latter cannot be at rest; this readily follows from the first law of dynamics. On the other hand, if a body is in a state of rest all the forces acting upon it are in an *equilibrium*, that is they *balance* each other and their sum is equal to zero. It should be noted that although a body is at rest under the action of mutually balancing forces it undergoes a *deformation*, that is it changes its form under that action. In the cases when there is a relationship between the deformation and the forces uniquely determined and governed by a certain law, the deformation of some bodies (such as a spring, a dynamometer, and the like) can serve as a measure of the forces.

The division of mechanics studying the laws of equilibrium of forces acting upon bodies is called *statics.* The laws of statics are a special case of the laws of dynamics, the former being essentially simpler. That is why the study of mechanics is sometimes begun with statics in accordance with the history of the development of mechanics since the laws of statics were discovered much earlier than the general laws of dynamics.

14. Measuring Constant Forces

Forces considered in statics do not vary and therefore their determination and measurement is simpler. In statics as a measure of a force is usually taken the force of weight of a certain body. *By the force of weight* or, simply, *weight is meant a force with which a body at rest acts on another body preventing the former from falling.*

If a body is at rest relative to the Earth it acts upon a post (a support) by a definite force characteristic of that body. As early as ancient times the man learned that the forces of weight of various bodies can be measured by compar-

ing them (with the aid of a balance) with the force of weight of a definite body whose weight was taken as a unit.

As is known, the simplest beam balance consists of an equal-arm lever (rotating freely about its midpoint) from whose ends the bodies whose weights should be compared are suspended (such a balance is schematically depicted in Fig. 29). If the equal-arm lever from which the bodies are suspended occupies the horizontal position and is in equilibrium the weights of the bodies suspended from the opposite ends of the lever are equal. In principle, this allows us to measure the force of weight of any body.

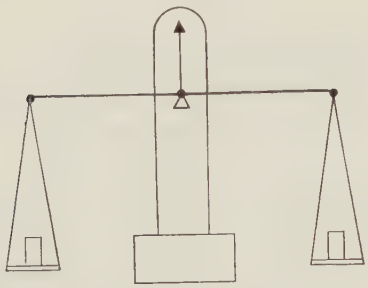

FIG. 29

By international agreement, as a unit of force was taken the weight of a definite standard rod made of platinum-iridium alloy which is preserved in Sévres (near Paris). Since the force of weight of a body depends on the geographical location of the place where that force is measured, the unit force is understood as the weight of the standard rod in the place where it is preserved. This unit of force is called the *kilogram* (*force*) and denoted kgf. In the SI system is used another unit of force called the *newton* (denoted N) which is 9.80665 times less than the kilogram. In the cgs system the unit of force is the *dyne* (denoted dyn) equal to 10^{-5} N (see Sec. 18).

The force of weight is a vector physical quantity. For instance, consider a pack of cotton suspended from a rope (Fig. 30). The force of weight of the pack is directed downwards.* Its magnitude (absolute value) and direction are represented by the directed line segment $\boldsymbol{Q}$ (equipped with an arrow indicating the direction). The length of the segment (measured with the aid of a chosen unit of length representing, conditionally, the unit of force) corresponds to the magnitude of the force. In the figure, besides the force of weight, is also shown the force of tension of the rope, that is the force with which the rope acts upon the pack of cotton. The latter force is represented by the line segment $\boldsymbol{F}$. The forces $\boldsymbol{F}$ and $\boldsymbol{Q}$ are equal in their absolute values and opposite in the directions.

If we measured precisely the length of the rope before the suspension of the load and after the suspension it would turn out that the rope has received an elongation. The deformation of the rope is in a certain relationship with the force of tension. The relationship between the force and the deformation depends on the properties of the rope and on the magnitude of the deformation. In the case when the magnitude of the deformation is *uniquely* determined by the magnitude of the force acting upon the body we speak of an *elastic* deformation.

As an example of a body admitting considerably large elastic deformations we can take a steel spring. Attaching one of the ends of the spring to the post (Fig. 31) and suspending loads of various weights from the other end we can measure the deformation (elongation) of the spring with the aid of a fixed scale. If the elongation Δl of the spring is determined solely by the weight of the suspended load (under the other conditions remaining invariable) the defor-

* This force is applied to the rope (see Secs. 46, 92).

mation of the spring is elastic. That is why a steel spring is the basic element of devices measuring the magnitudes of various forces. Such instruments are called *spring balances* (*dynamometers*).

The diagram representing the dependence between the force F extending the spring of the dynamometer and the elongation Δl of the spring is shown in Fig. 32*a*. The dependence between the force and the deformation is represented by a straight line in this figure irrespective of what force was applied to

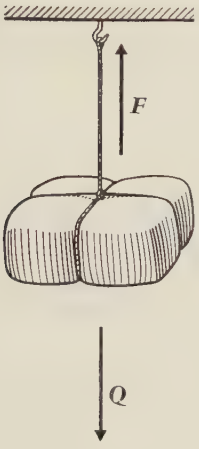

FIG. 30

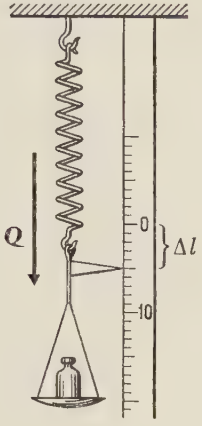

FIG. 31

the spring before the suspension of the load; its deformation Δl_1 is determined solely by the force F_1 acting at the given moment, the magnitude of the force F_1 being directly proportional to the deformation Δl_1.

A deformation of a body is called *inelastic* if the dependence between the force and the deformation cannot be represented by one line. In the case of an inelastic spring the investigation of the dependence of the elongation on the force, as the magnitude of the force is increased, results in a curve A (Fig. 32*b*); to the magnitude F_1 of the force there corresponds the elongation Δl_1 and to the

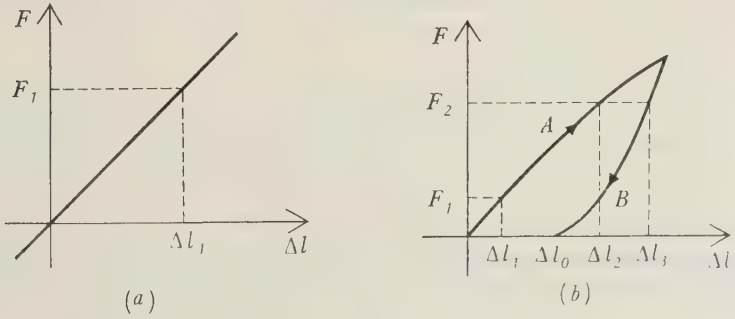

FIG. 32

magnitude F_2 the elongation Δl_2. Proceeding in this way we arrive at a maximum load, that is to a maximum magnitude of the force (it is not marked in Fig. 32*b*). On decreasing the load from its maximum value and marking the magnitudes of the force and the corresponding elongations, we receive an essentially different curve B showing that in this case to the force F_2 there corresponds an elongation Δl_3 and to the force F_1 an elongation Δl_2. After the load has been removed the elongation Δl_0 of the spring does not become equal to zero; the elongation Δl_0 is called the *residual* deformation. A spring of this kind cannot be used for measuring forces; therefore, in the dynamometers are always used elastic steel springs for which the dependence between the force and the deformation is determined uniquely and the residual deformation is negligibly small.

15. Equilibrium Conditions for Forces Acting on a Particle

With the aid of a set of loads (a set of weights) we can calibrate spring dynamometers. Unless some special measures are taken, the precision of spring balance measurements of forces is usually not very high and is much less than the precision of lever balance measurements. However, since it is easier and more convenient to deal with spring balances, they are frequently used both in engineering and in physical experiments.

Given one calibrated spring dynamometer, we can calibrate all the dynamometers we have. For this purpose, we join the ends of the springs of two dynamometers with a thread and then gradually stretch it. At any equilibrium state both dynamometers must show the same data. When the whole system is in an equilibrium state every part of the thread joining the dynamometers is acted upon by two forces $\boldsymbol{F}$ and $\boldsymbol{F}_1$ equal in the magnitudes and opposite in the directions. The vector equality expressing the equilibrium of the forces is written as

$$\boldsymbol{F} + \boldsymbol{F}_1 = 0 \tag{15.1}$$

Knowing the graduation mark of one dynamometer we can mark the values of the corresponding forces on the scale of the other dynamometer.

If one end of a dynamometer is attached to a fixed body with a thread (for instance, to a nail driven into the table as shown in Fig. 33) it is possible to determine the magnitude and the direction of the force with which the hand stretching the dynamometer acts on the nail (the direction of the force coincides with that of the thread). Thus, to specify a force one must indicate not only its magnitude but also the direction of its action in space; this means that *a force is a vector quantity.*

If there are a number of forces acting on one point in different directions their action can be replaced by the action of one force called the *resultant* of the given system of forces. As in the case of arbitrary vectors, the absolute value and the direction of the resultant are found according to the general addition rule for vectors.

What has been said can be confirmed by simple experiments with dynamometers. Let us take several calibrated dynamometers and a sheet of paper placed on the table from which a nail driven into the table rises. Let us put on the nail a small ring with three threads attached to it (see Fig. 34). On attaching the ends of two threads to the ends of the dynamometers B and C and stretching them we fix the opposite ends of the dynamometers. In this case the ring at

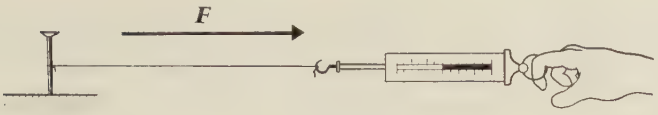

FIG. 33

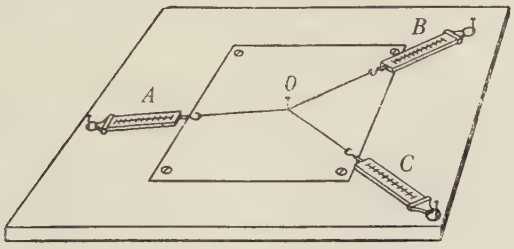

FIG. 34

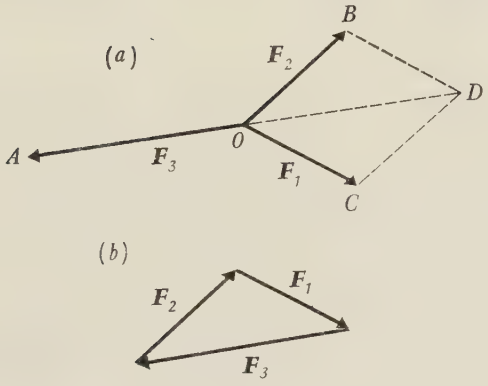

FIG. 35

the point O is acted upon by two forces forming a certain angle; such forces can never balance each other: if we remove the nail from the table the ring will no longer be in equilibrium.

We now draw on the sheet of paper the lines of action of the forces and set off along them line segments OB and OC (Fig. 35*a*) whose lengths (measured in certain units) represent the data shown by the corresponding dynamometers B and C. Now let us construct the line segment OD equal to the diagonal of the parallelogram with sides OB and OC and extend the line OD through the point O. Next we attach to the third thread the third dynamometer A, direct that thread along the line OD, stretch it with a force of magnitude equal to the segment OD and then fix the other end of the dynamometer A to the table (see Fig. 34). On removing the nail from the table we see that the ring remains in equilibrium. Consequently, the force measured by the dynamometer A balances the action of the forces measured by the dynamometers B and C. The three

vectors representing the forces F_1, F_2, and F_3 form a closed triangle shown in Fig. 35*b*. The fact that the resultant of the forces acting on the point O is equal to zero can be expressed in the vector form:

$$F_1 + F_2 + F_3 = 0 \tag{15.2}$$

When the sum of three forces acting on a point is equal to zero the force directed opposite to any of the three forces and equal to it in its absolute value is equal to the sum (the resultant) of the other two forces.

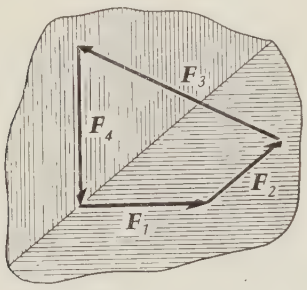

FIG. 36

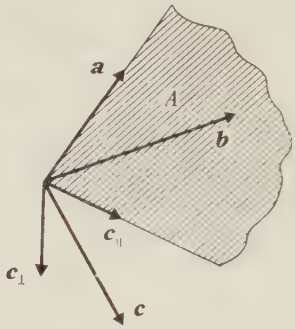

FIG. 37

It should be noted that an experiment completely analogous to the one shown in Fig. 34 can be performed with the aid of pulleys and weights replacing the spring dynamometers. In the latter experiment special attention should be paid to the quality of the pulleys: they must have small axial friction for the results to be sufficiently precise.

If a fourth force lying in the plane of the three former forces (see Fig. 34) is additionally applied to the point O the forces of tension of the threads can always be chosen so that the actions of the four forces acting on the ring are mutually balanced. In this case the vector diagram of the forces is represented by a closed quadrangle.

In just the same way we can add a fifth force and, generally, several forces; and number of forces acting on a point and lying in one plane can be chosen and arranged so that they are in equilibrium.

The general equilibrium condition reads: *in equilibrium of forces lying in one plane and acting on one point the vectors representing the forces form a closed polygon.*

The same condition is valid for forces acting on a point and not lying in one plane but in this case the vector polygon is a spatial figure (see Fig. 36).

Two forces acting on a point and not lying in one line can never balance each other. Similarly, three forces not lying in one plane can never be in equilibrium. Indeed, let three forces $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ (Fig. 37) act on one point and not lie in a common plane. Through the lines of action of any two of the forces a plane can be drawn. Let A be the plane containing the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Then the vector $\boldsymbol{c}$ can be resolved into two components $\boldsymbol{c}_\perp$ and $\boldsymbol{c}_{||}$, the former being perpendicular to the plane A and the latter lying in that plane: $\boldsymbol{c} = \boldsymbol{c}_\perp + \boldsymbol{c}_{||}$. The resultant (the sum) of the three forces $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}_{||}$ is in the plane A and

cannot be balanced by the force $\boldsymbol{c}_{\perp}$ directed perpendicularly to that plane. Four (and more) forces acting on one point and not lying in one plane can be in equilibrium. Generally, forces acting on a point are in equilibrium if and only if the vectors representing the forces form a closed (spatial, in the general case) polygon. The projection of a closed vector polygon on any axis is always equal to zero, therefore the former equilibrium condition for forces acting on one point (concurrent forces) can be restated in the following equivalent form: *in the case of equilibrium the sum of the projections of all the forces onto each of the three mutually perpendicular coordinate axes must be equal to zero.*

Let some forces $\boldsymbol{F}_1, \boldsymbol{F}_2, \boldsymbol{F}_3, \ldots, \boldsymbol{F}_n$ act on one point, their projections on the x-axis being $X_1, X_2, X_3, \ldots, X_n$ respectively. Analogously let the projections of the forces on the y-axis be $Y_1, Y_2, Y_3, \ldots, Y_n$ and the projections on the z-axis be $Z_1, Z_2, Z_3, \ldots, Z_n$. Then the equilibrium conditions can be written as

$$\left.\begin{aligned} X_1+X_2+X_3+\ \ldots\ +X_n=\sum_{i=1}^{n} X_i=0 \\ Y_1+Y_2+Y_3+\ \ldots\ +Y_n=\sum_{i=1}^{n} Y_i=0 \\ Z_1+Z_2+Z_3+\ \ldots\ +Z_n=\sum_{i=1}^{n} Z_i=0 \end{aligned}\right\} \tag{15.3}$$

where the sign $\sum_{i=1}^{n}$ symbolizes the summation of any quantities supplied with the index i as the index assumes all the values 1, 2, 3, . . ., n. The three conditions (15.3) are completely equivalent to one vector equality

$$\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 + \ \ldots\ + \boldsymbol{F}_n = 0 \tag{15.4}$$

If the equilibrium condition for all forces acting on a point is not satisfied the forces produce a motion of the point. The relationship between the action of given forces on a material point and the motion of the point will be studied in the following sections.

It should be noted that in the consideration of the action of a force on a body regarded as a material point we do not take into account any properties of the body. We only (tacitly) assume that the body is not destroyed under the action of the given forces. Although the body undergoes deformations under the action of the forces we do not take into account these deformations.

Now let us discuss in which way a constant force acts on a fixed body. The action of the force produces definite deformations of various parts of the body. During the time interval following the moment of the application of the force, as the deformations appear, various parts of the body receive some displacements, that is a certain motion is produced. After this transient process an equilibrium is established and all the parts of the body are at rest.

Thus, the action of a force on a fixed body results in deformations of various parts of the body. If the deformations are known the acting forces can be determined from them only when the deformations of the body are elastic. If otherwise, such a determination is impossible unless some special additional investigations are carried out because in the case of inelastic deformations the relationship between the forces and the deformations is extremely complex.

16. Force and Motion (Newton's First Law)

It was said in Sec. 13 that by a force is meant a physical cause producing a motion or changing a state of motion of a body and that such a cause always results from interaction of some bodies. Now, after the equilibrium conditions for forces (Sec. 15) have been established, we can give a more rigorous definition of a force: *a force is a physical quantity characterizing an interaction of at least two bodies and specifying a change of the state of motion of a given body or a deformation of the body or both.* A mechanical interaction between a given body and other bodies or the action of the other bodies on the given body can now be described with the aid of forces with which the other bodies act on that body.

The introduction of the notion of a force reduces a mechanical problem concerning the motions or the deformations of several interacting bodies to two problems: in the first problem we must determine the forces acting on each body and in the second problem it is required to determine the motions or the deformations of the bodies under the action of the known forces already found.

We know how a force can be measured and determined in statics but we do not know yet how a force specified by its absolute value and direction acting on a body is connected with the motion of the body and with the changes of the motion; the investigation of this connection is known as the *basic problem of dynamics.*

Before proceeding to the consideration of this problem we should mention that it is not known in advance that a force measured statically with the aid of a dynamometer (or with the aid of a beam balance) influences in a certain way the motion of the body it acts upon. Only experiments in which the forces are measured with the aid of dynamometers for moving bodies convince us that there exists a definite law connecting the forces acting upon a body and the motion of the body.

The basic problem of dynamics in which it is required to establish the law governing the relationship between forces and motions was completely solved by I. Newton on the basis of Galileo's inertia law (Sec. 12). Newton formulated this law as follows: *any body is in a state of rest or in a uniform rectilinear motion until some forces applied to it produce a change in the state of the body.* This law is referred to as *Newton's first law of dynamics.*

Newton's first law asserts that a state of rest and a uniform rectilinear motion of a body are basically the same mechanical state of the body. Besides, it dwells on the action of a force: it is only a force that can change a state of rest or a uniform rectilinear motion of a body. According to that law, in the absence of external actions any body moves along a straight line with a constant velocity. At first sight this appears strange because by far not always all the forces acting on a body are known in advance. Let us consider an example. When a car of a train is in a uniform rectilinear motion the sum of all forces applied to the car is equal to zero. Indeed, in this case the magnitude of the force pulling the car forward is exactly equal to that of the force acting on the car in the backward direction. We may not be interested in the fact that the pulling force is caused by the action of the preceding car and that the backward force is produced by the air drag, by the force of friction of the wheels and so on. The essence is that the resultant of the forward and backward forces is equal to zero since the car is in a uniform rectilinear motion. This conclusion is drawn from Newton's first law of dynamics.

Let us discuss another example. In a free fall of a body in vacuum the body is acted upon by a single force which is its gravity and therefore, by the same

first law, the body cannot move uniformly, that is its velocity cannot be constant; this conclusion is in a complete agreement with the results of observation.

A uniform rectilinear motion of a body is sometimes called an inertial motion. This must not be understood in the sense that inertia is the cause of that motion. For a body to remain in a state of a uniform rectilinear motion no causes are needed. A uniform rectilinear motion (inertial motion) of a body is a natural state of any body on which no forces act or which is acted upon by a system

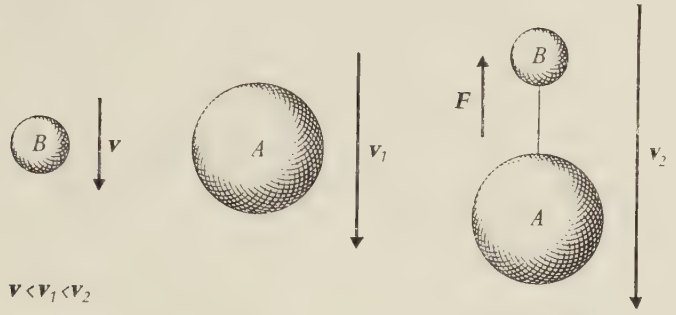

FIG. 38

of forces whose resultant is equal to zero. This is the case in the above example of a car of a train moving uniformly in a straight line: the sum of the forces acting on it is equal to zero and therefore the car retains its uniform rectilinear motion (inertial motion). It should be stressed that here we speak of forces acting upon a body at a given time and do not consider the forces which acted on the car when it started moving and which brought it to the state of the uniform rectilinear motion.

The Galileo-Newton law can also be formulated as follows: an inertial motion is a natural property of any material body. Thus, the inertia of a body is not a cause of its motion but a property of the body.

G. Galileo discovered his inertia law on the basis of experiments with bodies falling in the air; these experiments disproved the incorrect idea of Aristotle that heavier bodies fall faster than lighter ones. Galileo found experimentally the laws of the fall of bodies in the air, investigated the motion of bodies in inclined planes and drew the conclusion that in the *absence of air resistance all bodies must fall with the same speed.*

Later this conclusion was confirmed by Newton on the basis of the observation of the fall of bodies in a tube from which the air was pumped out. This experiment is often demonstrated now to students studying physics.

Galileo confirmed the validity of the conclusion drawn from the results of his experiments by the following additional argument. Suppose that, according to the idea of Aristotle, a heavy body A falls faster than a lighter body B (see Fig. 38). Now imagine that we connect the bodies with a thread; then we obtain a body which is heavier than A and, according to Aristotle's idea, it must fall faster than A. But the lighter body B must itself fall slower, and consequently in the fall of the system consisting of the two joint bodies the body B must lag behind and the thread must therefore be taut. Hence, the force of tension of the

thread F pulls the body A upwards and the body B downwards. The *upward* action of this additional force on the body A must result in an increase in the velocity of its *downward* motion; this contradicts Aristotle's idea which we assumed to be correct.

Consequently, Aristotle's idea is wrong.

In the absence of the air the fall of all bodies is a uniformly accelerated motion and the acceleration of the fall at a given place on the Earth's surface is the same for all the bodies. When a body falls from a small height the force of gravity remains practically the same, the acceleration is also constant and the velocity continuously increases. Experiments with bodies moving down an inclined plane show that the acceleration of the bodies is proportional to the magnitude of the component of the force of gravity directed along that plane. Consequently, in the case of a zero slope, that is on a horizontal plane, the body moves with a constant velocity (if the force of friction is negligibly small) or is in a state of rest. Hence, in the absence of acting forces the velocity of any body remains constant. It should also be mentioned that Galileo concluded, on the basis of the same experiments, that the gravity force whose magnitude (or the magnitude of its component along an inclined plane) had been determined only with the aid of balances was proportional to the acceleration of the body.

17. Newton's Second Law

As follows from the first law, a body acted upon by a force cannot be in a uni form rectilinear motion. Therefore there arises the following question: what is the motion of a body under the action of a force? The answer to the question is given by the *second law of dynamics.*

For the sake of simplicity we begin with a formulation of the second law which is not its most general statement (the general statement given by Newton himself will be presented later). This limited form of the second law of dynamics we are speaking of reads: *the product of the magnitude of the mass of a moving body by its acceleration is proportional to the magnitude of the force acting on the body, the directions of the force and of the acceleration coinciding.*

The second law of dynamics gives a relationship between three quantities: the force, the acceleration, and the mass. The physical meaning and the ways of measuring the magnitudes of a force and of an acceleration were elucidated before. Later on we shall give the definition of the third quantity, the mass of a body, by analysing experiments with motions of various bodies under the action of forces (see Sec. 18).

The application of forces of various magnitudes to the same body and the measurement of the accelerations produced by the action of the forces lead us to the conclusion that the acceleration is directly proportional to the force applied to the body. For example, consider the experiment whose scheme is shown in Fig. 39: a cart is moving on the rails under the action of the force of tension of the thread F. The thread is stretched by the weight G suspended from the other end of the thread passing around a pulley. The force of tension of the thread is measured with the aid of a dynamometer. Measuring the magnitude of the force and the distances travelled by the cart in its motion during definite time intervals we can establish the law connecting the acceleration and the force.

To begin with, let us analyse the character of the motion of the cart under the action of a constant force measured by the dynamometer. Suppose that the dynamometer shows a constant magnitude F_0 of the force (in this first experiment the load G remains invariable). Letting the cart move from a point A

we measure the distances S_1, S_2, S_3, . . . travelled by the cart during 1, 2, 3, . . . equal time intervals. The experiment implies that the distances are proportional to the squares of the whole numbers, that is $S_1 : S_2 : S_3 : \ldots = 1^2 : 2^2 : 3^3 : \ldots$. Such a variation of the distance as function of time corresponds to the formula for the path of a uniformly accelerated motion:

$$S = \frac{at^2}{2}$$

where a is the acceleration and t is time. Consequently, under the action of a constant force the cart is in a uniformly accelerated motion.

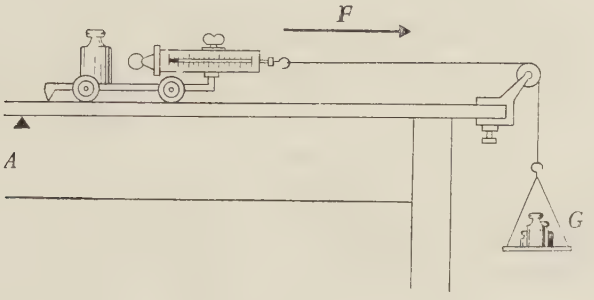

FIG. 39

Now let us investigate the dependence of the acceleration on the magnitude of the acting force. To this end we change the weight G so that, for instance, the magnitude of the force of tension of the thread measured by the dynamometer becomes one and a half times greater, that is equal to $^3/_2 F_0$. Then we again find that the motion is uniformly accelerated but in the new experiment the distances S_1', S_2', S_3', . . . travelled by the cart during 1, 2, 3, . . . equal time intervals become one and a half times greater than the corresponding distances S_1, S_2, S_3, . . . travelled by the cart in the former experiment under the action of the force F_0 during the same time intervals. The distance travelled by a body in a uniformly accelerated motion during a fixed time interval is proportional to the acceleration $\left(\text{this is seen from the same formula } S = \frac{at^2}{2}\right)$. Further analogous experiments show that the distances passed by the cart for various magnitudes of the force F are always proportional to F. The results of the experiments can be expressed by the general formula

$$F = ka \tag{17.1}$$

where k is a (constant) proportionality coefficient. Thus, *the magnitude of the force is proportional to the magnitude of the acceleration and, conversely, the latter is proportional to the former.*

Hence, these experiments prove the validity of the part of the second law of dynamics asserting that the magnitude of the force is directly proportional to that of the acceleration.

It should be noted that in the experiments described here we reproduce in fact the idea of Galileo's classical experiments with motion of bodies down

inclined planes; the only difference is that we measure the accelerating force with a spring dynamometer during the process of motion while in Galileo's experiments the force could only be measured in the state of rest.

In our experiments some measures should be taken to make the additional actions upon the cart produced by the other bodies (by the rails and the air) negligibly small. To make sure that this condition holds we can perform the following experiments. Let us remove the thread connecting the dynamometer with the load and then push the cart and measure the distances travelled by it during two equal time intervals. Let the velocity of the cart be approximately the same as in the above experiments in which the cart was accelerated. If it turns out that after the push the distances travelled by the cart during equal time intervals are approximately equal to each other then, according to the first law of dynamics, we can conclude that the additional actions upon the cart are small in comparison with the action of the force of tension of the thread. Hence, although there are additional actions their magnitudes are sufficiently small; therefore, the results of our experiments (provided that they have been performed with a sufficient accuracy) completely confirm the validity of the second law of dynamics. Getting ahead of what follows, we mention that if we increased the accuracy of the measurements of forces, distances and time intervals and measured the additional forces with which the rails and the air act on the cart, the results of these more precise experiments would only provide a further confirmation of the validity of the second law of dynamics.

18. Mass of a Body

Let us perform experiments with an accelerated motion of a cart under the action of a constant force F_0 in which the loads carried by the cart are changed and the accelerations of the cart are measured. The results of such experiments show that *the acceleration produced by the constant force decreases as the load is increased.* Consequently, we can draw the general conclusion that the acceleration of a body under the action of a given force depends not only on the magnitude of the force but also on a certain property of the accelerated body, the property varying as the quantity of matter the body consists of is changed. This property is referred to as the *inertness* of the body. The greater the inertness of the body, the smaller the acceleration it receives under the action of a constant force. The inertness increases as the load grows, that is as the quantity of matter the body contains is increased. The inertness of a body only appears in dynamical phenomena; for ordinary bodies it can be determined from dynamical experiments with accelerated motions of the bodies.

The physical quantity serving as a measure of the inertness of a body is called the *mass* of the body. When increasing the load on the cart we thus increase its mass, which results in a decrease of the acceleration of the cart produced by the action of the same force. The magnitude of the mass of a given body can be determined on the basis of the second law of dynamics by measuring the acceleration of the body produced by a definite force. The magnitude thus found can naturally be called the *inertial mass* of the body. In what follows we shall often omit the word "inertial" when speaking of a mass but the reader should keep in mind that, unless a special stipulation is made, by "mass" is usually meant the inertial mass.

The mass of a certain body can be chosen as the *unit of inertial mass* with which the masses of the other bodies should be compared. In the SI system as the unit mass is taken the mass of the standard rod mentioned in Sec. 14.

This unit of mass is called the *kilogram* (denoted kg). In physics we often use a unit which is equal to 0.001 kg and is called the *gram* (denoted g).

We now consider experiments in which a force of constant magnitude gives acceleration to a body whose mass is varied. As was said, the acceleration depends not only on the magnitude of the force but also on the inertness of the body, that is on the magnitude of its inertial mass. From the second law of dynamics it follows that the acceleration of the body in such experiments is inversely proportional to the inertial mass.

Let a body of unit mass receive an acceleration a_1 when it is acted upon by a force of magnitude F_0. Let another body whose mass is unknown have an acceleration a_2 under the action of the same force. Then, by the second law, we can write

$$F_0 = km_1a_1 = km_2a_2 \tag{18.1}$$

where k is a coefficient dependent solely on the choice of the system of units. From (18.1) it follows:

$$m_2 = m_1 \frac{a_1}{a_2} \tag{18.2}$$

In this way we can, in principle, measure the magnitude of the mass of any body. Later on we shall also discuss some other methods for measuring masses of bodies which, by the way, are in fact closely related to the above method.

Now let us imagine that analogous experiments are performed with a body which can slide freely on a smooth horizontal plane surface. Let us make the force, acting on the body in these experiments, change its direction relative to the body; this can be achieved by changing the direction of the action of the force or by turning the body itself in some way. The results of such experiments indicate that for any horizontal force applied to the body and for any orientation of the body relative to the direction of the force the magnitude of the acceleration is always proportional to that of the force. They also show that the directions of the acceleration and of the force coincide. Consequently, *a mass of a body is a scalar quantity.*

On denoting by m the inertial mass of an arbitrary body acted upon by a force $\boldsymbol{F}$ (as we know, a force is a vector) we can now write the relation expressing the second law of dynamics in the vector form

$$\boldsymbol{F} = km\boldsymbol{a} \tag{18.3}$$

where $\boldsymbol{a}$ is the acceleration (which is also a vector) and k is a coefficient dependent on the choice of the system of units. The presence of such a coefficient (dependent on the magnitudes and on the dimensions of the chosen units) in the relation expressing one of the fundamental laws of physics is extremely inconvenient. Therefore the magnitude of unit mass, its dimension and the magnitudes and the dimensions of the other units are usually chosen so that this coefficient become dimensionless and equal to unity: $k = 1$.

Following this convention, for the SI system in which the unit of mass is 1 kg and the unit of acceleration is 1 m/s^2 we put $k = 1$ in (18.3) and thus obtain the derived unit of force which is called the *newton* (denoted N):

$$1 \text{ N} = 1 \text{ kg} \cdot 1 \text{ m/s}^2$$

At present the SI system is considered preferable in all fields of science. Besides 1 kg, 1 m and 1 s, as basic units in this system are also taken 1 A (the international *ampere*, the unit of electric current intensity), 1 K (the *kelvin*, the

unit of thermodynamic temperature), 1 cd (the *candela*, the unit of luminous intensity) and 1 *mole* (the *gram-molecule*) which is the amount of substance containing the same number of grams as there are units of molecular weight of the substance.

In physics are also used some other systems of units, for instance, the cgs system in which the unit of mass is 1 g, the unit of length 1 cm, the unit of time 1 s (these are the basic units) and the unit of force is a derived unit chosen so that the coefficient k in formula (18.3) expressing Newton's second law is equal to 1. The dimension formula for the unit of force in the cgs system is written as

$$[F] = \text{g} \cdot \text{cm/s}^2 \tag{18.4}$$

This unit is called the *dyne* (denoted dyn).

In engineering is widely used the *technical* system of units in which the basic units are the *kilogram* (*force*) denoted kgf, the *metre* denoted m and the *second* (s). In this system the unit of mass is a derived unit and is again chosen so that the coefficient k in formula (18.3) expressing Newton's law is equal to 1. This *technical unit of mass* (*t.u.m*) is equal to the mass of a body which receives an acceleration of 1 m/s^2 when acted upon by a force of 1 kgf. The technical unit of mass is obviously equal to 9.81 kg.

The analysis of the results of the experiments considered in this section and the investigation of all mechanical phenomena in which the velocities of motions are small relative to the speed of light show that the mass of a body is a constant quantity. However, as was found in experiments and phenomena in which particles move with velocities close to the velocity of light, the mass of any body does not remain invariable and depends on the velocity of motion. In Einstein's modern relativistic mechanics it is shown that the mass is expressed by the formula

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{18.5}$$

where m_0 is the constant mass of the body which it has when its velocity $v \sim 0$. For ordinary motions $v \ll c = 3 \times 10^{10}$ cm/s, and therefore we can with a high accuracy consider the mass of a body as a constant quantity for such motions (this question will be discussed in more detail in Sec. 155).

19. General Form of Newton's Second Law

Newton stated the second law of dynamics in a more general form than the one considered in the foregoing sections. To proceed to this general statement we introduce one more physical quantity characterizing a moving body (a particle), namely, the *momentum* (or the *linear momentum*) of the body. The momentum of a given body is a vector quantity whose absolute value is equal to the product of the mass of the body by its velocity and whose direction coincides with that of the velocity. Denoting by $\boldsymbol{K}$ the momentum of a body having a mass m we can write

$$\boldsymbol{K} = m\boldsymbol{v} \tag{19.1}$$

where $\boldsymbol{v}$ is the velocity of the body. There are no special names for the units of momentum. In the SI system the dimension of the momentum is kg·m/s. For instance, a body having a mass of 10 kg and moving with a velocity of

2 m/s possesses the momentum

$$10 \text{ kg} \cdot 2 \text{ m/s} = 20 \text{ kg} \cdot \text{m/s}$$

Newton stated the second law of dynamics as follows: *the change of the momentum is proportional to the force producing the motion and its direction coincides with the direction in which the force acts.*

Using the notion of the derivative we can state this law in the following way: *the time derivative of the momentum of a body is equal to the acting force in its absolute value and coincides with it in its direction.**

If a body has a momentum $\boldsymbol{K}$ and is acted upon by a force $\boldsymbol{F}$ then at any time there holds the relation

$$\boldsymbol{F} = \frac{d\boldsymbol{K}}{dt}, \quad \text{that is} \quad \boldsymbol{F} = \frac{d}{dt}(m\boldsymbol{v}) \tag{19.2}$$

The only case when we can take the mass outside the differentiation sign is the one when the mass of the body remains constant; then we can write

$$\boldsymbol{F} = m\frac{d\boldsymbol{v}}{dt} \tag{19.3}$$

where $\frac{d\boldsymbol{v}}{dt}$ is the acceleration of the body. Thus, only in that special case (which however is most frequently encountered) the formulation of the second law of danymics given in the foregoing sections remains valid.

When considering a motion of a body whose mass varies as function of time we must use the general form of the second law (involving the momentum) because in this form the law describes adequately the dynamics of all the possible motions of a particle.

The general form of the second law of dynamics turns out to remain valid in Einstein's relativity theory. As was mentioned, according to that theory the mass of a body depends on the absolute value of the velocity of motion (formula (18.5)). Therefore, relation (19.2) remains valid while (19.3) does not. For a variable mass the law of dynamics is written as

$$\boldsymbol{F} = \frac{d}{dt}(m\boldsymbol{v}) = \frac{dm}{dt}\boldsymbol{v} + m\frac{d\boldsymbol{v}}{dt} = \frac{dm}{dt}\boldsymbol{v} + m\boldsymbol{a} \tag{19.4}$$

When the absolute value of the velocity varies the mass m also varies; therefore, in the general case the direction of the force $\boldsymbol{F}$ does not coincide with that of the acceleration $\boldsymbol{a}$ and the acceleration is not proportional to the force.

As will be shown later (Sec. 156), in the case of a variable mass the directions of the acceleration $\boldsymbol{a}$ and of the force $\boldsymbol{F}$ coincide only when the force is directed along the velocity $\boldsymbol{v}$ or when it is normal (perpendicular) to the velocity. In all the other cases the situation is approximately as shown in Fig. 40. In the Newtonian mechanics we have $\frac{dm}{dt}v \ll ma$ because the mass m differs very slightly from the constant m_0. Even for the velocities comparable with the Earth's velocity of rotation round the Sun which is very large (of the order of 30 km/s) the ratio $v/c \sim 10^{-4}$ is so small that $m/m_0 - 1 \sim 5 \times 10^{-9}$. For ordinary

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* Here, as usual, it is meant of course that all the quantities involved are measured in the same system of units.

"technical" velocities this quantity is still smaller. What has been said does not of course apply to the case of the velocity of an elementary particle in an accelerator whose magnitude v may be very close to c. In such a case the Newtonian mechanics is inapplicable and more precise Einstein's relativistic mechanics should be used.

Newton's first and second laws involve a force acting on one (given) body; other bodies whose actions result in the appearance of the force are not mentioned in them. However, as we know, a force always characterizes interaction of at least two bodies; the role of the other bodies in a dynamical process involving the given body is described by Newton's third law (which cannot be in fact separated from the first two laws).

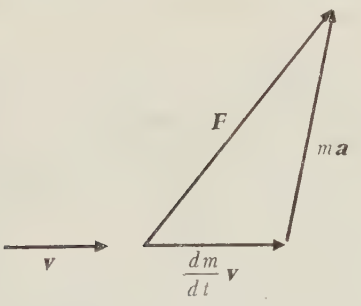

FIG. 40

20. Newton's Third Law

Newton stated the *third law of dynamics* as follows: *to every action corresponds an equal and opposite reaction*; in other words, *two bodies exert on each other actions whose magnitudes are equal and whose directions are opposite.*

The first and the second laws speak of a force acting on a given body and of the result of the action of that force completely irrespective of the other bodies exerting the force and of the phenomena generating the force. In reality any change of a state of motion of a body results only from an interaction of several bodies. In every concrete situation, when speaking about a force we always indicate at least two bodies: the one acted upon and the one exerting the force.

In what follows a force acting on a body A which is exerted on it by another body B will be designated as $\boldsymbol{F}_{AB}$.

The assertion of the third law can now be expressed in the following form: if a body B acts on a body A with a force $\boldsymbol{F}_{AB}$ then the body A, in its turn, must act on the body B with a force $\boldsymbol{F}_{BA}$ whose absolute value coincides with that of $\boldsymbol{F}_{AB}$ and whose direction is opposite to that of $\boldsymbol{F}_{AB}$, both forces lying in a common straight line. The third law means that *a force is always the result of interaction of two different bodies.*

The first two laws of dynamics abstract from the interaction and consider only one side of this phenomenon in order to simplify the analysis of the motion. In reality there always takes place an interaction, and there cannot exist a force without a reaction force.* It is evident that the distinction between the terms "action" and "reaction" is purely conditional; each of them can be replaced by the other.

As an instance, let us consider a weight lying on a man's palm (see Fig. 41). The force exerted by the palm on the weight is $\boldsymbol{F}_{\mathrm{wp}}$; it is applied to the weight and is directed upwards. The weight, in its turn, acts on the palm with the force $\boldsymbol{F}_{\mathrm{pw}}$ which is applied to the palm and is directed downwards. Now imagine that the man lifts his palm or lowers it. By the third law, in all the cases, we

* In the inertial frames of reference (see Sec. 44).

have

$$F_{wp} + F_{pw} = 0 \tag{20.1}$$

This equality *always* holds irrespective of whether the palm supporting the weight rests or moves.

The third law does not characterize the magnitudes of the forces and only asserts that they are equal. It is also important to stress that the forces of which the third law speaks *are always applied to different bodies.*

FIG. 41

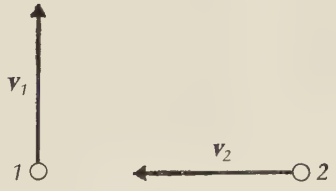

FIG. 42

In the case of an interaction between moving electric charges the situation is more complicated. For instance, let us consider the forces of interaction between charges *1* and *2* (Fig. 42) moving as shown in the figure. The electric interaction between the charges satisfies Newton's third law while their magnetic interaction does not satisfy it if that law is understood in its ordinary mechanical sense. Indeed, the magnetic field generated by charge *2* at the point where charge *1* is located is equal to zero, that is there is no force of magnetic action exerted by charge *2* on charge *1*; at the same time, the magnetic field generated by charge *1* at the point where charge *2* is placed is different from zero. Hence, there is a non-zero magnetic force exerted by charge *1* on charge *2* (its direction is perpendicular to the plane of the figure). However, there is no contradiction here because in the consideration of phenomena of this kind the momentum of the electromagnetic field should also be taken into account (this is done in electrodynamics).

The three fundamental laws of dynamics stated by Newton had been known before but Newton was the first to understand that these three laws form the foundation of mechanics. It was Newton who analysed the motions of various bodies and showed that all mechanical phenomena, however complex, are governed by the three laws of dynamics; and it was due to him that mechanics became a precise science. That is why the laws of dynamics are called Newton's laws.

21. Forces and Newton's Second and Third Laws

When investigating motions of bodies and solving mechanical problems one should carefully distinguish between consequences following from the second law and from the third law because their confusion often leads to misunderstanding. We shall demonstrate what has been said by examples.

The analysis of every mechanical problem should be started with establishing what forces act upon the given bodies, what bodies exert these forces and how the forces can be determined. On performing the preliminary analysis we then determine the acceleration of the body, whose motion we are interested in, taking into account, if necessary, the conditions imposed on the forces (acting on the body) by the third law. Let us consider a number of such problems.

(1) *A weight lying on the palm.* Let the palm move in a certain way. It is required to determine the forces acting on the palm and on the weight and find the acceleration of the weight.

Equality (20.1) written on the basis of the third law does not itself provide any means for the determination of the acceleration of the weight. To determine the acceleration we must know all the other forces exerted on the weight by the other bodies. Besides the force $\boldsymbol{F}_{wp}$ with which the palm acts on the weight, the weight is acted upon by the force of gravity, that is by the force generated by the interaction between the weight and the Earth*; we denote this force as $\boldsymbol{F}_{we}$. Now we can determine the resultant force acting on the weight which is the sum of the two forces $\boldsymbol{F}_{wp}$ and $\boldsymbol{F}_{we}$. According to the second law, the resultant is equal to the product of the mass of the weight by its acceleration:

$$\boldsymbol{F}_{we} + \boldsymbol{F}_{wp} = m_w \boldsymbol{a}_w \qquad (21.1)$$

Hence, if the magnitude of the force of gravity $\boldsymbol{F}_{we}$ is greater than that of the force of the palm $\boldsymbol{F}_{wp}$, the acceleration of the weight is directed towards the Earth; if otherwise, that is if the magnitude of the force of the palm exceeds that of the force of gravity, the acceleration is directed upwards.

The magnitude and the direction of the acting force determine only the acceleration but not the velocity and therefore we cannot find from (21.1) the direction of motion of the weight. For instance, when $F_{we} > F_{wp}$ the weight can be either in a downward accelerated motion or in an upward decelerated motion. More precisely, when the acceleration is directed downwards the velocity can have an arbitrary direction; it can go upwards or downwards or even form an angle with the vertical. The direction of the velocity at a given moment has no direct connection with that of the acceleration while the acceleration itself is completely and uniquely determined by the acting forces.

If the acceleration of the weight is equal to zero, the sum of the forces acting on it must be zero; in other words, in this special case the force $\boldsymbol{F}_{wp}$ of the action of the palm on the weight is equal in its magnitude and opposite in its direction to the force of gravity $\boldsymbol{F}_{we}$. In these circumstances the weight can be in a state of rest or in a uniform rectilinear motion with any constant velocity.

(2) *A weight is suspended from a spring attached to the post placed on the table* (see Fig. 43). We shall consider the post to which the spring is attached and the fixed table standing on the ground as one rigid body. Thus, we consider the interaction of three bodies: the weight, the spring, and the Earth (as has been said, the Earth together with the table and the post form one body). The forces

* The air resistance is very small and can be neglected in these considerations.

taking part in this interaction are shown in Fig. 44 where, for the sake of visuality, all the three bodies in question are (conditionally) depicted separately. The Earth acts on the weight with the force $\boldsymbol{F}_{we}$ (the force of gravity of the weight) and on the spring with the force $\boldsymbol{F}_{se}$ (the force of gravity acting on the spring). The weight acts on the spring with the force $\boldsymbol{F}_{sw}$ and the post (considered as one body together with the Earth and the table) acts on the spring with

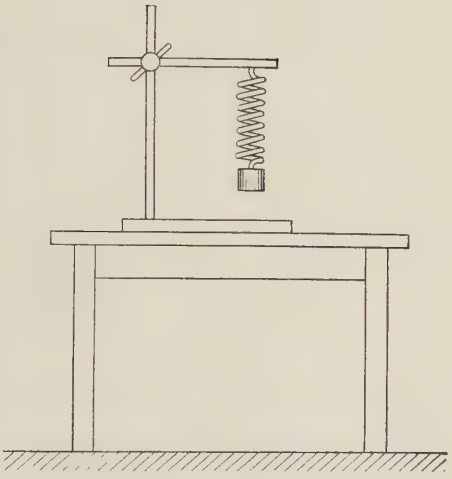

FIG. 43

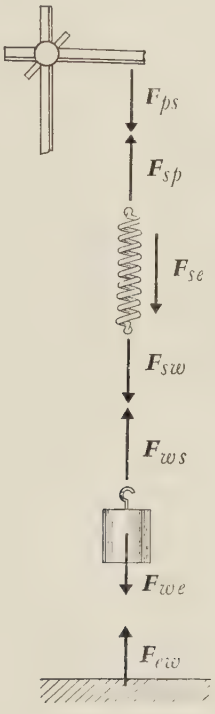

FIG. 44

the force $\boldsymbol{F}_{sp}$. According to the third law, we always have the equalities

$$\boldsymbol{F}_{we}+\boldsymbol{F}_{ew}=0, \quad \boldsymbol{F}_{sw}+\boldsymbol{F}_{ws}=0 \quad \text{and} \quad \boldsymbol{F}_{sp}+\boldsymbol{F}_{ps}=0 \tag{21.2}$$

and the second law implies

$$\left.\begin{aligned} \boldsymbol{F}_{we}+\boldsymbol{F}_{ws} &= m_w\boldsymbol{a}_w \\ \boldsymbol{F}_{sp}+\boldsymbol{F}_{sw}+\boldsymbol{F}_{se} &= m_s\boldsymbol{a}_s \\ \boldsymbol{F}_{ps}+\boldsymbol{F}_{ew}+\boldsymbol{F}_{es} &= m_e\boldsymbol{a}_e \end{aligned}\right\} \tag{21.3}$$

where m_w, m_s, and m_e are, respectively, the masses of the weight, the spring and the Earth and $\boldsymbol{a}_w$, $\boldsymbol{a}_s$, and $\boldsymbol{a}_e$ are their accelerations. Since the frame of reference we deal with is connected with the Earth we can put $\boldsymbol{a}_e = 0$.* The mass and the force of gravity of the spring are usually very small in comparison

* This does not mean that we can put $m_e\boldsymbol{a}_e=0$ because the mass m_e is very large ($m_e \to \infty$) and therefore $m_e\boldsymbol{a}_e$ may be a finite quantity for $\boldsymbol{a}_e \to 0$.

with the mass and the force of gravity of the weight; this allows us to put (in this and in many other problems) $m_s = 0$ and $F_{se} = F_{es} = 0$. Then equations (21.3) expressing Newton's second law can be rewritten as

$$F_{we} + F_{ws} = m_w a_w, \quad F_{sp} + F_{sw} = 0, \quad \text{and} \quad F_{ps} + F_{ew} = m_e a_e \tag{21.4}$$

Assuming that the magnitude of the mass of the spring is negligibly small (and only under this assumption) we can write, on the basis of the second law, the (approximate) equilibrium condition

$$F_{sp} + F_{sw} = 0 \tag{21.5}$$

Condition (21.5) shows that the force of tension of the ("massless", i.e. "inertia-free") spring is in all the circumstances the same at both ends of the spring. In this approximation the magnitudes of the forces acting on the ends of the spring are regarded as being precisely equal. Further, by the third law, these magnitudes are equal to those of the forces F_{ps} and F_{ws} with which the spring acts upon the bodies stretching it. Thus, an "inertia-free" spring "transmits" a force without changing the latter irrespective of whether that spring rests or moves*. That is why when speaking of a tension of a spring or of a thread we mean the magnitude of the stretching force which is considered the same for both ends of the spring or of the thread.

The force F_{we} with which the Earth acts on the weight (the force of gravity of the weight) is no longer equal to the force F_{ws} with which the spring acts on the weight. The difference between these forces determines the acceleration of the weight. It should be noted that if $F_{ws} > F_{we}$ at a certain time this does not necessarily mean that the weight moves upwards; this only implies that the acceleration of the weight is directed upwards. The force of the spring F_{ws} and the force of gravity F_{we} are not equal to each other (according to the second law). It is the difference between these forces that produces the acceleration of the weight.

When the weight and the spring are at rest their accelerations are equal to zero: $a_w = a_s = 0$. In this case we have zeros on the right-hand sides of the first two equations (21.3). Then the force F_{ws} with which the spring acts on the weight is equal in its magnitude to the force of gravity F_{we} and, by the third law, to the force F_{sw} with which the weight stretches the spring; in the state of rest the force F_{sw} coincides with the force of gravity of the weight. Thus, in the state of rest the absolute values of the three different forces F_{we} (the force of gravity of the weight), F_{ws} (the force of tension of the spring) and F_{sw} (the force with which the weight stretches the spring) are the same. The force of gravity of the weight and the force F_{sw} are simply equal to each other: $F_{we} = F_{sw}$.

It should be noted that if the force of gravity of the spring, that is F_{se}, is not negligibly small then even in statics the forces of tension acting on the opposite ends of the spring are different. By the second equation (21.3), in this case, for $a_s = 0$, we have

$$F_{sp} + F_{sw} = -F_{se} \tag{21.6}$$

The magnitude of the force F_{sp} pulling the spring upwards is then greater than that of the force F_{sw} pulling it downwards, the difference between these magnitudes being equal to the magnitude of the force of gravity of the spring F_{se}.

* Any body whose mass is negligibly small possesses this property; for instance, in our discussion we tacitly imply that the threads connecting the bodies in question are "massless", "inertia-free" and possess the indicated property.

When the force of gravity of the spring is very small in comparison with that of the weight the force $\boldsymbol{F}_{se}$ can be neglected, that is we can put $\boldsymbol{F}_{se} = 0$. Then in statical conditions (for the weight in a state of rest) all the forces under consideration are equal to each other in their magnitudes. Besides equalities (21.3) which always hold, we have in this case the equalities

$$\boldsymbol{F}_{we} + \boldsymbol{F}_{ws} = 0, \quad \boldsymbol{F}_{sp} + \boldsymbol{F}_{sw} = 0 \quad \text{and} \quad \boldsymbol{F}_{ps} + \boldsymbol{F}_{ew} = 0 \tag{21.7}$$

which follow from equations (21.3). Only in this single case all the six forces taking part in the interaction between the three bodies (the Earth, the weight, and the spring) are equal to each other in their magnitudes; three of them, namely, $\boldsymbol{F}_{we}$, $\boldsymbol{F}_{ps}$, and $\boldsymbol{F}_{sw}$ are directed downwards and the other three upwards.

We once again stress that, according to the third law, the equality of the magnitudes of the forces with which two different bodies act on each other always takes place; the resultant of the forces acting on one body is only equal to zero when the body in question is in a state of rest or in a uniform rectilinear motion (this follows from the second law).

The approximate equality of two forces applied to one body takes place in the special case when the mass of the body on which the forces act is very small or, more precisely, when the product of that mass by the magnitude of the acceleration of the body is negligibly small in comparison with the magnitude of each of the acting forces. For example, in the experiments with the cart in which the acceleration of the cart was measured (see Fig. 39) we neglected the mass of the spring of the dynamometer and therefore assumed that the data shown by the dynamometer are the same as in statics.

When considering impact phenomena in which there are very large accelerations we can by far not always neglect even very small masses.

(3) *Influence of inertness on transmission of force.* Let us consider a load with mass m suspended from a thread, another thread of the same width being attached to the load from below (Fig. 45). When the lower thread is pulled slowly with increasing tension it is the upper thread (carrying the load) that will finally break. If we pull the thread sharply then the lower thread will be ruptured.

It is evident that the sum of the force of tension $\boldsymbol{F}_l$ of the lower thread (i.e. the force with which the lower thread acts on the load), of the force of tension $\boldsymbol{F}_u$ of the upper thread, and of the force of weight $\boldsymbol{P}$ of the load equals the mass m of the load times its acceleration $\boldsymbol{a}$:

$$\boldsymbol{F}_l + \boldsymbol{F}_u + \boldsymbol{P} = m\boldsymbol{a} \tag{21.8}$$

The force $\boldsymbol{F}_u$ is directed upwards and the other forces go downwards. When the lower thread is being pulled (downwards) the acceleration $\boldsymbol{a}$ is also directed downwards. Consequently, equality (21.8) can be written in the scalar form

$$F_l - F_u + P = ma$$

whence

$$F_l - F_u = ma - P \tag{21.9}$$

Here the threads are considered weightless and therefore no other forces are taken into account. When the lower thread is pulled slowly the phenomenon in question is "almost statical", the acceleration of the load is negligibly small, and equality (21.9) reduces to

$$F_u - F_l = P \tag{21.10}$$

The magnitude of the force of tension $\boldsymbol{F}_u$ of the upper thread in this case is always greater by $\boldsymbol{P}$ than that of the force of tension $\boldsymbol{F}_l$ of the lower thread.

The phenomenon becomes quite different when the lower thread is pulled sharply (jerked). In this case the upper thread must almost instantaneously elongate to exert an action with the increased magnitude of the force F_u in order to balance the increase of the tension of the lower thread. Therefore in a short time of the duration of the jerk (resulting in the rupture of the lower thread) the load receives an acceleration $\boldsymbol{a}$ directed downwards; the greater the lower orce F_l, the greater the acceleration $\boldsymbol{a}$. In this case the force of tension F_l of he lower thread can be greater than the force of tension F_u of the upper thread.

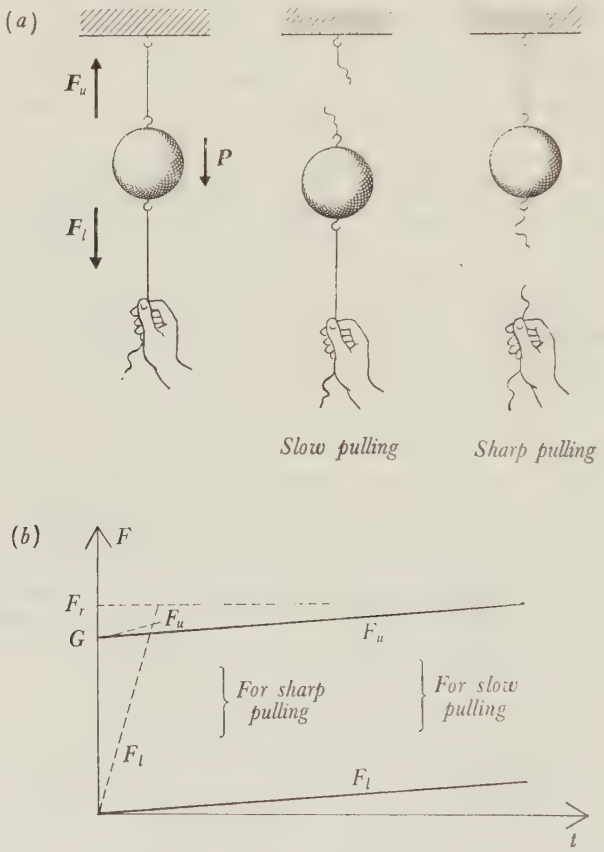

FIG. 45

As is seen from equality (21.9), if $ma > P$ then $F_l > F_u$. When the difference between the forces of tension of the threads together with force of gravity G give the load an acceleration $\boldsymbol{a}$ exceeding the acceleration of gravity g, the force of tension of the lower thread becomes greater than that of the upper thread.

It should be noted that although the displacement of the load is very small its acceleration can be very large since the time interval during which the load receives that acceleration is very short.

It is expedient to consider the graphs demonstrating the time variations of the forces for the two cases when the lower thread is pulled slowly and is pulled sharply (see Fig. 45*b*). For the case when the lower thread is pulled slowly both forces increase almost in the same manner although the force of tension F_u of the upper thread is all the time greater than F_l by the magnitude of the weight of the load. When the force of tension of the lower thread increases rapidly it very soon achieves the value F_r for which the thread is ruptured while the force of tension of the upper thread increases with a comparatively smaller rate.

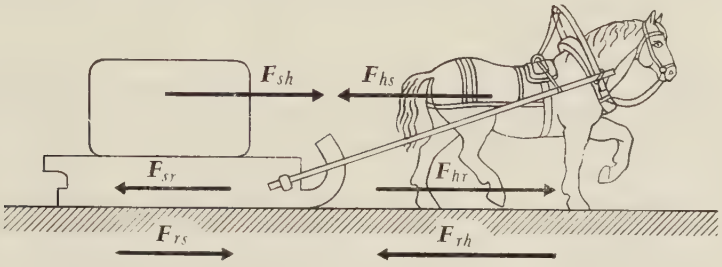

FIG. 46

It should be noted that in the explanation of these experiments the fact that the upper thread is elongated plays an essential role; the increase of the length of the thread, although very small, produces the necessary increase of the force of tension of that thread. In the case of the jerk the tension of the lower thread achieves the limiting value F_r during the time of the elongation of the upper thread; the inertness of the load prevents it from moving rapidly; more precisely, such a rapid motion of the load requires an action of a great force which the lower thread cannot exert on the load. Our considerations indicate that to succeed in carrying out these experiments we must take a sufficiently long upper thread.

(4) *Horse and sledge.* A traditional example for demonstrating the third law is that of a horse drawing a sledge. The test question is why the horse draws the sledge and not otherwise while the force action of the sledge on the horse is equal in its magnitude to that with which the horse pulls the sledge.

The examples considered above show that the third law is inapplicable to the problem of determining the acceleration of a body, and the question is stated incorrectly. The matter is that the forces whose equality is mentioned in the question are acting on different bodies and therefore this equality provides no means for the determination of the accelerations of the horse and of the sledge. It is necessary to determine all the other forces acting upon the horse and the sledge, and only then can the accelerations of the bodies (of the horse and of the sledge) be found.

Fig. 46 shows all the forces taking part in the phenomenon in question, that is in the interaction between the three bodies: the sledge, the horse, and the ground. The forces are denoted in the figure in the same way as in the first example. The forces $\boldsymbol{F}_{hs}$ and $\boldsymbol{F}_{sh}$ characterize the interaction between the horse and the sledge ($F_{hs} = F_{sh}$). The forces $\boldsymbol{F}_{rs}$ and $\boldsymbol{F}_{sr}$ arise because of the friction be-

tween the sledge and the road ($F_{rs} = F_{sr}$), F_{rh} and F_{hr} are the forces of interaction between the horse and the road ($F_{rh} = F_{hr}$). According to the third law, in each of these equalities are meant two forces whose magnitudes coincide and the directions are opposite; each of them acts upon one of the two different interacting bodies. If $F_{hr} > F_{sr}$, that is if the magnitude of the force F_{hr} with which the road acts on the horse exceeds the magnitude of the force of friction F_{sr} exerted by the road on the sledge, then, according to the second law, the horse and the sledge have an acceleration directed forwards. The forces F_{sh} and F_{hs} do not affect the resultant motion. If $F_{hr} < F_{sr}$ the horse and the sledge have an acceleration directed backwards. Only in the case of a uniform motion the magnitudes of all the forces shown in Fig. 46 are equal to one another.

Let the reader decide which of the forces taking part in this example are equal due to the third law and which are equal due to the second law.

All the examples we have considered show that the third law of dynamics speaks first of all of the fact that the cause of a change in a motion of a given body is always connected with at least one more body on which the former body, in its turn, exerts an action in the interaction phenomenon involving both bodies. This means that a motion of a body is produced or stopped by an interaction between that body and some other body (or a system of bodies); we can say that the motion is "transmitted" from one body to another.

22. Motion of a Body Under the Action of Given Forces

The simplest example of a motion of a body under the action of a given force is the free fall of the body in the field of gravity force. *A field of a force is a part of space at whose every point a particle (body) placed at that point is acted upon by a definite force.*

Generally, by a physical vector field is meant a region in space to whose every point there corresponds a definite value of a given vector (physical) quantity. For instance, a system of fixed bodies carrying electric charges generates a stationary field of the electric intensity vector $\boldsymbol{E}$. To each point in space there corresponds a vector $\boldsymbol{E}$ which specifies the force

$$\boldsymbol{F} = e\boldsymbol{E} \tag{22.1}$$

acting on a charge e placed at that point.

Another example is a system of conductors of constant electric currents generating in space a field of magnetic induction intensity vector $\boldsymbol{B}$ which determines the force

$$\boldsymbol{F} = e\,[\boldsymbol{v}\boldsymbol{B}] \tag{22.2}$$

acting on a charge e moving with velocity $\boldsymbol{v}$ and passing through the point in space at which the induction vector has the value $\boldsymbol{B}$. If the quantities involved are measured in the SI system then $[e] = \text{A}\cdot\text{s}$, $[E] = \frac{\text{kg}\cdot\text{m}}{\text{As}^3}$, $[B] = \frac{\text{kg}}{\text{A}\cdot\text{s}}$ and the force is measured in newtons.

When a charged particle e moves in a homogeneous electric field $\boldsymbol{E}$ it is acted upon by the corresponding constant force $\boldsymbol{F}$ which is the same at all the points because for a homogeneous field the vector $\boldsymbol{E}$ is the same at all

the points. Consequently, in this case the charge must be in a uniformly accelerated motion.

The motion of bodies in the gravitation field near the Earth's surface has an analogous character. If the motion is considered in a spatial region whose dimensions are small in comparison with the radius of the Earth the gravitation field can be considered homogeneous. In this case the moving body has a constant acceleration g ($g = 9.81$ m/s^2). Similarly, in a constant homogeneous electric field a charged particle e of mass m has the constant acceleration

$$\boldsymbol{w} = \frac{\boldsymbol{F}}{m} = \frac{e\boldsymbol{E}}{m} \qquad (22.3)$$

Such motions were studied in Sec. 10.

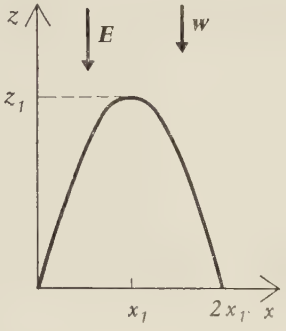

FIG. 47

In all such cases the trajectory of motion of the particle lies in the plane containing the point at which the particle is at the initial time $t = 0$ and the vectors $\boldsymbol{w}$ and $\boldsymbol{v}_0$ drawn from that point. The trajectory is a *parabola*. To show this let us take equation of motion (10.9) in vector form and then write it in terms of the projections of the vectors involved under the assumption that the origin (of the coordinate system) is at a point $\boldsymbol{r}_0$ and that the particle moves in the xz-plane. The vector equations are

$$\boldsymbol{r} - \boldsymbol{r}_0 = x\boldsymbol{i} + z\boldsymbol{k},\ \boldsymbol{v}_0 = v_x\boldsymbol{i} + v_z\boldsymbol{k} \text{ and } \boldsymbol{w} = -w\boldsymbol{k}$$

(the acceleration is directed along the negative z-axis). In terms of the projections on the x- and z-axes these equations are written as

$$x = v_x t \text{ and } z = v_z t - \frac{1}{2} wt^2 \qquad (22.4)$$

Equations (22.4) are the parametric equations of a parabola lying in the xz-plane and passing through the origin (for $t = 0$). On eliminating time t from the equations we obtain the equation of the trajectory in terms of the coordinates x and z (Fig. 47):

$$z = \frac{v_z}{v_x} x - \frac{w}{2v_x^2} x^2 \qquad (22.5)$$

This is the equation of a parabola with vertex at a point x_1, z_1 seen in the figure. The time period t_1 during which the particle attains the uppermost point (the vertex of the parabola) is readily found from the condition that $\frac{dz}{dt} = 0$ at time t_1:

$$\frac{dz}{dt} = v_z - wt_1 = 0 \text{ whence } t_1 = \frac{v_z}{w}$$

On substituting t_1 into equations (22.4) we find the coordinates of the vertex of the parabola:

$$x_1 = \frac{v_x v_z}{w} \text{ and } z_1 = \frac{v_z^2}{2w}$$

The particle whose motion starts at the point $x = 0$, $z = 0$ will again meet the x-axis at a distance $2x_1 = 2v_x t_1 = \frac{2v_x v_z}{w}$ from the origin (this is the *ballistic range*). It can easily be shown that for a given value of v_0 the maximum range is achieved when $v_x = v_z = \frac{1}{\sqrt{2}} v_0$. The maximum range is equal to v_0^2/w. The maximum height is equal to $v_0^2/2w$ and is attained when $v_x = 0$ and $v_z = v_0$.

The motion of a charged particle in a homogeneous magnetic field. An electric field acts on a resting charge in the same way as on a moving one: the acting force depends solely on the position of the point charge and is independent of its velocity. This force is always directed along the intensity vector $\boldsymbol{E}$ of the field.

The situation is completely different when a charged particle moves in a magnetic field. A magnetic field only acts on the particle when its velocity is different from zero ($\boldsymbol{v} \neq 0$) and is not parallel to the magnetic induction vector $\boldsymbol{B}$. On a fixed charge the magnetic field exerts no action.

By formula (22.2), the acting force $\boldsymbol{F}$ is equal to the charge e times the vector product of the velocity $\boldsymbol{v}$ by the magnetic induction vector $\boldsymbol{B}$. Hence, the force $\boldsymbol{F}$ is normal to the plane containing the vectors $\boldsymbol{v}$ and $\boldsymbol{B}$ and its magnitude is

$$F = evB \sin \alpha$$

where α is the angle between the vectors $\boldsymbol{v}$ and $\boldsymbol{B}$. The direction of the force $\boldsymbol{F}$ is specified by the *right-hand screw rule*: if a screw with right-hand thread is placed perpendicularly to the plane containing the vectors $\boldsymbol{v}$ and $\boldsymbol{B}$ and is turned in the direction of the shortest rotation from the vector $\boldsymbol{v}$ to the vector $\boldsymbol{B}$ then this screw is driven into the plane in the direction of the vector $\boldsymbol{F}$ (see Fig. 14).

Consequently, if the directions of the velocity $\boldsymbol{v}$ and of the vector $\boldsymbol{B}$ are parallel then $\boldsymbol{F} = 0$, that is, as was mentioned, in this case the field exerts no action on the moving particle. The force $\boldsymbol{F}$ is always perpendicular to the velocity $\boldsymbol{v}$ and therefore its action does not change the magnitude (modulus) v of the velocity. The velocity vector can only rotate under the action of $\boldsymbol{F}$ (as in the case of a uniform motion in a circle).

It is convenient to represent the motion of the particle in a homogeneous magnetic field $\boldsymbol{B}$ as a combination of two motions. Namely, let us resolve the velocity $\boldsymbol{v}$ of the particle into two components $\boldsymbol{v}_{\parallel}$ and $\boldsymbol{v}_{\perp}$ the first of which is parallel to the vector $\boldsymbol{B}$ and the other is normal to it:

$$\boldsymbol{v} = \boldsymbol{v}_{\parallel} + \boldsymbol{v}_{\perp}$$

Then

$$\boldsymbol{F} = e[\boldsymbol{v}\boldsymbol{B}] = e[(\boldsymbol{v}_{\parallel} + \boldsymbol{v}_{\perp})\boldsymbol{B}] = e[\boldsymbol{v}_{\perp}\boldsymbol{B}] \tag{22.6}$$

because $[\boldsymbol{v}_{\parallel}\boldsymbol{B}] = 0$.

The acceleration $\boldsymbol{F}/m$ is always normal to $\boldsymbol{B}$ and therefore the component $\boldsymbol{v}_{\parallel}$ of the velocity of the particle remains invariable. This means that in the direction of $\boldsymbol{B}$ the particle moves uniformly with constant velocity $\boldsymbol{v}_{\parallel}$. The component $\boldsymbol{v}_{\perp}$ of the velocity lies in the plane perpendicular to $\boldsymbol{B}$, according to (22.6); it varies so that the acceleration $\left(\text{which is equal to } \frac{e[\boldsymbol{v}_{\perp}\boldsymbol{B}]}{m}\right)$ is always

perpendicular to $\boldsymbol{v}_\perp$. The magnitude of the acceleration is

$$\frac{e}{m} v_\perp B \tag{22.7}$$

Consequently, the modulus of $\boldsymbol{v}_\perp$ remains constant. The particle moves in such a way that its projection onto the normal plane to $\boldsymbol{B}$ is in a uniform circular motion in a circle with linear velocity $v_\perp$. The radius R of that circle can be found from the known expression of the centripetal acceleration which must be equated to (22.7). This gives

$$\frac{v_\perp^2}{R} = \frac{e}{m} v_\perp B$$

whence

$$R = \frac{mv_\perp}{eB} \tag{22.8}$$

The period of one revolution is

$$\tau = \frac{2\pi R}{v_\perp} = \frac{2\pi m}{eB} \tag{22.9}$$

The period is independent of $v_\perp$ and, for the given particle, it is determined solely by the magnitude B of the field and by the ratio e/m.

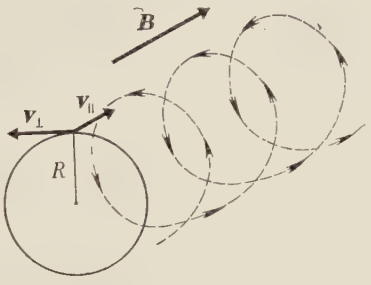

FIG. 48

The resultant motion of the particle is the combination of the two motions: of the uniform rectilinear motion with velocity $\boldsymbol{v}_\parallel$ along a line parallel to $\boldsymbol{B}$ and the uniform motion in a circle of radius R with velocity $\boldsymbol{v}_\perp$ whose magnitude is constant. The trajectory of the resultant motion of the particle is a a screw line (Fig. 48).

The component $\boldsymbol{v}_\parallel$ of the velocity of the motion varies neither in its magnitude (absolute value) nor in its direction while $\boldsymbol{v}_\perp$ has a constant magnitude but permanently changes its direction.

The equations of motion can be derived as follows. Let us suppose that $\boldsymbol{B} = B\boldsymbol{i}$ and $B_y = B_z = 0$ (this means that the vector field $\boldsymbol{B}$ is parallel to the x-axis). Then

$$\boldsymbol{v}_\parallel = v_x \boldsymbol{i} \text{ and } \boldsymbol{v}_\perp = v_y \boldsymbol{j} + v_z \boldsymbol{k}$$

The magnitudes v_x and $v_\perp = \sqrt{v_y^2 + v_z^2}$ are constant and are determined by the initial value $\boldsymbol{v}_0$ of the velocity.

If the particle is at the origin at time $t = 0$ and has an initial velocity $\boldsymbol{v}_0$ representable as

$$\boldsymbol{v}_0 = v_x \boldsymbol{i} + v_\perp \boldsymbol{j}$$

then the variation of the coordinate x of the particle is described by the equation

$$x = v_x t \tag{22.10}$$

The projection of the particle onto the yz-plane moves in a circle of radius R (see (22.8)) with velocity $v_\perp = \sqrt{v_y^2 + v_z^2}$. The component $\boldsymbol{v}_\perp$ of the velocity has the value

$$\boldsymbol{v}_\perp = v_y \boldsymbol{j}$$

at the initial moment $t=0$, that is the normal component of the initial velocity goes along the y-axis for $t = 0$. Therefore the projections of the component $v_\perp$ on the axes y and z (see Fig. 49) are expressed by the formulas

$$v_y = v_\perp \cos \frac{v_\perp}{R} t \text{ and } v_z = v_\perp \sin \frac{v_\perp}{R} t \tag{22.11}$$

The quantity $\varphi = \frac{v_\perp}{R} t$ is the angle through which the radius vector joining the centre A to the projection of the particle onto the yz-plane rotates during time t. Here it is expedient to introduce the notion of the numerical value ω of the *angular velocity* of the rotation which is defined as the rate of change of the angle φ and, according to the definition, is expressed by the formula

$$\omega = \frac{d\varphi}{dt} = \frac{v_\perp}{R} \tag{22.12}$$

Now we can write the expressions describing the time variations of the coordinates y and z of the particle. Taking into account (22.11) and (22.12) we obtain

$$\left.\begin{aligned} y(t) &= \int_0^t v_y \, dt_1 = v_\perp \int_0^t \cos \omega t_1 \, dt_1 = R \sin \omega t \\ z(t) &= \int_0^t v_z \, dt_1 = v_\perp \int_0^t \sin \omega t_1 \, dt_1 = R(1 - \cos \omega t) \end{aligned}\right\} \tag{22.13}$$

Thus, equalities (22.10) and (22.13) describe the motion of the point charge starting at moment $t = 0$ from the origin with the initial velocity $\boldsymbol{v}_0 = v_x \boldsymbol{i} + v_\perp \boldsymbol{j}$ in the homogeneous magnetic induction field $\boldsymbol{B} = B\boldsymbol{i}$.

It should be noted that, given the magnitude of the radius R and the initial velocity v_0, the ratio e/m is uniquely determined for the particle in question. This relation serves as the theoretical basis for the construction of devices meant for the determination of that ratio e/m. One of the simplest devices of this kind can be described schematically as follows.

Let a homogeneous magnetic field $\boldsymbol{B}$ be parallel to the z-axis and occupy a region $Oabd$ (see Fig. 50) adjoining the origin. Let a charged particle e enter that region at a point C with velocity v_0 parallel to the y-axis. On traversing a semicircle of radius R under the action of the field $\boldsymbol{B}$ the particle leaves the region occupied by the field at a point A. Measuring the distance $CA = 2R$ and knowing the value v_0 we find from (22.8) the ratio

$$\frac{e}{m} = \frac{v_0}{BR} \left[\frac{\text{C}}{\text{kg}}\right]^* \tag{22.14}$$

Motion of charged particle in homogeneous electric and magnetic fields whose vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ go in one direction at every point. Let a particle enter the region occupied by the electric and the magnetic fields with an initial velocity $\boldsymbol{v}_0$ perpendicular to $\boldsymbol{E}$. The action of the field $\boldsymbol{E}$ accelerates the particle in the direction of $\boldsymbol{E}$ (coinciding with that of $\boldsymbol{B}$) while the action of $\boldsymbol{B}$ gives the particle an acceleration lying in the normal plane to $\boldsymbol{B}$. This results in deviations of the particle from its initial rectilinear motion with velocity $\boldsymbol{v}_0$. Let us determine these deviations, under the assumption that the dimension l, along the direction of $\boldsymbol{v}_0$, of the part of the region of the field we are interested in is very small relative to the radius of curvature R of the trajectory of the particle

* C is the *coulomb*, the unit of electric charge in the SI system.—*Tr.*

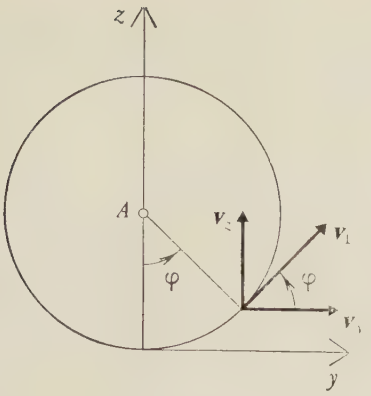

FIG. 49

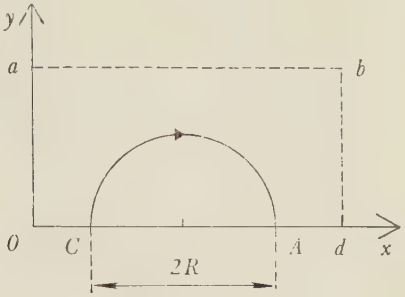

FIG. 50

moving in the magnetic field with induction $\boldsymbol{B}$: $l \ll R = \frac{mv_0}{eB}$. Let the particle enter the region of the field at the origin with initial velocity $\boldsymbol{v}_0$ along the negative x-axis, the vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ being parallel to the z-axis. It is required to determine the coordinates y and z of the particle when it is at a distance of l from the origin O in the normal plane to the x-axis.

Using (22.3) and (22.4) and taking into account that $t = l/v_0$ we can compute

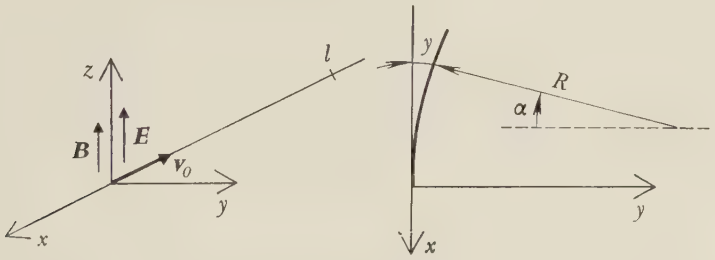

FIG. 51

the deviation of the particle along the z-axis under the action of the electric field $\boldsymbol{E}$:

$$z = \frac{1}{2} \frac{eE}{m} \frac{l^2}{v_0^2} \qquad (22.15)$$

The deviation along the y-axis under the action of the magnetic field $\boldsymbol{B}$ can be computed approximately if we take into account the condition $l \ll R$ (see Fig. 51):

$$y = R\,(1 - \cos\alpha)$$

Assuming that the angle α is sufficiently small we can put, approximately,

$$R\alpha \approx l \quad \text{and} \quad 1 - \cos\alpha \approx \frac{\alpha^2}{2}$$

Now, by (22.8), we obtain

$$y \approx \frac{l^2}{2R} = \frac{eBl^2}{2mv_0} \tag{22.16}$$

Formulas (22.15) and (22.16) give us the ratio

$$\frac{y}{z} = \frac{B}{E} v_0 = \cot\beta \tag{22.17}$$

which can be used for the determination of the velocity v_0 for given values of E and B.

On eliminating the velocity v_0 of the particle from equations (22.15) and (22.16) we find the relation

$$y^2 = \frac{eB^2l^2}{2mE} z \tag{22.18}$$

which is the equation of a parabola.

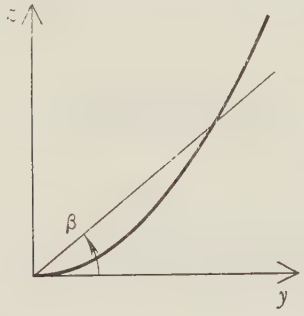

FIG. 52

From (22.18) we see that particles having the same ratio e/m and entering the region occupied by the given fields $\boldsymbol{E}$ and $\boldsymbol{B}$ with different velocities v_0 move in the same parabola (Fig. 52). As follows from (22.17), the points belonging to the straight line $z = y \tan \beta$ are common for any particles entering the field with the initial velocity $v_0 = (E/B) \cot \beta$.

These relationships also serve as the theoretical basis for the construction of devices meant for measuring the ratio e/m. Apparatuses based on this principle were used when isotopes of chemical elements were first discovered. The deviations from the parabolic law (22.18) observed for particles moving very fast indicate the relativistic variation of their masses dependent on the velocity of motion.

23. Constrained Motion of a Body

By a *free* motion of a body is meant a motion of the body under the action of given forces when there are no additional conditions imposed on the trajectory and on the velocity of the motion. A free motion starts from a given initial position with a given initial velocity.

However, there are such mechanical problems in which we encounter *constrained* motions for which the trajectory of motion or the velocity or both should satisfy some given conditions. For instance, we deal with such motions when considering a body sliding down inclined plane, a cart running along rails, a ball attached to a taut thread and moving in a circle, a ball rolling over a horizontal plane, two moving bodies connected with a string, etc. The body sliding down inclined plane must all the time remain on that plane, the ball rolling over the horizontal plane must remain on the plane, etc.

A constrained mechanical motion is subjected, irrespective of the magnitudes and the directions of the acting forces, to certain additional conditions called *constraints*. Usually, constraints imposed on a motion of a body are generated by some "non-deformable" bodies, most often, by the surfaces of such bodies. Although in a motion of one body over the surface of another body the

surface does in fact undergo some deformations, such deformations are usually very small and therefore they can be neglected, that is it is permissible to consider the trajectory (or surface) of motion as being, in a certain sense, predetermined and independent of the magnitudes and directions of the acting forces.

If a body is in a constrained motion then, besides the external forces (i.e. given forces which are known), it is subjected to the actions exerted by the bodies realizing the constraints. These forces are called *the reactions of the constraints.*

A mechanical problem involving constraints imposed on the motion of a body differs from problems dealing with free motions because the reactions of the constraints are not known in advance, and therefore the solution of such a problem requires some other methods. That is why the equations of motion of constrained bodies must include not only the given, known forces but also the unknown reactions of the constraints. On writing the equations of motion we must derive some additional relations from the conditions of the problem, for instance, from the known shape of the trajectory, which, when added to the equations of motion, allow us to determine both the unknown reactions and the unknown acceleration of the body.

In principle, the general method of the solution of all dynamical problems is rather simple: we must denote in some way the unknown quantities, write down the equations of motion using the second and the third laws of dynamics, and set the additional relations by taking into account the conditions imposed on the motion by the constraints. In this way we can always derive the required number of independent equations allowing us to determine the unknowns. Practical applications of this method can be demonstrated most distinctly by means of examples. Below are some typical examples involving constrained motions in which we start with some simple cases and then proceed to more complicated problems.

(1) *Circular motion of a body in horizontal plane.* Let a ball attached to a thread be in a circular motion as shown in Fig. 53. The force of action of the thread on the ball (the reaction of the constraint) $\boldsymbol{F}$ depends on the absolute value of the velocity v with which the ball moves in the circle.

When we consider such a motion the magnitude of the force F is usually unknown while the velocity v and the radius R can easily be measured. This makes it possible to determine the acceleration w of the ball. As is known (see Sec. 9), the centripetal acceleration is equal to v^2/R. We can therefore assert that the thread, unless it breaks, acts on the ball with a force F equal to mv^2/R. The force $\boldsymbol{F}$ (the reaction of the constraint) depends on the radius R of the trajectory of motion, on the mass m of the moving body and on the velocity v. The reaction of the constraint (in the case under consideration it is the force of tension of the thread) gives the ball the centripetal acceleration.

The reaction of a constraint is sometimes completely independent of the character of the motion but it can nevertheless be essential for understanding the nature of the forces of interaction taking part in the motion. For instance, when a body slides on a horizontal plane without friction the reaction of the constraint is always perpendicular to the plane, that is to the direction of the possible displacements of the body, and it is connected neither with the acceleration of the body nor with its velocity. In this case the force of gravity acting on the body and the components of the other forces normal to that plane are balanced by the reaction force of the constraint.

A similar situation occurs when a body slides down inclined plane without friction (Fig. 54). In the absence of the force of friction between the body and the plane the interaction forces between them are normal to the plane, that is the reaction of the constraint is perpendicular to the surface of the constraining body.

The constraints whose reactions are normal to the possible displacements of the body are called *ideal constraints.*

(2) *Motion of a body without friction down inclined plane.* Let us consider the forces acting upon a body lying on an inclined plane under the assumption

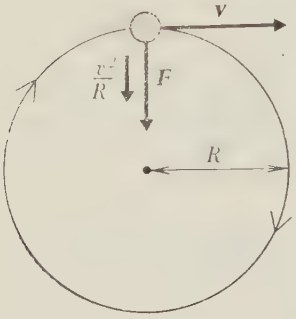

FIG. 53

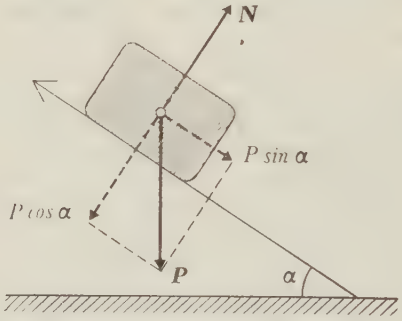

FIG. 54

that there is no friction between the body and the plane. These are the force of gravity $\boldsymbol{P}$ and the reaction $\boldsymbol{N}$ of the plane. Since, by the assumption, there is no force of friction (we deal with an ideal plane), the reaction $\boldsymbol{N}$ is normal to the plane. Both forces are applied to the body, and it is only these forces that cause the motion of the body down the plane. We must stress that here we only consider the case when during the time of motion the body permanently adjoins the inclined plane. It follows that the sum of the components of all the forces along the direction perpendicular to the plane of motion must be equal to zero.

Let us resolve the force of gravity $\boldsymbol{P}$ in two components: the normal component to the inclined plane with magnitude $P \cos \alpha$ (see Fig. 54) and the component parallel to the plane whose magnitude is $P \sin \alpha$. The acceleration being in the direction parallel to the plane of motion, we have

$$N = P \cos \alpha \tag{23.1}$$

Consequently, the reaction $\boldsymbol{N}$ does not affect the acceleration of the motion of the body sliding without friction down the plane; however, the presence of the reaction is necessary for creating the conditions in which this motion is possible.

The component of the force of gravity parallel to the plane and directed downwards, whose magnitude is $P \sin \alpha$, determines the acceleration of the body. Indeed, we have

$$M \frac{dv}{dt} = P \sin \alpha = Mg \sin \alpha$$

where M is the mass of the body whence

$$\frac{dv}{dt} = g \sin \alpha \tag{23.2}$$

It follows that, as in the case of the free fall, the acceleration is constant and the character of the motion without friction down an inclined plane is essentially the same, only the magnitude of the acceleration being reduced.

When the acceleration is sufficiently small the distance travelled by the body and the time of motion can be measured with a sufficient accuracy; that is why in the study of the laws of the free fall of the bodies it is convenient to use inclined planes as was done by Galileo.

In the case under investigation the reaction of the constraint whose magnitude is given by formula (23.1) remains invariable during the time of motion and depends solely on the force of gravity acting upon the body and on the angle of inclination of the plane.

(3) *Curvilinear motion of a body.* In more complicated cases the reaction of a constraint can be essentially dependent on the motion of the body; for instance, this is the case in the first example with the circular motion of a ball. As another instance of the same kind, let us determine the reactions of the constraints for a locomotive running along a curvilinear horizontal railway line. Let the velocity of the locomotive v have a constant absolute value. Generally, the direction of the velocity varies and consequently the locomotive has an acceleration w (Fig. 55). On determining this acceleration w and knowing the mass M of the locomotive we can find the accelerating force which is normal to the path of motion and results from the side pressure exerted by the rails on the rim flanges of the wheels of the locomotive. The magnitude of the velocity of the locomotive being invariable, the acceleration must be perpendicular to the velocity, that is to the rails. At any given time a small part of the path of the locomotive can be approximately replaced by a very small arc of a circle of a definite radius R and the real motion of the locomotive can be thought of as a motion along that arc of the circle (Fig. 56*a*). It follows that the acceleration w of the locomotive is directed to the centre of the circle and that its magnitude is equal to

$$\frac{dv}{dt} = v \frac{d\alpha}{dt} = v \frac{dS}{dt} \frac{1}{R} = \frac{v^2}{R} \tag{23.3}$$

where R is the radius of the circle and dS is the increment of the distance travelled during time dt.

The reaction of the constraint (of the rails) causing the variation of the direction of the velocity of the locomotive has the magnitude

$$F_A = M \frac{v^2}{R_A} \tag{23.4}$$

at the point A (see Fig. 55) where R_A is the radius of curvature of the path at that point. For the point B we have $R_B \to \infty$ because this part of the path is rectilinear and therefore, when passing this part, the locomotive is not acted upon by any horizontal force exerted by the rails, the horizontal component of the reaction of the constraint being equal to zero. At the point C the force of side pressure of the rails F_C has opposite direction (relative to F_A) and its magnitude depends on the radius of curvature of the path at that point.

It should be noted that these considerations also remain valid when the magnitude of the velocity of motion is no longer constant. The only distinction

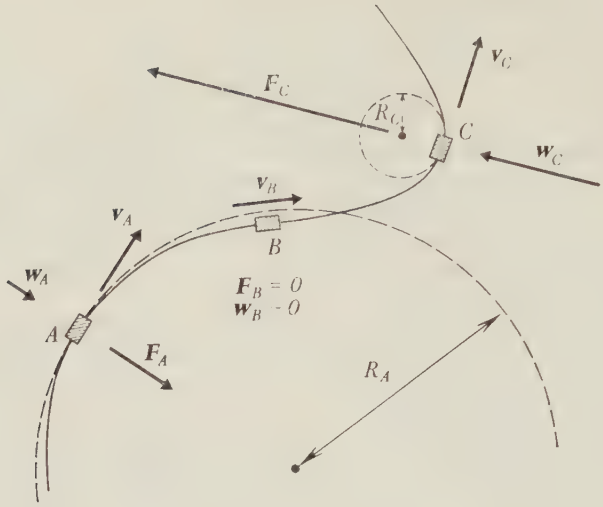

FIG. 55

between the latter and the former cases is that in the more general case of a variable velocity the acceleration of the locomotive is no longer normal to the rails. However, the relationship between the component of the acceleration normal to the rails and the velocity v remains the same at any given time t (see formula (23.3)), and consequently the force of side pressure of the rails upon the wheels is again determined by formula (23.4).

At any given moment the motion of the locomotive can be thought of as a motion along a circle whose radius is determined by the curvature of the

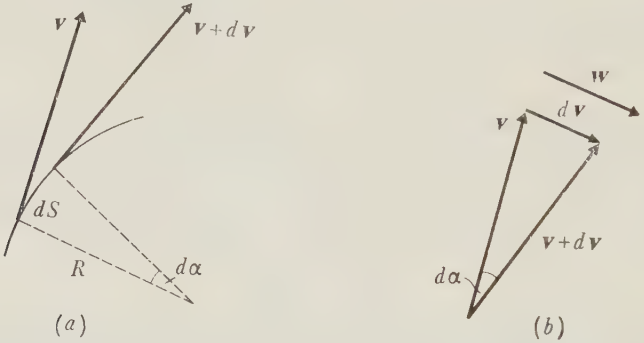

FIG. 56

path at the given point. (In mathematics by a curvature of a curve at a given point is meant the quantity inverse to the radius of the *osculating* circle of the curve at that point; this circle occupies the limiting position to which tends the circle passing through the given point and through two other points of the curve approaching indefinitely the former.) That is why force of side pres-

sure of the rails on the wheels of the locomotive which gives the locomotive an acceleration directed to the centre of curvature (that is to the centre of the circle mentioned above) is called, as in the case of a purely circular motion, the *centripetal force.* The distinction lies only in the fact that in a motion of a body in a circle the centre towards which the centripetal force is directed is fixed while in the general case of an arbitrary motion of a body along a given curve the centre to which that force is directed may change its position depending on the position of the moving point on the trajectory. This

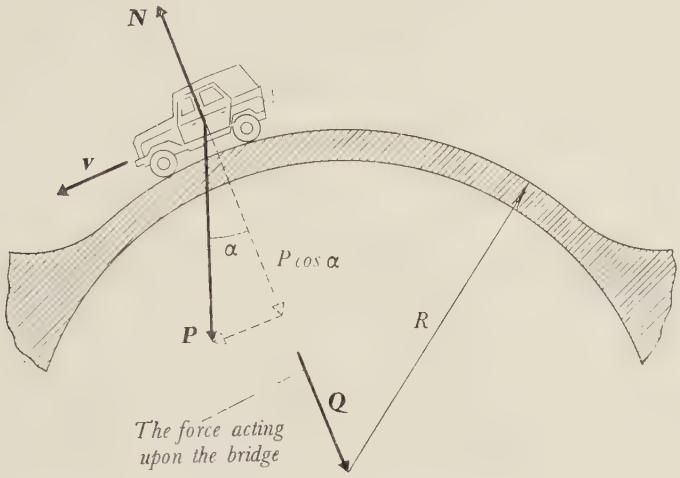

FIG. 57

force lies in a straight line perpendicular to the tangent line to the curve at the given point.* This is the case in the example of the locomotive we have discussed.

When we consider motions with given constraints it is supposed that the bodies realizing the constraints possess sufficient strength. If a constraining body cannot exert the necessary force on the moving body the necessary connection between the body and the constraint can no longer exist. For instance, when the rails are out of order the locomotive moving with a sufficiently large velocity can run off the rails on a curved part of its path.

When the motion of the locomotive is accelerated or decelerated there appear some additional forces with which the rails act upon the locomotive and produce the acceleration or the deceleration along the direction of the rails. These forces are caused by the friction between the wheels and the rails produced by the adhesion of wheels. They also result from the interaction between the rails and the locomotive. However these forces "do not make" the locomotive remain on the rails; that's why we do not consider them as reactions of constraints although, in principle, from the point of view of dynamics, there is no difference between these forces. The forces operating because of the adhesion of wheels are determined by the engine power of the locomotive but they can-

* This straight line is called the *normal* to the curve at that point.

not exceed a definite limiting value. This value depends on the weight of the locomotive and on the properties of the materials of which the rails and the wheels of the locomotive are made. When the force exerted by the engine on the wheels becomes too large the wheels start skidding and they no longer roll along the rails without sliding. The phenomena of friction and adhesion of wheels will be discussed in more detail in Chapter 8.

(4) *Motion of a car on a bridge.* Let a car be in a motion on a convex bridge (see Fig. 57). In this example the reaction of the constraint depends both on the velocity of motion and on the forces acting upon the moving body. For simplicity, we shall neglect the forces of friction between the car and the bridge, then the normal reaction N with which the bridge acts on the car and the component of the force of gravity normal to the bridge (its magnitude is $P \cos \alpha$), give the car the centripetal acceleration $\frac{v^2}{R}$ where v is the velocity of the car and R is the radius of curvature of the bridge. By the second law of dynamics, we have

$$\frac{mv^2}{R} = P \cos \alpha - N \tag{23.5}$$

where m is the mass of the car. It follows that the magnitude of the reaction of the constraint is

$$N = P \cos \alpha - \frac{mv^2}{R}$$

whence it is seen that it depends on the force $P \cos \alpha$, on the velocity v, on the shape of the bridge characterized by the radius R, and on the mass m of the moving body.

It should be noted that all the forces we have considered are applied to the moving body (the car); there is only one force applied to the constraint (the bridge), namely the force of pressure Q with which the car acts upon the bridge. By Newton's third law, the magnitude of that force is equal to the magnitude of N while the direction of Q is opposite to the direction of N.

An analogous argument can be used in the example with a man skiing downhill (Fig. 58). When the man is in the position shown in the figure the force of pressure Q exerted by the skier on the ground exceeds his weight at rest.

Thus, when investigating a problem involving a constrained motion of a body, we consider, besides the given forces (such as the forces of gravity in the above examples), the unknown forces (the reactions of the constraints) and write the equations of dynamics involving all the forces. The acceleration is found from some other conditions of the problem, for instance, from the velocity and from the shape of the trajectory of the body. When the mass of the body and the forces acting on it are known, then, after the acceleration has been determined, we can find the reactions of the constraints.

In the examples we have considered the given constraints determine the trajectory of motion. But there also exists another type of constraint imposed on the motion of a body, for instance, such as in the cases when all the bodies in question or some of them are simply joined with the aid of inextensible threads or non-deformable rods. In these cases the forces of tension of the threads connecting the bodies can also be regarded as reactions of constraints. When it is known that all the connected bodies have the same acceleration the whole system of bodies can be treated as one body; then we should only take into account the external forces applied to that system. However, even in such

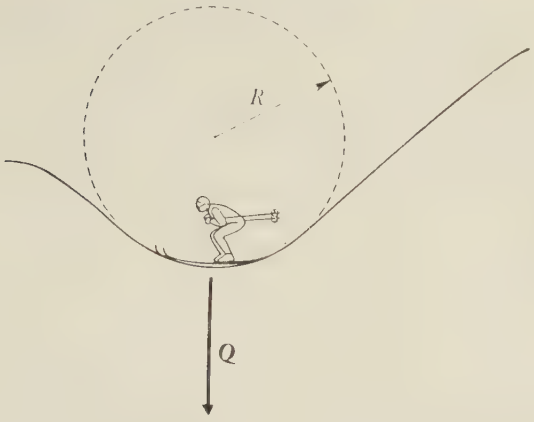

FIG. 58

a simple case, if it is required to find the tension of the threads (the magnitudes of the reactions), the equations of motion which we write must include the forces of tension.

Below are two examples of motions with constraints of that type; the first of them is rather simple while the other is more intricate.

(5) *Motion of connected carts.* Let two carts (see Fig. 59) connected with each other by means of a thread and accelerated by the force of weight P of a load move along rails. Let the masses of the carts be M_1 and M_2, the masses of

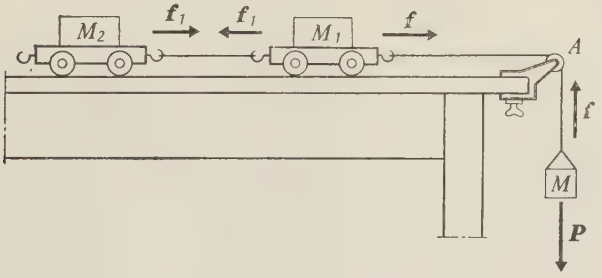

FIG. 59

the thread and of the pulley A being negligibly small. Since the rails are placed horizontally the forces of weight of the carts and the reactions of the rails acting on the carts are perpendicular to the direction of the possible motion of the carts (provided the forces of friction are so small that it is allowable to neglect them). Therefore, the forces of weight of the carts do not affect the accelerations of the carts. Similarly, the reactions of the rails do not affect the acceleration of the system either. The threads being considered inextensible, the velocities and the accelerations of all the three bodies (the two carts and the load) have the same magnitude. It is this condition which the constraints impose on the motion of the carts.

Let f denote the magnitude of the force of tension of the thread joining the first cart to the load, f_1 denote the tension of the thread connecting the two carts and P the weight of the load. It should be taken into account that the threads have zero masses and that they can only pull the bodies they are attached to. Next we must write the equations of dynamics for each of the bodies separately.

The equation for the load is written in the form

$$M\frac{dv}{dt} = P - f \tag{23.6}$$

the equation for the first cart is

$$M_1\frac{dv}{dt} = f - f_1 \tag{23.7}$$

and for the second cart we have

$$M_2\frac{dv}{dt} = f_1 \tag{23.8}$$

On the other hand, since the threads are inextensible we can also consider all the three bodies as a whole system. In this approach the forces of tension of the threads should be regarded as internal forces and do not play any role in the determination of the acceleration of the system, the mass of the whole moving system being $M_1 + M_2 + M$ and the external force acting on it being the force of gravity P. Indeed, on adding together the right-hand and the left-hand members of equations (23.6), (23.7), and (23.8) we obtain

$$(M_1 + M_2 + M)\frac{dv}{dt} = P \tag{23.9}$$

Since $P = Mg$ where g is the acceleration of gravity, the acceleration of the system consisting of the carts and the load is equal to

$$\frac{dv}{dt} = g\frac{M}{M + M_1 + M_2} \tag{23.10}$$

Note that the acceleration of the system of the three bodies is constant while the velocity depends on time and on the initial conditions, namely, on the velocity of the system at the initial moment of time.*

However, if it is required to determine the forces of tension of the threads (which are often important to know) it is necessary to substitute the magnitude of the acceleration found from equation (23.10) into equations (23.6), (23.7), and (23.8). It is evident that the tension of the thread from which the load is suspended is always greater than that of the other thread.

(6) *Motion of three loads suspended from pulleys.* Let us consider the system shown in Fig. 60. The threads are considered inextensible, the pulleys and the threads are assumed to be weightless and the friction is neglected. It is required to determine the forces of tension of the threads and the accelerations of the loads.

Let f_1, f_2, and f_3 denote the forces of tension of the threads and x_1, x_2, and x_3 the coordinates of the loads (see Fig. 60). Drawing the positive x-axis downwards we can write down the equations of dynamics for all the three loads in the form

$$m_1\frac{d^2x_1}{dt^2} = m_1g - f_1, \quad m_2\frac{d^2x_2}{dt^2} = m_2g - f_2, \text{ and } m_3\frac{d^2x_3}{dt^2} = m_3g - f_3 \tag{23.11}$$

* Here by the initial time we mean the moment at which the only external force P starts acting upon the system.

where m_1, m_2, and m_3 are the masses of the loads and g is the acceleration of gravity. The pulleys being considered weightless, the forces of tension of the threads must satisfy the relations

$$f_2 = f_3 \quad \text{and} \quad f_1 = f_1' = f_2 + f_3 \tag{23.12}$$

Now we must take into account the conditions imposed on the motion of the three loads by the constraints. Since the threads are inextensible we have

$$x_1 + x_0 + \pi r = l_1 \quad \text{and}$$
$$x_2 - x_0 + x_3 - x_0 + \pi r = l_2 \tag{23.13}$$

where l_1 and l_2 are the lengths of the threads, x_0 is the coordinate of the fixed pulley, and r is the radius of the pulleys. On eliminating the quantity x_0 from equations (23.13) we derive the condition imposed by the constraint on the coordinates of all the three bodies:

$$2x_1 + x_2 + x_3 = 2l_1 + l_2 - 3\pi r \tag{23.14}$$

It is this equation that determines the relationships between both the velocities and the accelerations of all the bodies. Let us differentiate equation (23.14) twice with respect to time; this results in the relationship connecting the accelerations of the loads:

$$2\frac{d^2x_1}{dt^2} + \frac{d^2x_2}{dt^2} + \frac{d^2x_3}{dt^2} = 0 \tag{23.15}$$

We now have six equations (23.11), (23.12), and (23.15) involving six unknowns: the three accelerations and the three reaction forces (the forces of tension in the threads). To find the accelerations we must solve these equations. On subtracting the last two equations (23.11) from the first one and using equalities (23.12) we obtain

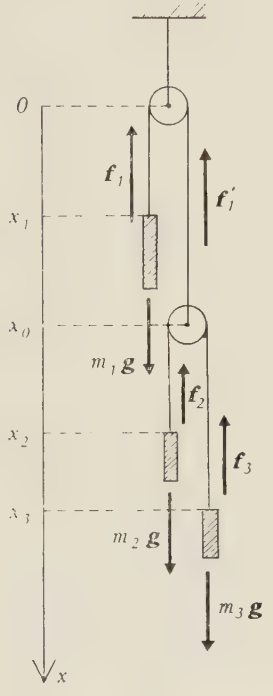

FIG. 60

$$m_1\frac{d^2x_1}{dt^2} - m_2\frac{d^2x_2}{dt^2} - m_3\frac{d^2x_3}{dt^2} = (m_1 - m_2 - m_3)g \tag{23.16}$$

Further, the subtraction of the third equation (23.11) from the second one results in

$$m_2\frac{d^2x_2}{dt^2} - m_3\frac{d^2x_3}{dt^2} = (m_2 - m_3)g \tag{23.17}$$

Finally, solving three equations (23.15), (23.16), and (23.17) containing three unknowns we obtain

$$\frac{d^2x_3}{dt^2} = \frac{(m_1m_3 - 3m_1m_2 + 4m_2m_3)g}{4m_2m_3 + m_1m_2 + m_1m_3} \tag{23.18}$$

To find $\frac{d^2x_2}{dt^2}$ and $\frac{d^2x_1}{dt^2}$ we substitute expression (23.18) into equations (23.17) and (23.15). If it is required to determine the tensile forces in the threads, the accelerations obtained should be substituted into equations of motion (23.11).

It is difficult to solve a problem of this kind without setting equations of dynamics (23.11) for every load because condition (23.15) connecting the acceleration, which is imposed by the constraints, is not known in advance. In the solution of such problems mistakes often occur when the signs + or — in front of the magnitudes of the forces and of the accelerations are set incorrectly. In this connection the reader should note that all the magnitudes (of the forces and of the accelerations) in the solution of the above problem are considered positive when the corresponding vectors are directed downwards and that the real directions of the accelerations can only be found after equations (23.18) have been solved and the signs of the unknowns have been determined.

CHAPTER 3

Momentum of a System of Bodies

24. Law of Conservation of Momentum

The application of the second and the third laws of dynamics to a system consisting of several interacting bodies leads to very important conclusions which imply the *conservation law* (the *law of constancy*) *for the momentum* of a system on which no external forces act and in which only internal forces may be present.

For the sake of simplicity, let us begin with a system consisting of only two interacting bodies, for instance, two balls lying on a horizontal plate (Fig. 61). Let there be a contracted spring between the balls which are connected with a thread (see the figure). We shall suppose that the friction forces between the balls and the plate are negligibly small. Let the masses of the balls be m_1 and m_2 and let the mass of the spring be very small so that it is allowable to put it equal to zero (we can also consider a spring having a non-zero mass but this case is more complicated since it involves the interaction of three bodies: the two balls and the spring). If at a certain moment the thread is cut the spring will push away the balls by acting with a force $\boldsymbol{F}_{12}$ on the ball m_1 and with a force $\boldsymbol{F}_{21}$ on the ball m_2. According to the third law, $\boldsymbol{F}_{12} = -\boldsymbol{F}_{21}$ because the mass of the spring is negligibly small and therefore the role of the spring in this example only reduces to the "transmission" of the action from one ball to the other. Thus,

$$\boldsymbol{F}_{12}+\boldsymbol{F}_{21}=0 \tag{24.1}$$

Indeed, by the third law of dynamics, we have

$$\boldsymbol{F}_{1s}+\boldsymbol{f}_{s1}=0 \quad \text{and} \quad \boldsymbol{F}_{2s}+\boldsymbol{f}_{s2}=0$$

where $\boldsymbol{f}_{s1}$ and $\boldsymbol{f}_{s2}$ are the forces with which the balls m_1 and m_2 act on the spring and $\boldsymbol{F}_{1s}$ and $\boldsymbol{F}_{2s}$ are the forces of action of the spring on the balls. Since the mass of the spring is put equal to zero, the second law of dynamics implies

$$\boldsymbol{f}_{s1}+\boldsymbol{f}_{s2}=0$$

Consequently,

$$\boldsymbol{F}_{1s}+\boldsymbol{F}_{2s}=0$$

and it is also apparent that

$$\boldsymbol{F}_{1s}=\boldsymbol{F}_{12} \quad \text{and} \quad \boldsymbol{F}_{2s}=\boldsymbol{F}_{21}$$

whence follows equality (24.1). The mass m_1 is acted upon by the force $\boldsymbol{F}_{12}$ and the mass m_2 by the force $\boldsymbol{F}_{21}$. The actions of these forces give the balls

some accelerations a_1 and a_2 which can be found from the equations

$$m_1 a_1 = F_{12} \quad \text{and} \quad m_2 a_2 = F_{21} \tag{24.2}$$

On adding together these equations and taking into account (24.1) we arrive at

$$m_1 a_1 + m_2 a_2 = 0 \tag{24.3}$$

Since $m_1 v_1$ is the momentum of the first ball, $m_2 v_2$ is the momentum of the other

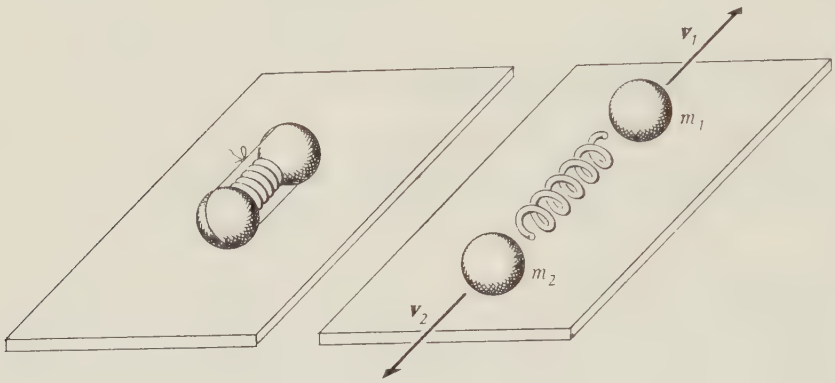

FIG. 61

ball, and the accelerations are expressed by the formulas

$$a_1 = \frac{dv_1}{dt} \quad \text{and} \quad a_2 = \frac{dv_2}{dt} \tag{24.4}$$

the substitution of these values of the accelerations into equations (24.3) results in

$$\frac{d}{dt}(m_1 v_1 + m_2 v_2) = 0 \tag{24.5}$$

It should be stressed that the magnitude and the character of the interaction forces between the balls play no role in the derivation of equation (24.5). It is only important here that these forces always satisfy condition (24.1). Equality (24.5) means that the sum of the momenta of the balls retains a constant value both during the period when the spring is pushing away the balls and after that period until some (new) external forces start acting on the balls.

This conclusion also applies to any two bodies, and the example with the balls was taken only for the sake of visuality.

Thus, *the total momentum of a system consisting of two bodies cannot change under the forces of interaction between the bodies* (when there are no external forces).

Now let us consider a number of bodies isolated from the other bodies and forming one mechanical system. The interaction forces between these bodies are called *internal* forces in the given system. When an isolated system consists of many bodies the law of conservation of momentum holds for the system as a whole. Indeed, let a system consist of three bodies with masses m_1, m_2,

and m_3 (Fig. 62). Then for the first body we can write the equation

$$m_1 a_1 = f_{12} + f_{13}$$

for the second

$$m_2 a_2 = f_{21} + f_{23}$$

and for the third

$$m_3 a_3 = f_{31} + f_{32}$$

On adding together all the three equations and taking into account the third law of dynamics we obtain

$$m_1 a_1 + m_2 a_2 + m_3 a_3 = 0$$

that is

$$\frac{d}{dt}(m_1 v_1 + m_2 v_2 + m_3 v_3) = 0 \quad (24.6)$$

where, as usual, a_1, a_2, and a_3 denote the accelerations of the bodies and v_1, v_2, and v_3 are the corresponding velocities.

Let K designate the *momentum of the system* (the *total* momentum), that is the sum of the momenta of the bodies in the system. Then the conservation law for the momentum can be written as

$$\frac{dK}{dt} = 0$$

or, which is the same,

$$K = \text{const} \quad (24.7)$$

Thus, *the momentum of a system of bodies cannot change under the action of the internal forces.*

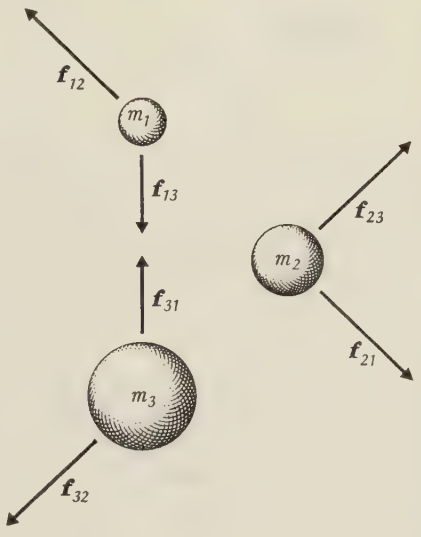

FIG. 62

In just the same way the law of constancy of the momentum under the action of the internal forces can be derived for any number of bodies forming an isolated system. The momentum of the system is the vector sum of the momenta of all the constituent bodies:

$$K = \sum_{i=1}^{n} m_i v_i \quad (24.8)$$

By the third law of dynamics, all internal forces satisfy the conditions

$$f_{ki} + f_{ik} = 0 \quad (24.9)$$

for any values of the indices k and i ($k = 1, 2, 3, \ldots, n$; $i = 1, 2, 3, \ldots, n$; n is the number of the bodies forming the system).

When no external forces act upon the system we can write for each of the bodies the equation

$$m_i \frac{dv_i}{dt} = \sum_{k=1}^{n} f_{ik} \quad (24.10)$$

where m_i is the mass of the ith body, v_i is its velocity, and $\sum_{k=1}^{n} f_{ik}$ is the sum of all those internal forces with which the other bodies of the system act upon

the ith body. Now we add together equations (24.10) for all the bodies, which results in

$$\sum_{i=1}^{n} m_i \frac{dv_i}{dt} = \sum_{i=1}^{n} \sum_{k=1}^{n} f_{ik} \tag{24.11}$$

By condition (24.9), the sum of all internal forces is equal to zero whence

$$\sum_{i=1}^{n} m_i \frac{dv_i}{dt} = 0$$

which implies

$$\frac{d}{dt} \sum_{i=1}^{n} m_i v_i = 0 \tag{24.12}$$

It is the last relation that expresses the law of conservation of momentum for an arbitrary system consisting of several bodies which move only under the action of the internal forces and are not acted upon by external forces.

Let us sum up the results of our discussion.

The law of conservation of momentum holds for the interaction of any number of bodies; the momentum of the system formed by the bodies remains *constant* when there are no *external* forces exerted on the system by the bodies not entering in the system.

The momentum is a *vector quantity*, and hence the conservation law means the constancy of both the absolute value and the direction of the momentum of the system. The momenta of the constituent bodies can vary but the momentum of the whole system remains invariable.

When the momentum of a body entering in a system changes the total momentum of the other bodies of the system *changes by the same quantity with opposite sign*. Thus, *the momentum of a body entering in an isolated system can only be transmitted to other bodies but can never disappear* or, which is the same, *the momentum of any isolated system is a vector quantity whose value always remains constant.*

The notion of a momentum was introduced by Descartes who also formulated the philosophical principle of the conservation (that is, the constancy) of the momentum. This was followed by the discussion among scientists as to which of the quantities mv or mv^2 should be considered as a measure of motion; the question was answered only after the law of conservation of energy was established (see Sec. 37).

25. Transfer of Momentum from One Body to Another

When there are forces of interaction between two bodies the momentum can be transferred from one body to another. The variation of the forces in an interaction process can be an extremely complicated phenomenon and, in the general case, its analysis is an intricate problem. The application of the law of conservation of momentum to these cases allows us to determine in a simple manner the result of the interaction without a detailed investigation of the forces acting on the bodies. What has been said is illustrated by some simple examples below.

(1) *A man running with a velocity* v_0, *jumps onto a cart standing on the rails* (Fig. 63). The momentum of the man is $K = m_1 v_0$ where m_1 is his mass. Before

the jump the cart is at rest, its momentum being equal to zero. After the man has jumped onto the cart he remains in a state of rest relative to the cart and there appear forces of interaction between the man and the cart. The force acting on the man decelerates his motion while the (reactive) force acting upon the cart produces the motion of the latter. These forces stop acting after the man and the cart obtain equal velocities, that is after the man occupies a fixed position relative to the cart. If the forces of friction between the rails and the cart and the resistance force produced by the air are negligibly small

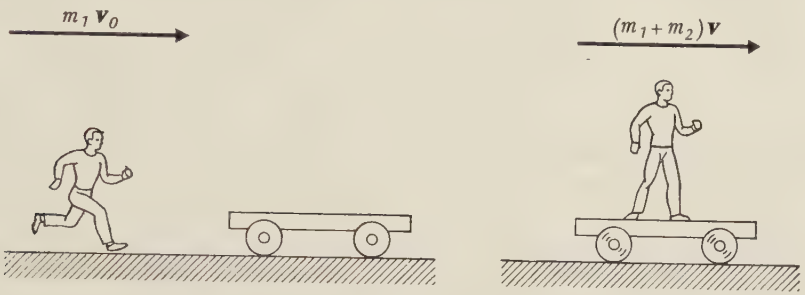

FIG. 63

the momentum of the system (consisting of the man and of the cart) is not changed in the interaction process. The initial momentum $m_1 v_0$ of the man is redistributed: the cart gains a part of the momentum which the man had before the jump. The velocity obtained by the cart and the man after they have formed one system can be determined from the law of conservation of the momentum. It is determined by the formula

$$v = \frac{m_1}{m_1+m_2} v_0 \tag{25.1}$$

The result is independent of the duration of the interaction between the feet of the man and the cart and of the specific variation process of the interaction forces. This phenomenon in which the man and the cart have taken part is an example of the *inelastic (perfectly plastic) impact.*

An inelastic (perfectly plastic) impact is a phenomenon in which an interaction between two bodies possessing different velocities results in a motion in which both bodies have the same velocity.

The specific mechanism of the interaction in the impact and the way in which the forces have varied do not affect the resultant velocity after the impact.

Those who study the theory of this phenomenon sometimes try to disprove the assertion that the momentum can only be transmitted from one body to another and cannot disappear on the basis of some examples which seem to contradict this assertion. All such examples contain incorrect arguments. Let us consider a characteristic example of this kind.

(2) *A boy throws a snowball which, after its impact against the wall, sticks to it or falls near the wall.* The boy "gives" a momentum $K = mv$ to the snowball (m is the mass of the snowball and v is its velocity). The question is whether the momentum of the ball disappears after its impact with the wall or to which body the momentum has been "transferred".

To elucidate the process of transfer of the momentum let us first imagine that both the boy and the wall are on a boat and the boy throws a clay ball which sticks to the wall, all the bodies being at rest before the ball was thrown (see Fig. 64). In this example the boat plays a role analogous to that of the Earth in the previous example. The momentum the ball possesses after it has been thrown is "obtained" from the system "the boy and the boat", and consequently during the flight of the ball this system has a momentum of opposite direction and of magnitude $K = mv$. At the same time, the momentum of the system "the boat, the boy, and the ball" is constant and equal to zero.

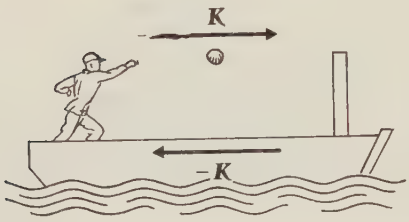

FIG. 64

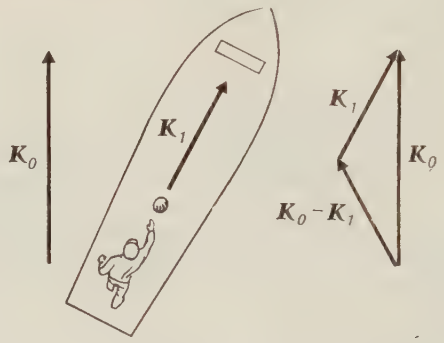

FIG. 65

During the process of inelastic impact of the ball against the wall an interaction between the ball and the boat takes place, the wall playing the role of a body "providing conditions" for the phenomenon to take place. After the impact the ball and the boat have momenta of equal magnitudes and opposite directions, the sum of the momenta being equal to zero. Thus, after the jerk given by the boy to the boat the latter starts moving and then stops when, after the impact of the clay ball and the wall, the ball sticks to the wall. Thus, no "disappearance" or "creation" of a momentum has taken place.

It should also be noted that our argument remains valid in a more general case when the boat is in a motion before the ball is thrown. In this case the whole system has a momentum before the process we consider. After the jerk the momentum of the system "the boy and the boat" decreases by an amount equal to the momentum of the ball, the same amount being gained after the impact. Thus, the total momentum remains the same as before. The directions of the motions of the boat and of the ball thrown by the boy can be arbitrary in this phenomenon. Suppose that the momentum of the system before the jerk is K_0. Let us represent it by the corresponding vector (Fig. 65); if the momentum of the ball thrown by the boy is K_1 the remaining momentum of the boat is equal to $K_0 - K_1$, and after the impact of the ball against the wall the whole system again has a momentum equal to K_0.

The system "the boy and the boat" obviously loses some momentum if the clay ball does not stick to the wall but falls into the water. Then the remaining momentum of the system is $K_0 - K_1$ and the momentum K_1 will be "transferred" to the water. However, if we include the Earth into the system it can again be readily shown that the total momentum is preserved.

Proceeding from the latter example we can interpret the phenomena in the former example when a snowball thrown by the boy sticks to the wall and "transmits" its momentum to the wall (in this case the wall is regarded as being rigidly connected with the Earth and forming one system with it); the transmitted momentum "appears" as the result of the interaction of the boy and the Earth.

For the sake of simplicity we supposed that the impact between the ball and the wall was perfectly plastic since the ball did not bounce off the wall, it either stuck to it or fell into the water near the wall. An analogous situation takes place in an elastic impact when the ball bounces off the wall in the backward direction. Although in this case the boat will move forward, the total momentum of the system "the boat, the boy, and the ball" will remain the same. Indeed, after the ball bounces off the wall, falls into the boat, and stops moving the whole system is at rest.

In these considerations we assumed that there was no friction between the boat and the water (in reality, the friction exists but is negligibly small). However, if we took into account these interaction forces as well and considered the water as a part of the system it would be possible to show in just the same manner that the momentum of the whole system remains constant.

(3) *Measuring masses.* The experiments with the balls moving under the action of the extending spring indicate the possibility of using such a scheme for comparing (that is, measuring) masses of two bodies. Indeed, let us observe the motion of two bodies under the forces resulting from the interaction between them. On determining experimentally the velocities obtained by the bodies after the interaction process we can find, using the law of conservation of momentum, the ratio of the masses of the bodies which is equal to the reciprocal of the ratio of the magnitudes of their velocities. For instance, if one of the colliding balls has unit mass the reciprocal of the ratio of the absolute values of the velocities of the balls is equal to the mass of the other ball measured in the same units.

It is convenient to perform such experiments with carts on the rails which are moved apart by spring buffers (Fig. 66). The carts are first pressed together and then the buffers are let to push them apart. On measuring the distances S_1 and S_2 travelled by the carts during equal time intervals we take into account that these distances are proportional to the velocities of the carts and thus inversely proportional to the masses m_1 and m_2 of the carts. Therefore

$$\frac{m_1}{m_2} = \frac{S_2}{S_1} \tag{25.2}$$

(4) *Perfectly plastic impact of two balls.* Let two balls with masses m_1 and m_2 moving, respectively, with velocities v_1 and v_2 collide at a certain moment, "stick together" and then move as one body (Fig. 67). Here we have the phenomenon of a (completely) inelastic (perfectly plastic) impact. Given the momenta of the balls before the impact, we can readily determine the motions of the balls after the impact. Let $K_1 = m_1 v_1$ be the momentum of the first ball and $K_2 = m_2 v_2$ that of the second ball before the impact; then after the impact the momentum of the system consisting of the two balls stuck together is $K_1 + K_2 = K$. The directions of motion of the balls may change after the impact but the total momentum of that system does not change. The velocity v of the system of the balls after the inelastic impact is therefore given by the formula

$$v = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \tag{25.3}$$

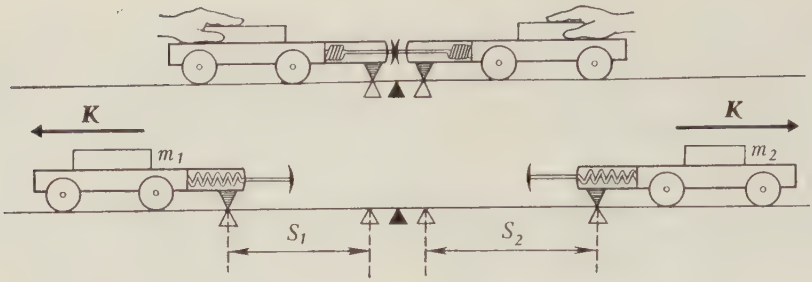

FIG. 66

Let us consider another example which can be treated in just the same way. Imagine a shell which bursts into two parts during its flight. In this case the total momentum cannot change either: the vector sum of the momenta of the parts must be equal to the initial momentum of the shell (provided that the force of air resistance is negligibly small). The same argument applies to the case when the shell bursts into any number of parts.

There are many examples of this kind in everyday life and in engineering. Let the reader give such examples.

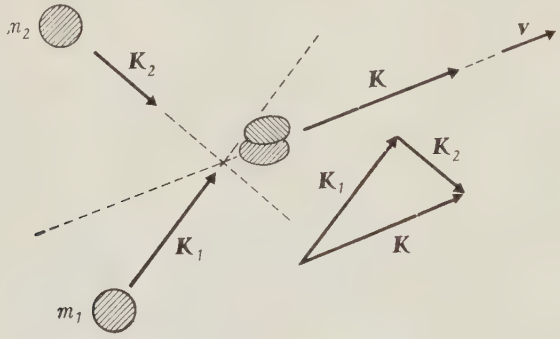

FIG. 67

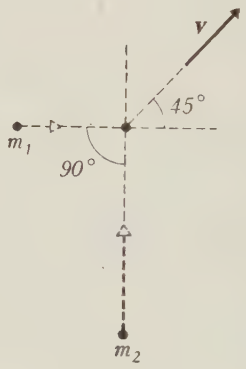

FIG. 68

It should be noted that, when solving a mechanical problem, one should check whether the given problem admits of the application of the law of conservation of momentum. If this is the case it is unnecessary to know the character of variation of the forces of interaction between the bodies during the interaction period and the way in which they act on them, and the application of the law of the constancy of the momentum considerably simplifies the solution of the problem.

As an exercise, let the reader consider Fig. 68 schematizing an inelastic impact of two material points; the distances between the masses m_1 and m_2 at a certain moment before the impact are shown in the figure with a certain length scale and the directions of their velocities before and after the impact are indicated by the arrows. It is required to find the ratio of the masses.

26. Impulse of a Force

Newton's second law (Sec. 19) can be written in the form

$$d(m\boldsymbol{v}) = \boldsymbol{F}\,dt \tag{26.1}$$

It reads: *the increment of the momentum during time interval dt is equal to the impulse* $\boldsymbol{F}$ *of the force during that time.* The vector physical quantity $\boldsymbol{F}\,dt$ is called an (elementary) *impulse* of the force during time dt. The impulse of the force $\boldsymbol{F}\,dt$ is an infinitesimal quantity characterizing the action of the force during the infinitesimal time interval dt.

If the action of a variable force is considered not during an infinitesimal time dt but during a finite interval lasting from t_1 to t_2 then in order to obtain the corresponding impulse $\boldsymbol{P}$ we should split that finite time interval into infinitesimal subintervals dt, find the impulses $\boldsymbol{F}\,dt$ for each subinterval dt and then add them all together. As is known, such a sum is called an integral and is denoted

$$\int_{t_1}^{t_2} \boldsymbol{F}\,dt = \boldsymbol{P} \tag{26.2}$$

The vector quantity $\boldsymbol{P}$ is the impulse of the force $\boldsymbol{F}$ during time interval from t_1 to t_2. If the force $\boldsymbol{F}$ is constant both in magnitude and direction then

$$\boldsymbol{P} = \boldsymbol{F}(t_2 - t_1) \tag{26.3}$$

Thus, the impulse of a constant force is equal to its product by the time during which that force acts.

On integrating both sides of equality (26.1) we obtain

$$\int_{t_1}^{t_2} d(m\boldsymbol{v}) = \int_{t_1}^{t_2} \boldsymbol{F}\,dt = \boldsymbol{P} \tag{26.4}$$

If the velocity of motion of the body is equal to $\boldsymbol{v}_1$ at time t_1 and to $\boldsymbol{v}_2$ at time t_2 expression (26.4) can be written as

$$m\boldsymbol{v}_2 - m\boldsymbol{v}_1 = \boldsymbol{P} \tag{26.5}$$

Indeed, $d(m\boldsymbol{v})$ is an infinitesimal increment of the momentum during time dt; on adding together all these increments for time interval from t_1 to t_2 we obviously obtain the total increment $m\boldsymbol{v}_2 - m\boldsymbol{v}_1$ of the momentum during that finite time interval.

Thus, *the increment of the momentum of the body for some time interval is equal to the impulse of the acting force during that interval.*

For instance, let a stone of weight of 1 N fall on the ground. The magnitude of the impulse of the force of gravity of the stone during 5 s is equal to $P = 5$ N·s. The same increment has been gained by the momentum of the stone (which is directed towards the ground). Consequently within 5 seconds the velocity of the stone gains an increment which, by formula (26.5), is equal to

$$|\boldsymbol{v}_2 - \boldsymbol{v}_1| = \frac{\boldsymbol{P}}{m} = \boldsymbol{e}\,\frac{5\ \text{N}\cdot\text{s}}{\frac{1}{9.81}\ \text{kg}} \approx \boldsymbol{e}\cdot 50\ \text{m/s}$$

where $\boldsymbol{e}$ is unit vector directed vertically downwards. The result we have obtained is completely independent of whether the stone falls vertically or moves

in a parabola; it is completely specified by the fact that during these 5 seconds the stone has been acted upon only by the force of weight. When considering this example the reader should recall that the mass of a body weighing 1 N is equal to $\frac{1}{9.81}$ kg.

Although the increment of a momentum is precisely equal to the impulse of the acting force (provided that this increment is produced by that very force solely, we must nevertheless distinguish between the notion of the increment of a momentum and that of an impulse of a force. The matter is that there exist cases when the body in question can be acted upon by some other forces. For example, let a worker push away a cart, his own position remaining unchanged. The increment of the momentum of the cart is equal to the impulse $\boldsymbol{P}$ of the force exerted by the worker (on condition that the force of friction is very small and can be neglected). In its turn, the cart gives the worker the impulse $-\boldsymbol{P}$. However, the worker retains his position and thus his momentum receives no increment. Consequently, although the reaction force of the cart applied to the worker has a non-zero impulse, the worker gains no increment of the momentum. The explanation of this phenomenon lies in the fact that the worker is acted upon by one more force, namely, by the force with which the Earth acts on his feet. This force gives him an impulse of the same magnitude in opposite direction. If the worker were not acted upon by the force exerted by the Earth he would receive the same momentum. That is why the impulse of a force is sometimes spoken of as being equal to a possible increment of a momentum of a body. It becomes equal to the real increment of the momentum when the body is acted upon by that force solely.

In what follows we shall always speak of a *real* momentum but not of a possible (*virtual*) momentum and of a *real* impulse of the force or of the resultant of the forces acting on the body. However, as has been indicated, there is an essential distinction between these quantities which lies in the fact that the impulse of one of the forces acting on a body is not necessarily equal to the increment of the momentum the body receives. The only case when we can say that the increment of the momentum is equal (both in its magnitude and direction) to the impulse of the force during a certain time is the one in which by the impulse of the force is meant the *impulse of the resultant* of all the forces acting on the body during the same time. In this sense only the increment of the momentum and the impulse of the force (of the resultant of all the forces) can be identified.

In the next section we shall apply the conservation law for momentum to the investigation of motions of bodies with variable masses.

27. Motion of a Body with Variable Mass

There are phenomena in which the masses of bodies taking part in the motion vary during the time of motion. A moving body can lose a part of its mass, that is some particles it consists of can be discharged from it and then its mass decreases or some new particles can be added to it and its mass increases. Examples of this kind are the motion of a sprinkling truck spraying the street, loading or unloading a moving freight car, the motion of a rocket issuing a jet stream of burned gases and the like. In this section we shall study phenomena of this kind.

In such cases not only the mass of a moving body can change but also its velocity can vary. The general laws of motion of bodies with variable masses

were investigated dy I. V. Meshchersky and K. E. Tsiolkovsky; the latter applied them to projecting a reactive spaceship.

Let us consider the motion of a sprinkling truck (Fig. 69) spraying the street. Let the velocity of the water spray relative to the truck be a constant vector $\boldsymbol{c}$ coinciding in its direction with the velocity $\boldsymbol{v}$ of the truck relative to the ground. Let $\boldsymbol{F}$ be the force acting on the truck due to the adhesion of wheels as the engine of the truck is working. This force is directed along the street. The force $\boldsymbol{F}$ is applied to the truck and is an external force exerted on the truck by the Earth.

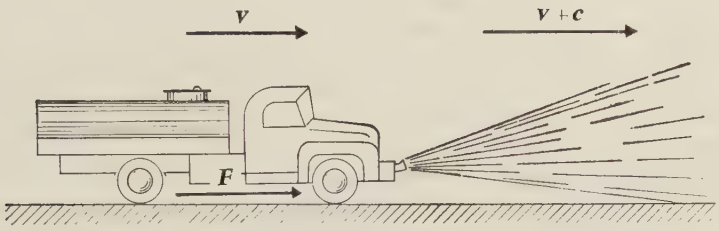

FIG. 69

The per-second expenditure of water will be invariable if the discharge velocity $\boldsymbol{c}$ and the cross-section of the nozzle ejecting the water remain constant.

Let us denote by μ the per-second expenditure of the mass of the ejected water and by M the mass of the truck together with the water. The mass M will uniformly decrease as function of time and its rate of change will be

$$-\frac{dM}{dt} = \mu \tag{27.1}$$

It follows that the mass conservation condition can be written in the form

$$dM + \mu\, dt = 0 \tag{27.2}$$

where dM is the decrease of the mass of the truck filled with the water during time dt and $\mu\, dt$ is the mass of the water flowing out during that time interval. The minus sign in equation (27.1) indicates that the mass M decreases; the per-second expenditure of water μ is assumed to be positive.

Let us apply the law of variation of the momentum to the system "the truck and the ejected water"; according to this law, the increment of the momentum of the system equals the impulse of the external forces*.

Let the truck have a mass M and a velocity v at time t, its momentum at that moment being Mv. After an infinitesimal time interval dt, that is at the moment of time $t + dt$, the mass of the truck becomes $M - \mu\, dt$ and its velocity becomes $v + dv$. The mass of water ejected during time dt is $\mu\, dt$ and its velocity relative to the Earth is equal to $v + c$. It follows that the momentum at time $t + dt$ is equal to

$$(M - \mu\, dt)\,(v + dv) + \mu\, dt\,(v + c)$$

* In this example all the vectors are parallel to one straight line. On projecting them on that line we can use the scalar form of the equations.

On subtracting the quantity Mv from the last expression we obtain the increment of the momentum corresponding to time interval dt. It must be equal to the impulse $F\,dt$ of the external force. Neglecting the term $\mu\,dt\,dv$ which is of the second order of smallness relative to dt, we obtain

$$M\,dv + \mu c\,dt = F\,dt$$

whence

$$M\frac{dv}{dt} = F - \mu c \tag{27.3}$$

Relation (27.3) is known as *Meshchersky's equation*; it holds for any body moving with velocity v and issuing a part of its mass with velocity c in the direction of motion with per-second expenditure μ. Equation (27.3) reads: *the product of the mass of the body by its acceleration equals the acting force F minus the reactive force of the ejected mass*, the latter being equal to per-second expenditure of mass μ times the velocity c of relative motion of the discharged particles.

The reactive force is due to the fact that the ejected particles are given velocity c. These particles (together with the whole mass of the water) are at rest relative to the truck until a certain force gives them an acceleration relative to the moving truck; this results in the velocity c of the particles relative to the truck. The reaction force (whose existence is asserted in the third law of dynamics) corresponding to the former force is applied to the truck and its direction is opposite to the direction of the velocity of the ejected water. It is this force that we call the reactive force.

In the most general case we can similarly prove that a body ejecting a part of its mass with per-second expenditure μ and velocity $\boldsymbol{c}$ directed quite arbitrarily is acted upon by a reactive force $\boldsymbol{f}_r$ expressed by the formula $\boldsymbol{f}_r = -\mu\boldsymbol{c}$. Let us present the proof. During an infinitesimal time interval dt the body loses a part of its mass equal to $\mu\,dt$ which has a velocity $\boldsymbol{v}_1$ relative to the Earth (Fig. 70). At time t the body had the momentum

$$M\boldsymbol{v}$$

where $\boldsymbol{v}$ is the velocity of the body at that moment. At the moment of time $t + dt$ the remaining part of the mass of the body has the momentum

$$(M - \mu\,dt)\,(\boldsymbol{v} + d\boldsymbol{v})$$

During time dt the velocity of the body receives an increment $d\boldsymbol{v}$, and the ejected mass $\mu\,dt$ possesses the momentum

$$\mu\,dt\,\boldsymbol{v}_1$$

Consequently, the change of the momentum of the whole system during time interval dt is equal to

$$(M - \mu\,dt)\,(\boldsymbol{v} + d\boldsymbol{v}) + \mu\,dt\,\boldsymbol{v}_1 - M\boldsymbol{v} = M\,d\boldsymbol{v} + \mu\,dt\,(\boldsymbol{v}_1 - \boldsymbol{v}) - \mu\,dt\,d\boldsymbol{v}$$

By the second law of dynamics, the change of the momentum equals the impulse of the external force $\boldsymbol{F}$ acting on the body during that time interval. Therefore

$$M\,d\boldsymbol{v} + \mu\,dt\,(\boldsymbol{v}_1 - \boldsymbol{v}) = \boldsymbol{F}\,dt$$

since the term $\mu\,dt\,d\boldsymbol{v}$ is of the second order of smallness and can be neglected. On dividing both members of the last equality by dt and transposing the term involving μ to the right we obtain

$$M\frac{d\boldsymbol{v}}{dt} = \boldsymbol{F} + \mu\,(\boldsymbol{v} - \boldsymbol{v}_1)$$

Denoting by $\boldsymbol{c}$ the relative velocity of the particles separated from the body we can write $\boldsymbol{v} + \boldsymbol{c} = \boldsymbol{v}_1$, whence $\boldsymbol{v} - \boldsymbol{v}_1 = -\boldsymbol{c}$, and we finally obtain

$$M\frac{dv}{dt} = \boldsymbol{F} - \mu\boldsymbol{c} \tag{27.4}$$

The direction of the reactive force is opposite to that of the velocity with which the ejected particles leave the body.

For instance, when a rocket projectile moves near the Earth's surface (see Fig. 71) the resultant external force is the sum of the force of gravity $\boldsymbol{P}$ produced

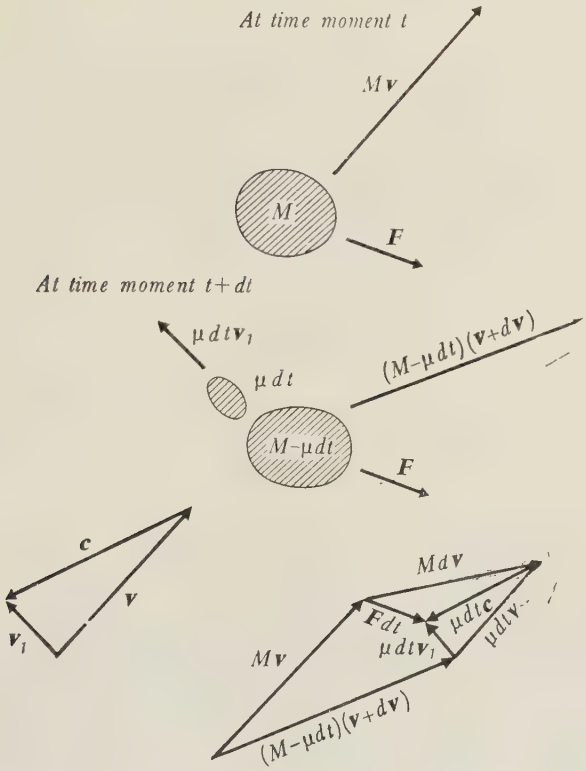

FIG. 70

by the Earth and the air drag (resistance) $\boldsymbol{X}$, the acceleration of the rocket being additionally dependent on the magnitude and the direction of the reactive force $-\mu\boldsymbol{c}$. Changing the magnitude and the direction of the reactive force during the motion we can control the flight of the projectile.

It should be noted that if we considered the addition of particles to the moving body (instead of their separation) resulting in an increase of its mass equation (27.4) would remain valid with the only distinction that the sign of the reactive force must be opposite, namely

$$M\frac{dv}{dt} = \boldsymbol{F} + \mu\boldsymbol{c} \tag{27.5}$$

where c is the velocity of the particles (which are added to the body) relative to the body. As an exercise, let the reader derive this equation.

Using formula (27.4) we can compute the thrust force of a turbojet engine (or of an air-breathing engine) of an airplane which intakes the air in its flight, compresses it, burns the fuel, and ejects the combustion product as a gas stream with large velocity. A turbojet engine has an air compressor driven by a turbine placed in the stream of the ejected gas (Fig. 72).

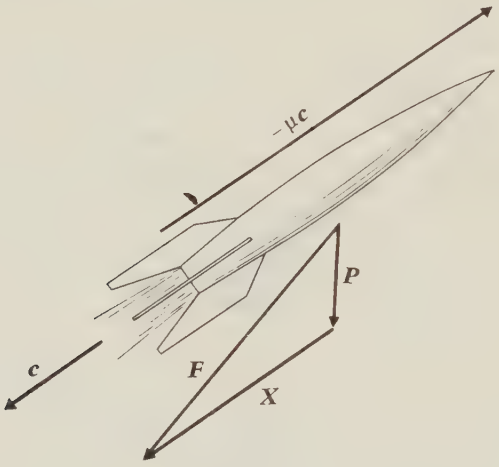

FIG. 71

An air-breathing engine has neither a compressor nor a turbine. It is a tube with variable cross section in which the fuel is burned in the air which is sucked in and compressed because of the motion of the airplane (Fig. 73). Such an engine can only produce a thrust force when the aircraft is already in motion while a turbojet engine can produce it even when the airplane does not move.

It is evident that for a reaction engine the per-second inlet of the air equals the per-second outlet of the air. Let the airplane fly with velocity v, the per-second flow-out of the air being μ_a. The working engine ejects the gases (the combustion product consisting of the air and the burned fuel). Let the per-second expenditure of the fuel be μ_f. Since the atmospheric air is considered to be at rest relative to the Earth, the intake of the air produces a backward reactive force equal to $\mu_a v$. The ejection of the combustion product produces a forward reactive force equal to $(\mu_a + \mu_f)\, c$. Hence, the resultant reactive force of a turbojet engine (or of an air-breathing engine) directed forward is equal to

$$\mu_a (c - v) + \mu_f c$$

Practically we always have $\mu_f \ll \mu_a$, which allows us to put, approximately, the reaction force of a turbojet or of an air-breathing engine equal to

$$\mu_a (c - v)$$

The motion of an airplane with such an engine can be considered, approximately, as a motion of a body with constant mass provided that the mass

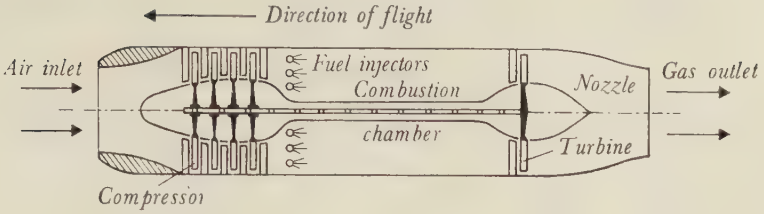

FIG. 72

of the burning fuel is negligibly small. The last expression of the reaction force shows that for a thrust force to be obtained it is necessary that the velocity c of the ejected particles should exceed the velocity v of the flight. In order to increase the thrust of the engine it is necessary to increase the velocity of the ejected gases or the per-second air expenditure in the engine or both. Irrespective of the complicated character of the processes in the engine (the air intake, the compression of the air, the fuel burning, the operation of the gas turbine

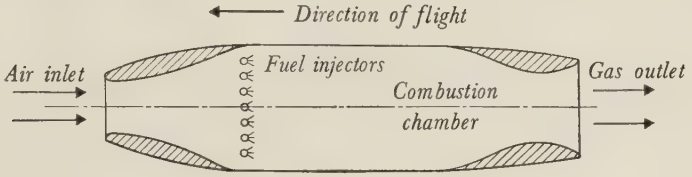

FIG. 73

etc.), to determine the thrust force giving the airplane the velocity v it is only required to know two quantities, namely μ_a and the velocity c.

The thrust can also be estimated as follows: the mass of the air μ_a receives every second a momentum equal to $\mu_a (c - v)$ (relative to the Earth) under the action of the forces with which the airplane acts upon the air. Consequently, by the law of conservation of momentum, the momentum of the airplane (relative to the Earth) must gain a per-second increase equal to the latter quantity. Therefore $\mu_a (c - v)$ is equal to the impulse of the thrust force of the reaction engine during one second, the impulse of a force in unit time being equal, numerically, to the magnitude of the force.

What has been said also applies to any reaction engine intaking and ejecting water (instead of the air).

The situation becomes quite different in the case of a (powder or liquid-propellant) *rocket* engine. In this case the fuel and the rocket oxidizer are both in the flying vehicle and therefore *the thrust of a rocket is always equal to* $-\mu c$ for any velocity of the flight of the vehicle moved by the rocket engine. The thrust force is independent of the properties of the surrounding medium and appears in a flight in vacuum as well.

The dependence of the velocity of a rocket on time and the relationship between the velocity and the mass can be derived from formula (27.4) on condition that the motion is only caused by the reactive force. Let the initial mass

of the rocket (for $t = 0$) be M_0, then at time $t > 0$ it will be equal to $M = M_0 - \mu t$. By formula (27.4), we can write

$$M \frac{dv}{dt} = \mu c, \text{ that is } (M_0 - \mu t)\, dv = \mu c\, dt \tag{27.6}$$

because the velocity of the ejected gases is always directed along the tangent

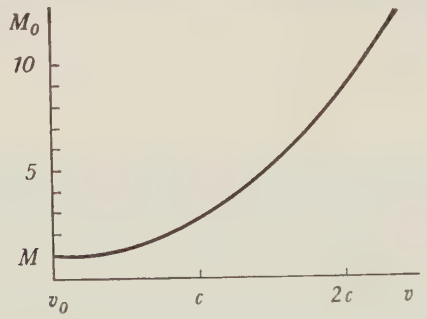

FIG. 74

to the trajectory of the rocket. On rewriting the last equation in the form

$$\frac{dv}{c} = \frac{\mu\, dt}{M_0 - \mu t}$$

we can readily integrate its right-hand and left-hand members, which results in

$$\frac{1}{c}(v - v_0) = \int_0^t \frac{\mu\, dt}{M_0 - \mu t} = \ln M_0 - \ln (M_0 - \mu t) = \ln \frac{M_0}{M_0 - \mu t} = \ln \frac{M_0}{M}$$

It follows that

$$v - v_0 = -c \ln \left(1 - \frac{\mu t}{M_0}\right)$$

whence

$$M_0 = M \exp \left(\frac{v - v_0}{c}\right) \tag{27.7}$$

Relation (27.7) is known as *Tsiolkovsky's formula*. The dependence descrided by it is represented in the diagram in Fig. 74. If a mass of magnitude M should be given a velocity v the initial mass of the rocket must be equal to M_0 when the initial velocity is v_0. In a special case v_0 can be equal to zero. When the given velocity v is increased the initial mass M_0 grows exponentially. When $(v - v_0) \gg c$ the ratio M_0/M becomes very large; this creates technical difficulties for the construction of cosmic rockets whose velocities are of the order of 10 km/s. The increase in the velocity of the ejected gases makes it possible to reduce considerably the initial mass and thus to facilitate the solution of this construction problem.

When a rocket moves in a dense atmospheric layer it is necessary to overcome the air drag and therefore the initial mass of the rocket must be in fact greater than that computed above; however, when the velocity of the rocket, as it passes through the dense layer of the atmosphere, is not very large the drag is comparatively small and therefore the corresponding increase of the initial mass M_0 is not very large either.

CHAPTER 4

Work and Energy

28. The Concept of Energy

The momentum of a body is an important characteristic of its mechanical motion. However, this quantity by far not always characterizes completely the changes in the motion of the body. For instance, after a perfectly plastic impact of two identical balls moving towards each other with equal velocities the motion of the balls "disappears" in the sense that before the impact each of the balls moved while after the impact they are at rest. The law of conservation of momentum remains of course valid in this case as well; the momenta of the balls before the impact were equal in their magnitudes and opposite in the directions so that the momentum of the whole system was equal to zero. The total momentum retains the same value after the impact but the law of conservation of momentum can be applied to neither of the two balls because during the impact process each of the balls is acted upon by an external force, namely, by the force of pressure produced by the other ball.

In this example the states of the two balls underwent essential changes after the impact. The balls moved before the impact and stopped moving after the impact; their motions "disappeared" and they lost their momenta. Besides, as experiments show, the temperature of each ball increases after the impact. We see that the law of conservation of momentum cannot be used for a complete description of all mechanical phenomena, which means that the momentum cannot serve a unique and exhaustive measure of all the changes in the mechanical states of the bodies. The fact that the temperature of the ball increased during the impact process is of principal importance: the mechanical motion of the balls "disappeared" but instead of it a new form of motion of substance appeared, namely, heat. Experiments show that the amount of heat resulting from the impact of the balls is not proportional to the magnitude of the momentum that every ball possesses before the impact. The momentum of a body and an amount of heat are of different nature: the former is a vector quantity while the latter is a scalar. The vector sum of the momenta of the balls is equal to zero. One may think that since an amount of heat is a scalar it would be more expedient to compare it with the sum of the moduli of the momenta of the balls but, as experiments show, the sum of the moduli is not proportional to the amount of heat obtained either. Hence, besides the momentum, we must introduce another measure of a mechanical motion of a body, which is particularly needed for the description of phenomena accompanied by the conversion of the mechanical motion of a body into other forms of motion of substance.

It turns out that a measure of this kind, applicable in all the cases, is the *energy*. It remains invariable in all the conversions of substance. Thus, *the energy serves as a universal measure for all the forms of motion of substance.*

The fundamental question concerning the connection between the numerical value of the energy of a moving body and other physical characteristics of its mechanical motion and of other forms of motion of substances was answered completely about a hundred years ago. Since then the *law of conservation of energy* has been understood as the *fundamental* law of nature.

Before proceeding to the discussion of the measure of the energy of a mechanical motion we must consider an important physical quantity, namely *work*, which plays a basic role in those processes where the mechanical motion and the mechanical energy are transmitted from one body to another.

29. Work and Energy

We shall begin with two examples in which some forces act while the momenta do not vary. As the first example let us consider a body lying on the table. In the second example we consider an electric locomotive producing a tractive force applied to the train in a uniform rectilinear motion with a constant velocity. In both cases there are external forces: in the first example the force of gravity acting on the resting body is an external force and in the second example the tractive force of the locomotive is an external force acting upon the train. In both cases there are no obvious changes of the momenta since the body is in a state of rest and the train is in a uniform rectilinear motion. The explanation of the constancy of the momenta is that the body and the train are subjected to the actions of some other forces so that the resultant of all the forces is equal to zero in either case.

However, there is an essential distinction between the situations described in these examples. In the first example the body lies on the table, the force of gravity acts on it permanently, and there are no changes both in the body itself and in the surrounding bodies. In the second example we also deal with a permanently acting force and there are no changes of the momentum either but the action of the force is in this case in a certain connection with some rather complex processes involving the surrounding bodies. The force of gravity acting upon the body on the table is produced by the Earth but the latter does not undergo any changes. Conversely, for the tractive force of the electric locomotive to act on the train a certain amount of electric energy must be spent, which is connected with a number of complicated processes. Hence, the continuous action of the tractive force involves complex phenomena in which the surrounding bodies take part.

Now let us discuss the basic distinction between the phenomena described in these examples from the point of view of mechanics. The distinction lies solely in the fact that in the first case the point of application of the force (of gravity) is *fixed* while in the second case the point of application of the (tractive) force *moves* with a certain velocity. Experiments show that if all the other conditions remain invariable the amount of fuel expended on the electric supply during some time is directly proportional to the product of the magnitude of the tractive force by the distance travelled by the locomotive during that time. It also turns out that in all phenomena of this kind an essential role is played by an important physical quantity called the *work which is measured by the product of the acting force and the distance travelled*; generally,

the work is a measure of the transfer of motion from one body to another, or, in other words, *a measure of the transfer of energy from one body to another.*

Substance is always in motion and it is only the forms of that motion which may vary. In natural processes there is a permanent conversion of one form of motion into another. There exists a measure of motion of substance common for all the phenomena in the world and for all forms of motion of substance, and, as has already been said, the *energy* of a body or of a system of bodies serves as that measure.

The ancients had been familiar with the idea that there cannot be an annihilation of the motion of substance and this idea lay in the foundation of the philosophic systems of such great scientists as R. Descartes, M. V. Lomonosov, and others. However, only in the nineteenth century was the general law known as the *law of conservation of energy* universally recognized by all scientists and, first of all, by physicists as the basic law of nature.

Every body or a system of bodies possesses a certain amount of energy. In every natural phenomenon there is a transfer of energy from one body to another or from some parts of a body or of a system of bodies to its other parts. In physics we distinguish between various forms of motion (of substance) when classifying the subjects of different sciences. In this sense we speak of mechanical, thermal, electromagnetic motions, etc. but *the energy is a unified quantitative measure for the motion of substance common for all the forms of motion.*

There are such forms of motion of substance in which the form of the motion is not changed: for instance, a hot body heats a colder one, a stone thrown upwards moves and then falls on the ground, etc. But there are also such phenomena in which one form of motion is converted into another. For instance, a bullet hitting a board and sticking in it is heated, that is the mechanical motion is converted to the thermal one. When a locomotive moves the train the thermal motion resulting from the burning of the fuel at the power electric station is converted into the electric motion and the latter is then converted into the mechanical motion of the train. When a ball rolls on a table and then stops moving the mechanical form of motion is converted into the thermal one, etc. However, in all the cases the amount of energy of any form transmitted from one body or from a system of bodies to another body or another system of bodies is exactly equal to the amount of energy which the latter body or system of bodies receives.

We sometimes say "the mechanical energy", "the thermal energy", "the electromagnetic energy" and the like; these expressions should be understood in the sense that they indicate certain amounts of energy corresponding to the forms of motion mentioned, that is to the mechanical form of motion, to the thermal form of motion, etc. But there are in fact no different kinds of energy; there only exist various forms of motion of substance, the energy being a unified measure for all kinds of motion. However, for the sake of simplicity, we shall often conditionally speak of "the mechanical energy", "the thermal energy", etc. bearing in mind what has been said here.

As to the cases when we only deal with the mechanical form of motion or with phenomena in which the mechanical form of motion is converted into some other form and vice versa the quantity we considered above, that is the product of the magnitude of the force by the distance travelled which is called the mechanical work, serves as the measure of the amount of energy transferred. That is why the unit of energy is usually chosen equal to the unit of work.

In the SI system the unit of work and energy is 1 *joule* (1 J); it is equal to the work performed by a force of 1 N (one newton) over a distance of 1 m

(one metre). In the cgs system often used in physics the unit of work is the work performed by a force of 1 dyn (one dyne) over a path of 1 cm (one centimetre); this unit is called the *erg* (1 erg). It can readily be shown that

$$1 \text{ J} = 10^7 \text{ erg}$$

30. Work of a Force

When a force acts on a moving body and the directions of the force and of the velocity of the motion coincide then the transfer of energy from the body which exerts the action to the body on which the force acts takes place. In such a case the work performed by the force is considered *positive*. A positive value of the work corresponds to the transmission of energy from the acting body to the body which is moved. When the directions of the force and of the displacement are opposite then, conversely, the energy is transmitted to the body exerting the force. In this case the work of the force is considered *negative*.

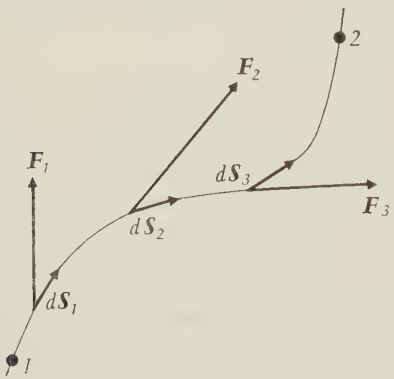

FIG. 75

In the general case when the directions of the force and of the displacement are different the magnitude of the work is, by definition, equal to the product of the projection of the force on the direction of the displacement by the magnitude of the displacement (by the distance travelled) or, in other words, the work is equal to the *scalar product* of the vector of force by the displacement vector.

Thus, if body *1* acts on body *2* with a force $\boldsymbol{F}_{21}$ and body *2* receives a displacement $d\boldsymbol{S}_2$ then body *1* performs the work

$$dA = \boldsymbol{F}_{21} \cdot d\boldsymbol{S}_2 \tag{30.1}$$

equal to the scalar product of the vector of the force by the displacement vector. If $dA > 0$ then body *1* transmits energy to body *2*; if $dA < 0$ then body *2* transmits energy to body *1*. In case the displacement $d\boldsymbol{S}_2$ is perpendicular to the force $\boldsymbol{F}_{21}$ we have $dA = 0$, that is there is no exchange of energy between the bodies. The work is performed only by the component of the force along the direction of the displacement; hence, the amount of work performed can be expressed as

$$dA = F_{21} \cdot dS_2 \cdot \cos \alpha \tag{30.2}$$

where α is the angle of shortest rotation from the direction of the force $\boldsymbol{F}_{21}$ to the direction of the displacement $d\boldsymbol{S}_2$.

When an acting force varies both in its magnitude and direction as the point of its application moves then in order to compute the total work corresponding to the whole trajectory of motion we must find the elementary work dA for every infinitesimal part dS of the path of motion which is equal to $F\,dS$ and then add together all these elementary magnitudes of work along the whole

trajectory (for instance, from point *1* to point *2* as shown in Fig. 75). In other words, we must integrate F with respect to dS from point *1* to point *2*.

Thus, the work performed by the force F along the trajectory joining point *1* to point *2* can be expressed in the form

$$A = \int_1^2 dA = \int_1^2 F\,dS \tag{30.3}$$

The magnitude of the work A can only be found by formula (30.3) when the variation of the force F along the trajectory of motion is known.

31. Potential Energy of Strain

When a body undergoes a deformation produced by the action of an external force the point of application of the deforming force gains a displacement, and hence the system exerting that force performs an amount of work which is the measure of the energy transmitted to the body subjected to the deformation. If an *elastic* body is being deformed the work performed is expended on the increase of the amount of energy of that body; this is the *strain energy* (the *potential energy of strain*).

In the case of an elastic body the magnitudes of the deformation and of the force are connected in a one-to-one manner irrespective of the direction of the deformation. For instance, imagine that such a body is contracted by dS under the action of a force F; in this case an amount of energy equal to $dA = F\,dS$ is expended. If the body is then allowed to extend by dS and to restore its original shape and size it exerts over the path dS a force $-F$ of the same magnitude and opposite direction and performs the work $-dA$. In this process the contracted body transmits its strain energy to the body on which it acts in the extension process. It is obvious that the strain energy of an elastic body can be completely transformed into an equal amount of mechanical work.

The magnitude of deformation is usually connected with that of the acting force by a definite law. The magnitude of the force changes as the deformation varies and therefore the amount of work is not simply equal to the product of the force by the distance because the force is variable and depends on the displacement of the point of its application. The work can readily be computed when the deformation is proportional to the displacement. For example, suppose that it is required to elongate the spring shown in Fig. 76 which obeys the law[1]

$$F = kx \tag{31.1}$$

where F is the magnitude of the force applied to the spring, x is the elongation of the spring under the action of that force, and k is a constant proportionality coefficient called the *stiffness factor* of the spring. The dimension of k is N/m and its numerical value is equal to the force under which the deformation of the spring is equal to unit length. The work that must be expended for the spring to elongate from x to $x + dx$ is obviously equal to $F\,dx$, that is to $kx\,dx$. Consequently, the whole work A corresponding to the extension from $x = 0$ to $x = x_0$ is expressed by the formula

$$A = \int_0^{x_0} F\,dx = k\int_0^{x_0} x\,dx = \frac{1}{2}kx_0^2 \tag{31.2}$$

This work is equal to the strain energy of the spring, with stiffness factor k, extended (or contracted) by x_0.

The force exerted by the contracted spring can be made to act on a body which starts moving while the spring elongates. The force of action of an elastic spring is determined by formula (31) and therefore the work performed by the extending spring is equal to the work expended on its contraction, the

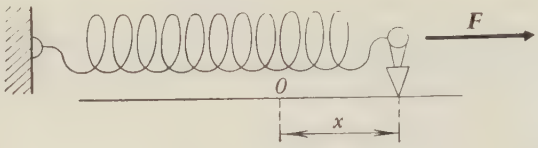

FIG. 76

strain energy of the contracted spring being completely transmitted to the body on which the spring acts when it is extending.

32. Kinetic Energy of a Body

When a force is applied to a body and the latter moves under the action of the former the point of application of the force is displaced and the system exerting that force performs some work. The energy of the moving body gains an increase equal to that work. A mechanical motion of a body in space is the simplest form of motion of substance; as the measure of this motion serves the energy stored by the body; it is called the *kinetic energy*.

The magnitude of the kinetic energy, that of the energy connected with a motion of a body, can be determined from the amount of work which should be expended on the creation of that motion. Let a force $\boldsymbol{F}$ act upon a body of mass m and produce the motion of the body with velocity v_0, the initial velocity being zero (a state of rest). Then the work of the force $\boldsymbol{F}$ along the whole path travelled by the body during the time of the variation of its velocity from zero to the value v_0 is expended on the increase of the kinetic energy of the body. By the second law of dynamics, we always have

$$m \frac{d\boldsymbol{v}}{dt} = \boldsymbol{F} \tag{32.1}$$

On multiplying scalarly both members of equality (32.1) by $d\boldsymbol{S}$ ($d\boldsymbol{S}$ is the increment of the distance travelled by the body) we obtain

$$m \frac{d\boldsymbol{v}}{dt} d\boldsymbol{S} = \boldsymbol{F}\, d\boldsymbol{S} = dA \tag{32.2}$$

As we know, $\boldsymbol{v} = \frac{d\boldsymbol{S}}{dt}$, and therefore equality (32.2) can be rewritten in the form

$$m\boldsymbol{v}\, d\boldsymbol{v} = \boldsymbol{F}\, d\boldsymbol{S} \tag{32.3}$$

The last relation involves the scalar product of the vector $\boldsymbol{v}$ by the vector $d\boldsymbol{v}$ which is equal to $v\, dv \cos \alpha$. The quantity $dv \cos \alpha = dv_S$ is the projection of the increment $d\boldsymbol{v}$ of the vector $\boldsymbol{v}$ onto the direction of the vector $\boldsymbol{v}$ (i.e. on the direction of the tangent line to the trajectory of the body) or, which is the same, the projection of the increment of the velocity on the direction of the vector $d\boldsymbol{S}$ drawn from the point of the trajectory we deal with. Hence, dv_S

is the increment of the modulus of the velocity vector during time dt. Therefore equality (32.3) can also be written as

$$mv\,dv_S = m\,d\left(\frac{v^2}{2}\right) = \boldsymbol{F}\,d\boldsymbol{S} \tag{32.4}$$

We now integrate the right-hand and the left-hand sides of equality (32.4) over the path of integration corresponding to the increase of the magnitude of the velocity from zero to v_0:

$$m\int_0^{v_0} d\left(\frac{v^2}{2}\right) = \int_0^t \boldsymbol{F}\,d\boldsymbol{S} = A = \frac{mv_0^2}{2} \tag{32.5}$$

Thus, the kinetic energy of a body of mass m moving with velocity v_0 is equal to

$$\frac{mv_0^2}{2}$$

The system exerting the force $\boldsymbol{F}$ which produces the motion of the body *performs* some work; some part of the energy of that system is transmitted to the body. Conversely, when a body moving with a certain velocity is stopped or decelerated the decelerative force also performs some work but the magnitude of that work is negative (since the displacement of the body and the force have opposite directions). In this case the decelerative body gains an amount of energy equal to the decrease of the kinetic energy of the moving body. Thus, the decelerated moving body *transfers* a part of its kinetic energy (or the whole kinetic energy) to the system producing the deceleration. The form into which the kinetic energy is transformed depends on the concrete physical conditions.

When, for instance, a bullet hits a board and sticks in it the board undergoes a jerk and starts moving. In this case a part of the kinetic energy of the bullet is transformed into heat and the rest goes into the kinetic energy of the motion of the board after the impact. This is an example of an inelastic (perfectly plastic) impact; this phenomenon is accompanied by the transformation of a part of the kinetic energy into the thermal energy. If both bodies are at rest after an inelastic impact the whole kinetic energy goes into heat.

33. Perfectly Plastic Impact of Two Bodies

Let a body of mass m_1 moving with velocity $\boldsymbol{v}_1$ collide with another body of mass m_2 having (before the impact) a velocity $\boldsymbol{v}_2$. If both bodies have the same velocity $\boldsymbol{v}$ after the impact (and move as one system) we speak of an *inelastic* (or *perfectly plastic*) impact. As was indicated in Sec. 25, the velocity $\boldsymbol{v}$ of the bodies resulting from the inelastic impact can be found with the aid of the conservation law for momentum. The absolute value and the direction of the velocity after the impact are determined (see (25.3)) by the formula

$$\boldsymbol{v} = \frac{m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2}{m_1 + m_2} \tag{33.1}$$

where $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are the velocities of the bodies before the impact and m_1 and m_2 are their masses. The resultant value E_r of the kinetic energy of the system

consisting of the two bodies (after the impact) has the value*

$$E_r = \frac{1}{2}(m_1 + m_2)v^2 = \frac{(m_1 v_1 + m_2 v_2)^2}{2(m_1 + m_2)} \tag{33.2}$$

Before the impact the initial kinetic energy E_{in} of the two bodies is

$$E_{in} = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \tag{33.3}$$

We can now evaluate the "loss" of the mechanical energy, that is the part

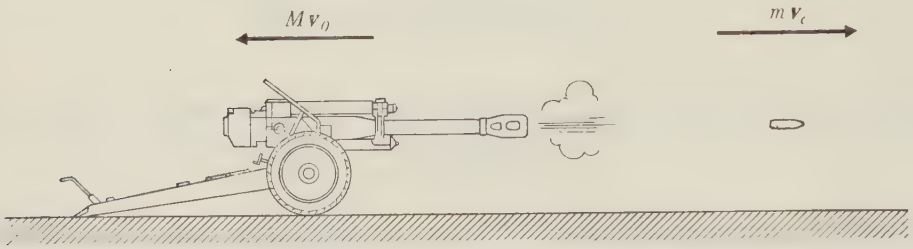

FIG. 77

of the energy which has been transformed into the thermal energy. To this end we subtract equality (33.2) from (33.3), which gives

$$E_{in} - E_r = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_1 v_2^2 - \frac{(m_1 v_1 + m_2 v_2)^2}{2(m_1 + m_2)} = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (v_1 - v_2)^2 \tag{33.4}$$

The vector difference $v_1 - v_2$ is the velocity of the relative motion of the bodies before the impact. That is why the part of the kinetic energy transmitted into heat depends both on the factor $\frac{m_1 m_2}{m_1 + m_2}$ involving the masses of the colliding bodies and on the relative velocity of their motion before the impact. The "energy loss" $E_{in} - E_r$ can be conditionally regarded as the kinetic energy of an "effective" mass

$$m_0 = \frac{m_1 m_2}{m_1 + m_2}$$

moving with velocity $v' = v_1 - v_2$. In this form the formula determining the "loss" of the mechanical energy in a perfectly plastic impact can easily be memorized; this form is also convenient for computations and analysis.

Let us consider the special case when the mass of one of the colliding bodies is very large relative to that of the other; for definiteness, let $m_2 \gg m_1$. Then we have

$$m_0 = \frac{m_1}{1 + (m_1/m_2)} \approx m_1$$

and the loss of the mechanical energy is practically equal to the kinetic energy of the relative motion of the smaller body.

The amount of heat separated after a dud shell hits the armour of a moving tank (or of a combat airplane) and gets stuck in it is practically equal to the

* It should be noted that $v^2 = v^2$.

kinetic energy of the shell. When computing the kinetic energy of the shell we must take its velocity relative to the armour.

The mechanical phenomena taking place when, on firing, the gun recoils (see Fig. 77) or when a shell blows up are essentially of the same nature as the phenomenon of an inelastic impact; the only distinction is that the course of the processes is opposite. In both cases two (or more) bodies possess the same velocity and a certain amount of total kinetic energy before the shot and the blast. During a (very short) period of the explosion of the powder charge the shell and the gun receive different velocities and some amounts of kinetic energy. If the masses of the gun and of the shell and also the amount of the mechanical energy resulting from the shot are known the laws of conservation of energy and momentum allow us to compute the velocities of the shell and the gun after the shot.

34. Elastic Impact

In an (entirely) elastic impact of two bodies, for instance, of two ivory balls or of two extra-hard steel balls, the bodies undergo elastic deformation, their surfaces are pressed in and the forces resulting from the deformation change their velocities.

A thorough analysis of the phenomena taking place in an impact of arbitrary elastic solids is rather intricate; that is why we begin with the simplest case of a *central* impact of two homogeneous balls. By a central impact is meant the one in which the velocities of the colliding balls before the impact are parallel to the line joining their centres (Fig. 78*a*).

The impact process can be described as follows. During the period following the beginning of the impact when the centres of the balls come together (Fig. 78*b*) the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ of their interaction grow as the deformation increases until both balls obtain the same velocity (Fig. 78*c*). At this moment the deformation achieves its maximum. After that moment the forces begin decreasing and their action pushes the balls apart (Fig. 78*d*). This lasts until the balls are separated, after which they move with different velocities (Fig. 78*e*).

A detailed investigation of the deformation phenomenon of the balls during the impact is an extremely complicated problem. However, the velocities of the balls after the impact can easily be found without such an investigation provided that the masses and the velocities of the balls before the impact are known and there is no transformation of the mechanical energy into heat. During the compression period (when the balls are being brought together) the kinetic energy is transformed into the strain energy of the balls, after which during the *restitution period* (as the balls are separated) the strain energy is completely transformed back into the kinetic energy.

This allows us to use both the energy conservation law and the law of conservation of momentum. Therefore, we have

$$m_1 v_{10} + m_2 v_{20} = m_1 v_1 + m_2 v_2 \tag{34.1}$$

and

$$\frac{1}{2}(m_1 v_{10}^2 + m_2 v_{20}^2) = \frac{1}{2}(m_1 v_1^2 + m_2 v_2^2) \tag{34.2}$$

where m_1 and m_2 are the masses of the balls, v_{10} and v_{20} are their velocities before the impact and v_1 and v_2 the velocities after the impact.

The vectors representing the velocities of the balls after the elastic impact lie in the centre line of the balls because, by the symmetry, the interaction forces acting during the impact lie in that line. Equations (34.1) and (34.2) can

be rewritten in the form

$$\left.\begin{aligned} m_1(v_{10}-v_1) &= m_2(v_2-v_{20}) \\ m_1(v_{10}^2-v_1^2) &= m_2(v_2^2-v_{20}^2) \end{aligned}\right\} \tag{34.3}$$

On performing the term-by-term division of the first of these equations by the other we obtain

$$v_{10}+v_1=v_2+v_{20} \tag{34.4}$$

Next we multiply equation (34.4) by m_1, add the resultant equation to the

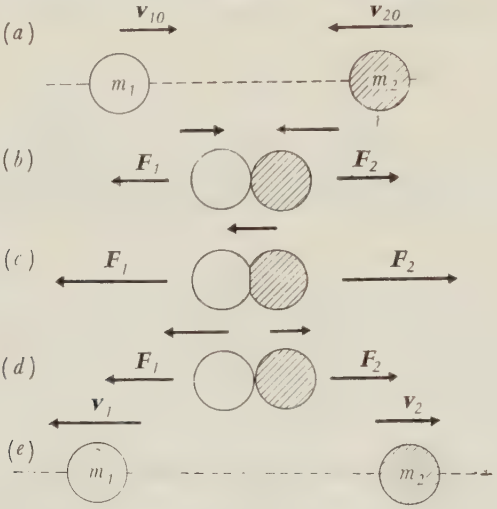

FIG. 78

first equation (34.3), and find the velocity of the second body after the impact:

$$v_2=\frac{2m_1v_{10}-(m_1-m_2)\,v_{20}}{m_1+m_2} \tag{34.5}$$

The formula determining the velocity of the first body after the impact can readily be obtained from formula (34.5) by interchanging the subscripts 1 and 2:

$$v_1=\frac{2m_2v_{20}-(m_2-m_1)\,v_{10}}{m_2+m_1} \tag{34.6}$$

If the balls are of the same mass and one of them is at rest before the impact, for example, $v_{20}=0$, then the velocity of the first ball after the impact becomes zero while the second ball starts moving with the velocity v_{10} which the first ball had before the impact.

This result can also be obtained directly from the energy and momentum conservation laws. Indeed, the masses of the balls being equal, the law of conservation of momentum implies that the sum of the velocities of the two balls after the impact must be equal to the velocity of the first ball before the impact. The energy conservation law requires that the sum of the squares of the velocities of the balls after the impact should be equal to the square

of the velocity of the first ball before the impact. These two conditions can hold simultaneously if and only if the velocity of one of the balls after the impact is zero, and it is only the velocity of the first ball which moved before the impact that can turn into zero after the impact since only the first ball is decelerated during the impact process.

In a central impact of two homogeneous balls with equal masses the exchange of the velocities between the balls takes place. For, putting $m_1 = m_2 = m$ in formulas (34.5) and (34.6), we obtain

$$v_2 = v_{10} \text{ and } v_1 = v_{20} \tag{34.7}$$

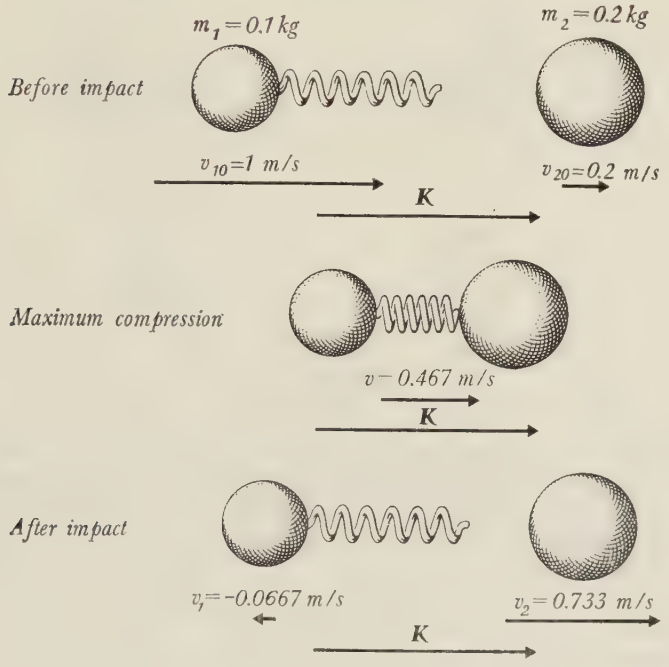

FIG. 79

The essence of the phenomenon of an elastic impact can be explained if we consider a conditional model of two (perfectly) rigid balls with a spring placed between them which collide and interact with the aid of that spring.

Let the spring be attached to one of the balls, the mass of the spring being negligibly small relative to the mass of each of the balls (a "weightless" spring). Let the initial velocities of the balls be in the direction of their centre line so that the force exerted by the spring, when it is contracted, also goes along that line. This model is shown in Fig. 79. For definiteness, the concrete values of the quantities involved are indicated in the figure.

The results of the action of the elastic forces in the impact of two elastic deformable balls are the same as the results of the action of the spring in our model since the mass of the spring is put equal to zero. The kinetic energy of the system in Fig. 79 before the impact is $\frac{m_1 v_{10}^2}{2} + \frac{m^2 v_{20}^2}{2} = 0.0540$ J. In

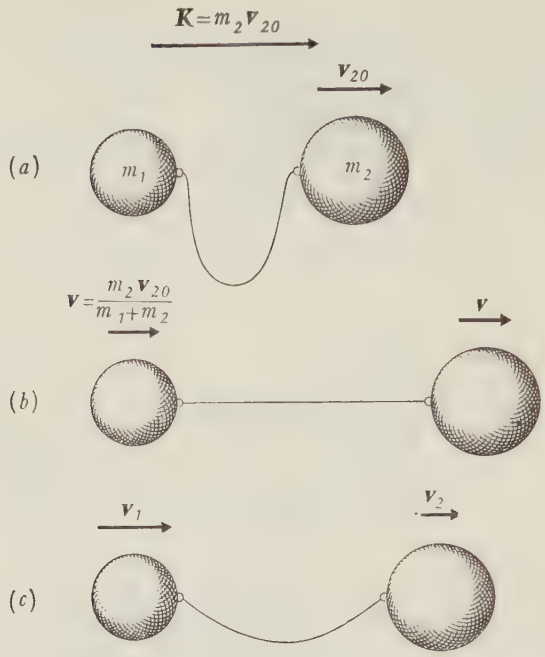

FIG. 80

the impact process the potential energy of the spring increases and the spring contracts until the velocities of the balls become equal; during this period the first ball undergoes deceleration and the other undergoes acceleration. At the moment the velocities of the balls become equal their (common) velocity is

$$v = \frac{K}{m_1 + m_2} = \frac{0.14}{0.3} \approx 0.467 \text{ m/s}$$

and the maximum strain energy of the spring is equal to

$$0.0540 \text{ J} - \frac{0.3 \times 0.467^2}{2} \text{ J} \approx 0.0215 \text{ J}$$

After that moment the spring starts elongating. The forces exerted by the extending spring obviously decelerate the first ball changing the direction of its velocity to the opposite and bringing the magnitude of the velocity to —0.0667 m/s. These forces also accelerate the second ball up to the velocity of 0.733 m/s; the indicated numerical values can readily be found from formulas (34.5) and (34.6).

The terminal velocities of the balls after the impact are independent of the stiffness of the spring; in this model it is only important that the spring is regarded as a perfectly elastic body having no (residual) strain energy after the impact.

However, the duration of the impact process, the time periods during which the spring contracts and then extends to achieve its original length are essentially dependent on the stiffness of the spring and on the masses of the balls. The greater the stiffness factor of the spring, the shorter is the duration of

the impact process. It can be shown that for fixed values of the masses of the balls the impact duration is proportional to $\frac{1}{\sqrt{k}}$ where k is the stiffness factor of the spring.

Since there are only repulsive forces (not attractive) in the impact model we have considered, the increment of the velocity of each ball resulting from the impact is always directed away from the other ball.

In an impact involving attractive forces the situation is reverse, that is the

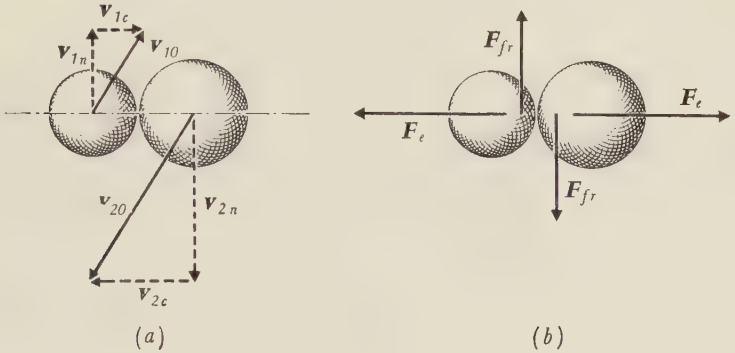

FIG. 81

increment of the velocity of each ball must be directed towards the other ball. Such an impact can be realized, for instance, with the aid of a "weightless" extensible elastic thread. Let two balls be connected with such a thread and let one of them be given an initial velocity (see Fig. 80*a*). The moment the thread is rectified the impact takes place in which attraction forces act. If the thread is perfectly elastic the further motion develops in the following way. As the thread attains its maximum elongation (Fig. 80*b*), the velocities of both bodies equal, after which the thread starts contracting until its length achieves its original value, the ball with mass m_1 being accelerated and the ball with mass m_2 being decelerated in that process. At the moment the thread attains its original length (and after it) the velocity of the mass m_1 is greater than that of m_2 and the thread does not affect the motion of the balls which draw together with a constant (relative) velocity (Fig. 80*c*).

We have considered in detail the central impact of two balls. The situation remains the same when two arbitrary bodies collide provided that the initial velocities are parallel to the line joining their centres of mass and the interaction forces go along the same line. If otherwise, the impact phenomenon becomes extremely complicated and its analysis falls outside the limits of our course.

Let us proceed to an *oblique* impact of balls in which the situation is essentially different. In such an impact phenomenon the centres of the balls come together because of their deformation and the surface of one ball slides relative to the surface of the other ball. To investigate the mechanism of the impact let us resolve the vectors representing the velocities of the balls along their centre line and along the direction perpendicular to that line (Fig. 81).

It appears obvious that the sliding of the surfaces of the balls produce forces of friction $\boldsymbol{F}_{\text{fr}}$ and $\boldsymbol{F}'_{\text{fr}}$. These friction forces together with the forces $\boldsymbol{F}_{\text{e}}$ and $\boldsymbol{F}'_{\text{e}}$

of elastic interaction will determine the course of variation of the velocities of the balls. Besides, the forces of friction will result in rotation of the balls about their centres. Only in the case when the magnitudes of the friction forces F_{fr} and F'_{fr} are very small in comparison with the magnitudes of the elastic forces F_e and F'_e, that is $F_{fr}, F'_{fr} \ll F_e, F'_e$, the friction forces can be neglected.

Under the condition that $F_{fr}, F'_{fr} \ll F_e, F'_e$ the problem of an oblique impact can be solved comparatively simply. For, this condition allows us to assume

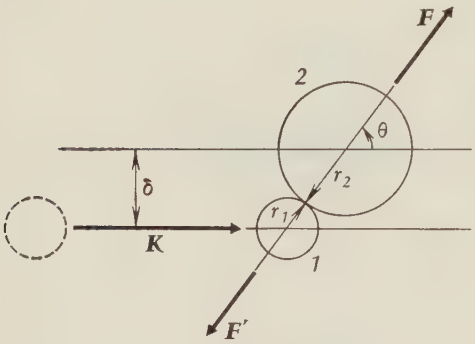

FIG. 82

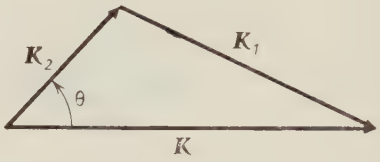

FIG. 83

that, with a high accuracy of the approximation, the magnitudes v_{1n} and v_{2n} of the normal components of the velocities of the balls will not change during the impact while the two components of the velocities along the centre line which result from the impact can be determined on the basis of the law of conservation of momentum and the law of conservation of energy, as was done when we studied the central impact. The equations expressing these conservation laws are written in the form

$$\left.\begin{aligned} m_1 \boldsymbol{v}_{1c} + m_2 \boldsymbol{v}_{2c} &= m_1 \boldsymbol{v}'_{1c} + m_2 \boldsymbol{v}'_{2c} \\ m_1 (v_{1n}^2 + v_{1c}^2) + m_2 (v_{2n}^2 + v_{2c}^2) &= m_1 (v_{1c}'^2 + v_{1n}^2) + m_2 (v_{2c}'^2 + v_{2n}^2) \end{aligned}\right\} \quad (34.8)$$

involving only the two unknowns $\boldsymbol{v}'_{1c}$ and $\boldsymbol{v}'_{2c}$.

The general laws of the impact of the balls in the case under consideration can be established as follows. Let us suppose that ball *2* (see Fig. 82) is at rest before the impact while ball *1* moves*. Since the forces of friction have been neglected (according to the hypothesis we have assumed), the interaction forces in the impact process pass through the centres of the balls and their direction depends on the distance δ between the centre of the resting ball and the (rectilinear) trajectory of motion of the centre of the other ball before the impact. The plane of the figure is meant to pass through the centres of the balls and to contain the vector representing the velocity of ball *1*.

The impact takes place on condition that $\delta < r_1 + r_2$ where r_1 and r_2 are the radii of the balls. The angle θ depends on δ and on $r_1 + r_2$. The com-

* This condition does not restrict the generality of our argument since it is always possible to choose a frame of reference moving with velocity equal to that of ball *2* before the impact and consider the motion relative to that frame (see Sec. 43).

ponent of the momentum of ball *1* normal to the interaction forces $\boldsymbol{F}$ and $\boldsymbol{F}'$ remains invariable. The components of the momenta of the balls in the direction of the forces $\boldsymbol{F}$ and $\boldsymbol{F}'$ vary in accordance with the laws governing the central impact studied before.

By the constancy law for the total momentum, we have

$$\boldsymbol{K} = \boldsymbol{K}_1 + \boldsymbol{K}_2 \tag{34.9}$$

where $\boldsymbol{K}$ is the momentum of ball *1* before the impact and $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are, respectively, the momenta of balls *1* and *2* after the impact.

The law of conservation of energy can be written as

$$\frac{K^2}{m_1} = \frac{K_1^2}{m_1} + \frac{K_2^2}{m_2} \tag{34.10}$$

since $\boldsymbol{K} = m\boldsymbol{v}$ and $mv^2 = K^2/m$ for any body.

Let us suppose that the vector $\boldsymbol{K}_2$ forms an angle θ with the vector $\boldsymbol{K}$ (this means that the ball which is at rest before the impact bounces aside after the impact at an angle of magnitude θ reckoned from the initial velocity of ball *1*). Observing the triangle shown in Fig. 83 we conclude that

$$K_1^2 = K_2^2 + K^2 - 2KK_2 \cos\theta$$

Taking into account the constancy of the energy (formula (34.10)) and eliminating K_1 we obtain the expressions

$$K_2 = \frac{2m_2}{m_1 + m_2} K \cos\theta = \beta K \cos\theta \text{ and } \beta = \frac{2m_2}{m_1 + m_2} \tag{34.11}$$

Formulas (34.11) present the general relationship between K_2 and K depending on the angle θ and the ratio m_1/m_2 of the masses.

We must distinguish between the two cases $m_1 > m_2$ and $m_1 < m_2$. In the former case we have $\beta < 1$, that is the heavier ball strikes the lighter one, the connection between $\boldsymbol{K}_2$ and $\boldsymbol{K}$ being as shown in Fig. 84. We see that the terminus of the vector $\boldsymbol{K}_2$ describes a circle with diameter βK. After the impact both balls move in the direction of the initial motion of the first ball. The magnitude of the angle θ varies from 0 to $\pi/2$. The deflection angle φ of the first ball may vary from 0 to a certain value φ_{max}. In the general case to a given value of φ there correspond two values of θ. The point B corresponds to the case of a central impact after which both balls move in one direction (this case was studied earlier). The point A corresponds to a "miss" (the balls do not touch each other).

In the second case when $m_1 < m_2$ the lighter ball strikes the heavier one. The possible distribution of the momenta of the balls after the impact is shown in Fig. 85. Here we have $\beta > 1$ and ball *1* may move backwards after the impact. The deflection angle φ of the striking ball may vary from 0 to π. The point B again corresponds to the case of a central impact. To every admissible value of φ there corresponds a single value of θ.

In the "intermediate "case when $m_1 = m_2$ (the masses of the balls are equal) the possible distribution of the momenta is shown in Fig. 86. The angle φ varies from 0 to $\pi/2$. In the case of a central impact ball *1* stops moving while ball *2* starts moving with the same velocity (this corresponds to the point B). The "dispersion" angle $\theta + \varphi$ of the balls is always equal to $\pi/2$.

We can readily derive the relation between the angle θ and the distance δ (Fig. 82); it follows that

$$(r_1 + r_2) \sin\theta = \delta \tag{34.12}$$

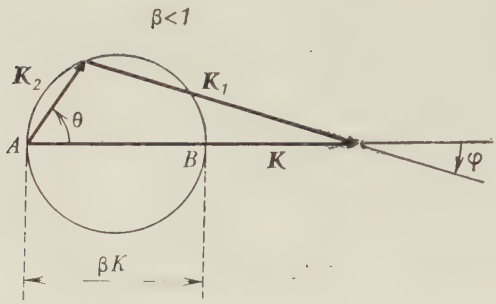

FIG. 84

Given δ, the diameters of the balls and their masses, we can find θ and β and then, from a given vector $\boldsymbol{K}$, determine $\boldsymbol{K}_2$ and $\boldsymbol{K}_1$ which, in their turn, specify the velocities and the directions of motion of the balls after the impact. The phenomenon of an elastic impact of two balls has thus been completely investigated.

It should be stressed that the solution of the problem we have presented is based on the two conservation laws for the energy and the momentum. Therefore all the conclusions drawn from formula (34.11) and demonstrated

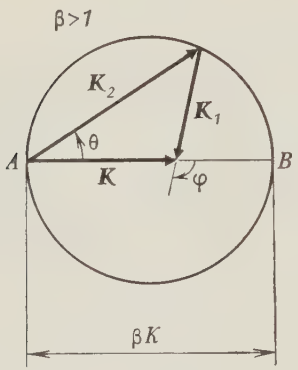

FIG 85.

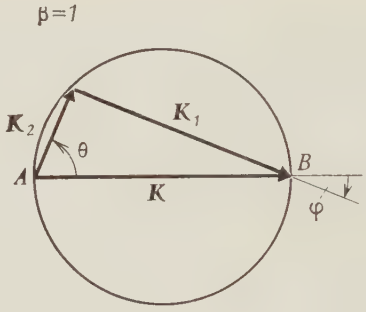

FIG. 86

in the above figures are applicable to an elastic impact of two particles (that is two bodies considered as material points). Without knowing the specific mechanism of the interaction of the particles in their impact we simply assume that their kinetic energy after the impact is equal to that before the impact. The constancy of the kinetic energy can thus be regarded as the condition specifying the phenomenon of an elastic impact. For bodies thought of as material points this condition holds when the repulsive forces depend in a unique manner on the distance between the points and are directed along the line joining them (see Sec. 36).

Although relation (34.12) holds only for balls the magnitude of the angle θ can nevertheless be found from a given value of δ provided that the depen-

dence of the repulsive forces on the distance between the particles is known. The general representation of all possible courses of the impact processes involving particles is given by formula (34.11) and is clearly demonstrated by the figures. There are a number of problems in which such a representation is quite sufficient. For example, when considering the collision of a very heavy particle with a lighter one ($m_1 \gg m_2$), the deflection angle φ_{max} can

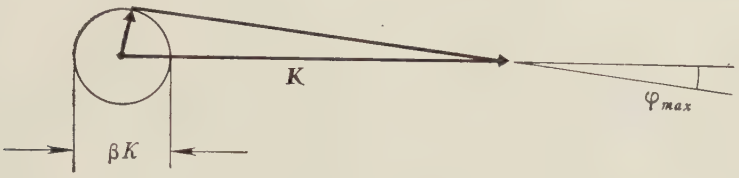

FIG. 87

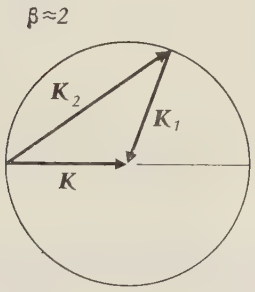

FIG. 88

be used for the determination of the ratio of the masses. It can easily be seen from Fig. 87 that

$$\varphi_{max} \approx \frac{\beta}{2} \approx \frac{m_2}{m_1}$$

When a very light particle strikes a heavier one ($m_1 \ll m_2$) the momentum distribution is as shown in Fig. 88 (where $\beta \approx 2$). In the case of a central impact ($\theta = 0$) we can approximately write $K_2 \approx 2K$ and $K_1 \approx -K$. For any impact with $K_1 \approx K$ the velocity of the lighter particle remains almost invariable in its magnitude, only its direction changing. The magnitude of the velocity of the heavier particle after the impact does not exceed

$$v_2 = \frac{2K}{m_2} = \frac{2m_1}{m_2} v$$

where v is the initial velocity of the striking particle. In the limit when $m_1/m_2 \to 0$ and $v_2/v \to 0$ we obtain from the last case the situation appearing in the impact of a particle with a fixed wall.

The analogy between the impacts of balls and of particles applies when the forces of friction arising as the balls touch one another are negligibly small

and when the balls are in a translatory motion after the impact and do not rotate. It would be more precise to state the latter condition as the requirement that the rotation of the balls, provided it takes place before the impact, should not be changed in the impact process.

35. Impact of Non-elastic Bodies

The type of impact studied in the foregoing section which was called an elastic impact and which was not accompanied with "losses" of mechanical energy should have been more precisely referred to as an *entirely elastic* impact because in reality such losses (connected with the transformation of some part of the mechanical energy into heat) are inevitably present. However, when the losses are very small the situation we described can be regarded as a sufficiently accurate approximation to the real processes.

In the general case when ordinary non-elastic bodies collide the losses of mechanical energy may be considerably large; then the course of the impact phenomenon is of a different kind.

The principal distinction between various types of impact can be elucidated by analysing the model of two (perfectly) rigid balls with a spring between them (see Fig. 79). A perfectly plastic (inelastic) impact is analogous to the case when the spring in the model breaks on achieving its maximum compression or possesses the property that it can only contract but cannot gain a backward extension. Such a model can be realized practically with the aid of a latch not preventing the spring from contraction and not allowing it to extend. Then the strain energy of the compressed spring will be equal to the "loss" of the kinetic energy.

An impact of real non-elastic bodies is in a certain sense an intermediate case between entirely elastic and perfectly inelastic impacts (and is spoken of as an *imperfectly elastic* impact). An adequate model of an impact of this kind is the one with two rigid balls and a spring which is neither perfectly plastic nor perfectly elastic. Such a spring undergoes a contraction during the part of the duration of the impact when the balls come together (and when the interaction forces increase; this is the *compression period*) and then extends (this part of the impact duration is the *restitution period*) but does not achieve its original length. The same occurs if the repulsive force acting during the compression of the spring exceeds the force produced when the spring extends. A part of the potential energy of strain of the spring will be transformed into heat and will not go into the kinetic energy. It follows that the energy conservation law is inapplicable to that case. The equality condition for the velocities after the impact will no longer hold either (this was the case in a perfectly plastic impact) because after the impact the bodies may move with different velocities.

An impact of non-elastic bodies can be characterized by the ratio of the part of the strain energy which is transformed into heat during the impact to the whole strain energy. But, as was shown by Newton, the relative velocities of two colliding non-elastic balls before and after the impact are in a constant ratio for given materials of the balls and it is therefore more expedient to characterize such an impact by the latter ratio called the *coefficient of restitution*. Denoting this coefficient by e we can write

$$e = \frac{|v_2 - v_1|}{|v_{20} - v_{10}|} \tag{35.1}$$

where $v_{20} - v_{10}$ is the relative velocity before the impact and $v_2 - v_1$ is that after the impact. Experiments show that, to within an appropriate accuracy, the magnitude of e can be regarded as a constant depending solely on the materials of the colliding balls.

It can readily be shown that for an entirely elastic impact the relative velocity retains its absolute value and changes the direction to the opposite. Indeed, as follows from equation (34.4), in this case we have

$$v_{10} - v_{20} = -(v_1 - v_2) \tag{35.2}$$

The coefficient of restitution is always less than unity and greater than zero in real (imperfectly elastic) situations because it is equal to 1 for an entirely elastic impact and to zero for a perfectly plastic impact since in the latter case $v_2 - v_1 = 0$. Newton found experimentally that $e = 15/16$ for glass, $e = 5/9$ for iron, etc. Knowing the coefficient e we can compute the velocities of the balls after the impact and the energy "losses" as was done when we studied a perfectly plastic impact in Sec. 33.

36. Potential Energy

A body placed near the Earth's surface is permanently under the action of the force of gravity directed towards the centre of the Earth. Consequently, when the distance from the body to the Earth's surface (or, more precisely, to the centre of the Earth) changes the gravity force, that is the force of weight, performs some work.

If a certain device lifts up a body it performs positive work. Conversely, if a body is in a free fall its distance from the ground decreases and the force of gravity performs positive work. In this case the work is equal to the increase of the kinetic energy of the body. A body moving near the Earth's surface gains a displacement in the gravitation field of the Earth. In the general case a displacement of a body in the gravitation field is connected with the performance of work by the force of gravity. When we move a body from one point to another we always expend or gain some energy. Hence, the displacement of the body is connected with a change of energy.

To find the increase or decrease of energy we must compute the work performed by the forces of gravity as the body is moved from one point in space to another and establish the relationship between this work and the variation of the position of the body relative to the Earth.

We begin with the simplest case: let a body of mass m be lifted under the action of a force of constant magnitude F directed upwards, the increment of the height being h. Then the work of the acting force is equal to Fh. The corresponding increment of the kinetic energy of the body can be found on the basis of the equation

$$F - F_g = m\frac{dv}{dt} \tag{36.1}$$

where $F_g = mg$ is the force of weight, g is the acceleration of gravity, and v is the velocity. On multiplying both members of (36.1) by the increment dS of the distance and integrating from 0 to h with respect to S or, accordingly, from v_{in} to v_t with respect to v where v_{in} is the initial velocity and v_t is the

terminal one, we obtain*

$$F\int_0^h dS - mg\int_0^h dS = m\int_{v_{in}}^{v_t} v\,dv$$

whence

$$Fh = mgh + \frac{mv_t^2}{2} - \frac{mv_{in}^2}{2} \tag{36.2}$$

Thus, the work performed by the force F along the path h is equal to the increment $\frac{mv_t^2}{2} - \frac{mv_{in}^2}{2}$ of the kinetic energy plus the work performed by the force of gravity over that path which is equal to mgh. The same conclusion can be drawn in a more general case when the external force varies along the path according to an arbitrary law, the only distinction being that the work performed by the external force should be computed by the formula

$$\int_0^h F\,dh$$

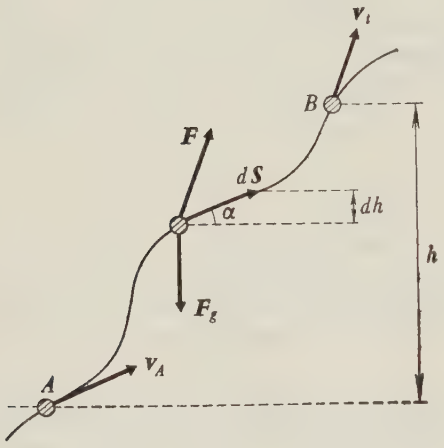

FIG. 89

If the initial value of the kinetic energy $mv_{in}^2/2$ at the beginning of the path is equal to its terminal value $mv_t^2/2$ at the end of the path, that is if $v_{in} = v_t$, then the work of the external force is equal to that of the force of gravity. This is also the case when $v_{in} = v_t = 0$, and consequently *when any external action results in a vertical upward displacement of the body by a distance of h the corresponding work performed by the external force has a definite value equal to mgh and dependent solely on the distance h and on the magnitude mg of the gravity force.*

Now let us find the work to be expended on the lifting of a body of mass m along an arbitrary path, not necessarily vertical (see Fig. 89), so that the terminal point B lies above the initial point A by a distance (reckoned along the vertical direction) of h. If the external force F varies along the path connecting the points A and B the work performed by it is equal to $\int_A^B F\,dS$. This amount of work must be equal to mgh plus the variation of the kinetic energy.

Indeed, using the notation in Fig. 89 we can rewrite the dynamical equation in a form similar to (36.1) whence, on multiplying it scalarly by dS, we obtain

$$\boldsymbol{F}\,d\boldsymbol{S} + \boldsymbol{F}_g\,d\boldsymbol{S} = m\frac{d\boldsymbol{v}}{dt}\,d\boldsymbol{S} \tag{36.3}$$

* Taking into account that $\frac{dS}{dt} = v$.

that is

$$\boldsymbol{F}\,d\boldsymbol{S} = -\boldsymbol{F}_g\,d\boldsymbol{S} + m\boldsymbol{v}\,d\boldsymbol{v} = -\boldsymbol{F}_g\,d\boldsymbol{S} + d\left(\frac{mv^2}{2}\right)$$

The work performed by the force $\boldsymbol{F}$ along the path AB is equal to the integral of the last expression:

$$\int_A^B \boldsymbol{F}\,d\boldsymbol{S} = -\int_A^B \boldsymbol{F}_g\,d\boldsymbol{S} + \frac{mv_t^2}{2} - \frac{mv_{in}^2}{2} \tag{36.4}$$

Taking into account that $\boldsymbol{F}_g\,d\boldsymbol{S} = -mg\,dh$ we find

$$\int_A^B \boldsymbol{F}\,d\boldsymbol{S} = mg\,(h_B - h_A) + \frac{mv_t^2}{2} - \frac{mv_{in}^2}{2} \tag{36.5}$$

where $h_B - h_A = h$.

Thus, we have proved that if the initial and terminal values of the kinetic energy coincide (for instance, when the velocities at the points A and B are equal to zero) the work performed by the external force is equal to that performed by the force of weight as the body is moved downwards by h.

When a body of mass m travels along *any* path from a point A to any point B (whose horizontal level is above that of the former by a distance of h (reckoned along the vertical) there must be a system exerting an external force which performs an amount of work equal to mgh. Conversely, when a body of mass m is displaced from a point B to any point A whose horizontal level is below that of B by a distance h the body (or, more correctly, the system "the body and the Earth") performs an amount of work equal to mgh, or, which is the same, expends an amount of energy equal to mgh.

Thus, for a body to receive a displacement in the gravitational field of the Earth a certain system must expend or gain some energy; the amount of that energy depends solely on the horizontal levels of the initial and terminal positions of the body and on its mass and is *independent of the shape of the path* along which the body goes from one level to another. This means that the system "the body and the Earth" possesses a *potential energy* U whose value is equal to mgh + const: $U = mgh$ + const. The potential energy is determined to within an arbitrary constant equal to its value for $h = 0$.

The potential energy of a body in the gravitational field* depends on the position of the body relative to the Earth, namely on its *height* over the Earth's surface. More precisely, the potential energy of gravitation of the system of the two bodies "the Earth and the body of mass m" depends on the *mutual position* of these bodies, namely on the distance between the centres of mass of the bodies. The latter statement also remains valid when h is large, that is comparable with the radius of the Earth. However, for such large values of h the acceleration of gravity g can no longer be considered constant; it decreases as h increases and therefore the expression of the potential energy assumes another form (see Sec. 78).

The strain energy (the potential energy of strain) of an elastic body, for example, the potential energy of an elastic spring, is also dependent only on the mutual position of the parts of that body.

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* For more detail concerning gravitational forces see Sec. 76.

In the general case the potential energy of gravitation is determined by equality (36.5); on comparing (36.5) with (36.4) we can write the expression for the potential energy in a more convenient form:

$$U_B - U_A = mg(h_B - h_A) = -\int_A^B F_g \, dS \tag{36.6}$$

Thus, the work of the gravitational force is equal to minus the increment of the potential energy.

Expression (36.6) is valid for any interaction forces between two bodies (particles) if F_g is replaced by the corresponding force of interaction F_{int} provided (this should be stressed) the work $\int_A^B F_{\text{int}} \, dS$ is independent of the shape of the path joining the points A and B.

The work of internal forces (interaction forces) is positive when the potential energy of the systems decreases and, conversely, it is negative when the potential energy increases.

It can be shown that any system of particles (that is of bodies whose sizes are sufficiently small) possesses a potential energy if the interaction forces depend only on the distances between the particles and are directed along the straight line joining them with each other.

Let us consider a system of two particles. Let one particle be at the origin of a vector $\boldsymbol{r}$ and the other at its terminus. Suppose that the former particle acts upon the latter with a force $f(r)\,\boldsymbol{r}$ and the latter acts on the former with a force of the same magnitude and opposite direction.

Let us assume that the former particle is at rest while the latter moves from a point $\boldsymbol{r}_A$ to a point $\boldsymbol{r}_B$ along a certain path. Then if the work

$$-\int_A^B f(r)\,\boldsymbol{r}\,d\boldsymbol{S}$$

does not depend on the shape of the path it must be equal to $U_B - U_A$. Let us show that this work is in fact independent of the path. The product $\boldsymbol{r}\,d\boldsymbol{S}$ is equal to $r\,dr$ where dr is the increment of the modulus of $\boldsymbol{r}$ (Fig. 90), and therefore the integral

$$\int_A^B f(r)\,\boldsymbol{r}\,d\boldsymbol{S} = \int_A^B f(r)\,r\,dr \tag{36.7}$$

is a function dependent solely on $\boldsymbol{r}_A$ and $\boldsymbol{r}_B$ and thus dependent only on the distance between the two particles; it assumes the same value for any trajectory of motion connecting the points A and B (see Fig. 91). Integral (36.7) is obviously equal to zero for $\boldsymbol{r}_A = \boldsymbol{r}_B$. If the function $f(r)$ is known the value of integral (36.7) can readily be computed and, according to (36.6), it is equal to the variation of the potential energy of the system consisting of the two particles:

$$-\int_A^B f(r)\,r\,dr = U_B - U_A \tag{30.8}$$

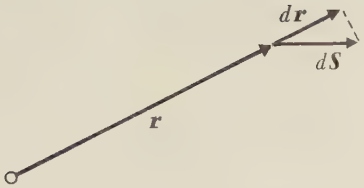


FIG. 90

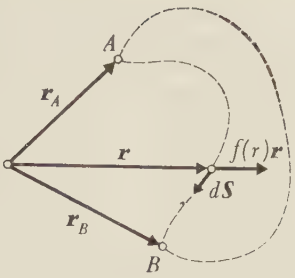


FIG. 91

Thus, the system of the two particles possesses a potential energy. If the force $f(r)\,r$ decreases for large values of r and tends to zero sufficiently fast as $r \to \infty$ the potential energy at infinity ($U(\infty)$) may have a finite value; in such a case $U(\infty)$ is usually put equal to zero (since the potential energy is determined to within an arbitrary constant and the proper choice of the latter make

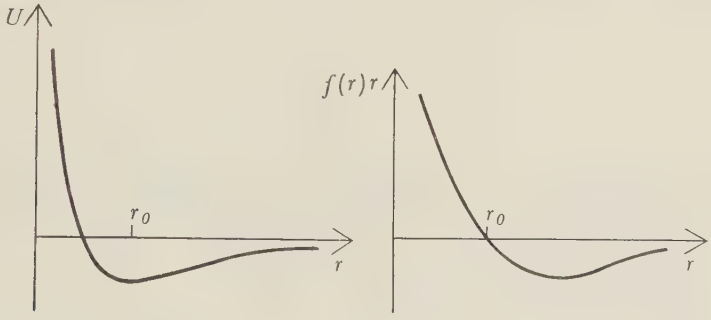


FIG. 92

it possible to obtain $U(\infty) = 0$). In such a case we derive from (36.8) the expression

$$U(r_1) = \int_{r_1}^{\infty} f(r)\,r\,dr$$

determining the potential energy of the system of the two particles when they are at a distance of r_1 from each other.

If $f < 0$ the interaction forces are attractive and therefore $U_B > U_A$ when $r_B > r_A$. In this case the potential energy increases as the distance grows. If $f > 0$ the interaction forces are repulsive: $U_B < U_A$ for $r_B > r_A$ and the potential energy decreases as the distance between the particles increases.

In physics we often encounter a situation when the interaction forces are attractive for large distances and repulsive for small distances. In this case the dependence of the potential energy of two particles on the distance between them is of the type shown in Fig. 92. The part of the diagram in the figure

corresponding to the interval $0 < r < r_0$ describes the repulsion of the particles ($f > 0$); the potential energy decreases on that interval and assumes zero for $r = r_0$. As r receives a further increase, that is when $r > r_0$ ($f < 0$), the potential energy increases. All these conclusions readily follow from the important relation

$$\frac{dU}{dr} = -f(r)\,r \tag{36.9}$$

implied by (36.8).

A system consisting of an arbitrary number n of interacting particles and possessing the property that every two particles interact according to a law of the type $f(r)\,\boldsymbol{r}$ also possesses a definite potential energy. Let U_{ik} ($i, k = 1, 2, \ldots, n$) be the potential energy of the interaction between the ith and the kth particles of the system: $U_{ik} = U_{ik}(|\boldsymbol{r}_k - \boldsymbol{r}_i|)$ where $\boldsymbol{r}_i$ and $\boldsymbol{r}_k$ are the position vectors of the particles. Then the potential energy of the interaction between the ith particle and all the other particles is equal to

$$U_i = \sum_{k \neq i}^{n} U_{ik}$$

(the summation extends over $k = 1, 2, \ldots, i-1, i+1, \ldots, n$). The potential energy of the whole system of all the particles is obviously equal to

$$U = \frac{1}{2}\sum_{i=1}^{n} U_i = \frac{1}{2}\sum_{i=1}^{n}\sum_{k \neq i}^{n} U_{ik} \tag{36.10}$$

where 1/2 in front of the sum appears because the double sum itself involves the potential energy of every pair of particles twice since $U_{ik} = U_{ki}$.

The potential energy U of the entire system depends solely on the mutual position of the particles, that is on the configuration of the system in space.

37. Change of Energy of a Body in the Gravitation Field. Law of Conservation of Energy

Let us suppose that a given body of mass m is in the field of gravitation and that there are no other forces acting upon it; then the acceleration of the body will always be equal to g and will be directed downwards*.

The motion of such a body on which only the force of gravity acts can be called a *free* motion in the field of gravitation of the Earth. The kinetic energy of the body can only change when the potential energy of gravitation varies and vice versa; the sum of the kinetic and potential energies retains a constant value, that is the mechanical energy remains constant.

The law of conservation of energy can be expressed mathematically in this case with the aid of (36.5) by putting the external force to be zero: $F = 0$. Indeed, we have

$$mgh + \frac{mv_t^2}{2} = \frac{mv_{in}^2}{2} \tag{37.1}$$

where h is the increment of the height which can be expressed as $h = h_t - h_{in}$ where h_t and h_{in} are, respectively, the terminal value and the initial value

* When $h \ll R$ where R is the radius of the Earth and h is the height of the body above the Earth's surface, the acceleration g can practically be considered constant.

of the height. This allows us to rewrite (37.1) in the form

$$mgh_t + \frac{mv_t^2}{2} = mgh_{in} + \frac{mv_{in}^2}{2} = E_0 \tag{37.2}$$

where $E_0 = \text{const}$.

Formula (37.2) expresses mathematically the law of conservation of mechanical energy for a body moving in the field of gravitation. The sum of the kinetic energy and the potential energy remains constant at any moment and

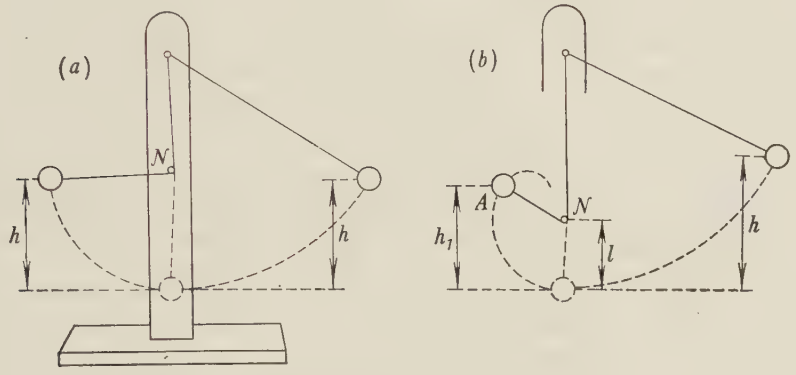


FIG. 93

is equal to E_0. It should be noted that this law is only applicable when the body moves under the action of the gravity forces solely. In the general case when some other forces are present (such as resistance forces and the like) the mechanical energy (under which is meant the sum of the potential energy and the kinetic energy) does not necessarily retain a constant value.

The law of conservation of mechanical energy in the gravitation field was known to Galileo. He demonstrated its validity with the aid of his well-known experiments with a pendulum (Galileo's pendulum).

Galileo's pendulum is a ball suspended from a flexible inextensible thread (see Fig. 93*a*). Under the point to which the other end of the thread is attached there is a nail driven into the post at a point N lying on the vertical passing through the lower equilibrium position of the ball at a height h above it.

The pendulum is first given a deviation from the vertical (to the right, in the figure) so that the ball is at a height h above its lowest position, after which the ball is allowed to move. At a certain moment the thread meets the nail N and the ball continues to move rotating about the point N. On achieving its highest level (on the left of the point N in the figure) the ball goes down, etc. The experiment shows that the height h is almost the same for the deviations of the ball in both directions; if there were no resistance forces (produced by the air friction and the like) the height h would be precisely the same.* The moment the ball achieves the highest point at the level h its velocity and its kinetic energy become equal to zero.

* It is obvious that the tensile force in the thread performs no work (since the thread is assumed to be inextensible) and therefore we need not take the action of this force into account in the analysis of the phenomenon.

In this connection we note that if the nail N is placed sufficiently low or the initial height h is sufficiently increased there appears a situation when the ball does not attain the original height h. The explanation lies in the fact that the thread the ball is suspended from is flexible. It can easily be verified that if $h > l$ the ball achieves a height h_1 smaller than h. In this case, when moving upwards (see Fig. 93*b*), the ball reaches a point at which the tension of the thread becomes zero before the ball stops (that is before its velocity becomes zero). At that moment the ball leaves the circular orbit of radius l and then passes through the highest point with a non-zero horizontal component of its velocity. At the highest point the kinetic energy of the ball is thus different from zero and that is why the height h_1 must be smaller than h.

The work of a force specifies the amount of energy transmitted from one body to another. The energy is a unified quantitative measure for various forms of motion of the substance. The motion of the substance is transformed from one form to another and never disappears.

We have subjected to a thorough analysis the *energy of the mechanical form of motion*, that is the mechanical energy which, in its turn, can exist in two forms: the kinetic form and the potential one. The *kinetic energy* depends on the velocities and the masses of the moving bodies. The *potential energy* is determined by the mutual positions of the bodies (or on the positions of various parts of one body). It thus depends on the configuration of the system in space, that is on the coordinates of the bodies.

However, there are phenomena in which the mechanical energy can be transformed into the thermal form of energy; for instance, as we already know, in an inelastic impact a part of kinetic energy is transformed into heat. The amount of the thermal energy evolved in the impact is equal to the "loss" of the mechanical energy.

In physical and chemical processes the energy is transmitted from one body (or from a system of bodies) to another body (or to another system of bodies) but there exist no processes in which the energy disappears or is created. The motion of the substance may change its form but *the energy retains constant value in all the changes of the form of motion.* This is the *law of conservation of energy*; it is one of the fundamental laws of nature.

CHAPTER 5

Friction

38. Various Types of Friction Forces

In almost all mechanical phenomena there arise forces of friction whose action, as a rule, is connected with a transformation of one form of energy into another; most often under the action of friction the mechanical energy is transformed into heat. In their action the forces of friction do not differ from other types of forces such as forces of gravity, pressure, deformation, etc. but their nature possesses some specific features which we shall demonstrate by examples.

Let a body slide on an even and smooth horizontal plane; for instance, let a small board slide on the ice. The observation shows that the motion of the body will undergo a deceleration and the body will finally stop. Thus, the velocity of the board decreases and the direction of its acceleration is opposite to that of the velocity. The deceleration of the body is caused by the forces of friction exerted on it by the ice and the air.

Let us consider another example of this kind (see Fig. 94): a body lies on the table and we start pushing it, with the aid of a thread attached to it, along the horizontal surface of the table. When the tensile force F of the thread is not sufficiently large the body remains at rest; this means that the table acts on the body with the force $-F$ whose magnitude is equal to that of F while its direction is opposite. It is the force $f = -F$ that results from the friction between the body and the table. The force of gravity P and the reaction force N of the table are vertical; they balance each other and do not affect the magnitude of the horizontal acceleration.

The physical character of the friction forces in the first and the second example is different: in the former case the force of friction arises in the motion of the body or, more precisely, because of the motion, while in the latter case the force of friction acts when the body is at rest; the latter force is due to the action of the external force F and in this case we speak of the *force of friction of rest* (or of *static friction*).

In the first example the direction of the force of friction is opposite to that of the velocity of motion and its action is connected with the transformation of the kinetic energy into heat. Since the velocity specifies the direction of the displacement of the body, the displacement and the friction force are of opposite directions and consequently the work performed by the force of friction is *negative*. This means that the energy is transmitted from the body on which the friction force acts. Hence when a body is acted upon by a friction force solely its kinetic energy can only decrease.

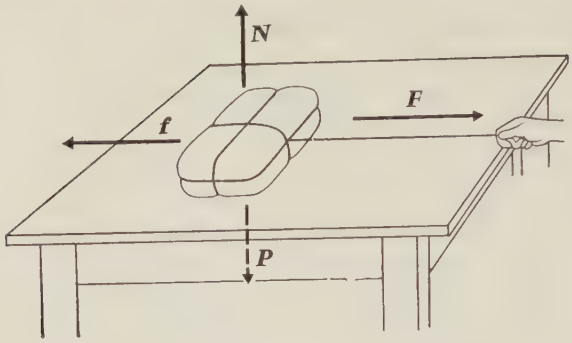


FIG. 94

Indeed, for a body of mass m moving with a velocity $\boldsymbol{v}$ under the action of a friction force $\boldsymbol{f}_{\text{tr}}$ we can write, according to the second law of dynamics, the equation

$$m\frac{d\boldsymbol{v}}{dt} = -\boldsymbol{f}_{\text{tr}}$$

On multiplying this equation scalarly by $d\boldsymbol{S}$ we arrive at the formula

$$d\left(\frac{mv^2}{2}\right) = -\boldsymbol{f}_{\text{tr}}\, d\boldsymbol{S} \tag{38.1}$$

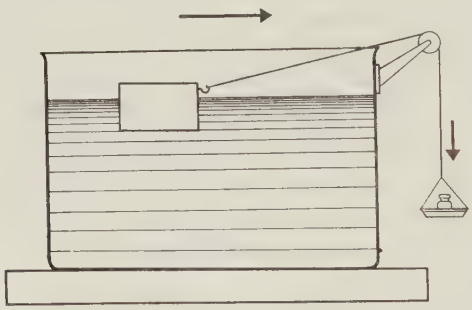


FIG. 95

analogous to formula (32.4). Equation (38.1) reads: the decrease of the kinetic energy is equal to the work of the friction force (this conclusion can of course be drawn directly as a consequence of the energy conservation law).

The investigation of the interaction connected with the friction between a moving body and surrounding bodies shows that the kinetic energy of the body is transformed into the thermal form of energy.

In the phenomenon of static friction there is no motion of bodies and therefore no work is performed, that is no energy is converted from one form into another. When a body moves the magnitude of the force of friction depends

on the physical properties, on the shape of the moving body, on the properties of the surrounding medium and bodies and also on the velocity of the motion.

We distinguish between the following two types of friction: (1) the friction between dry surfaces of rigid bodies and (2) the friction arising in a motion of a body in a fluid or gas. In the first case we simply speak of *dry friction* and in the second case of *viscous friction.*

In the case of dry friction there may arise a force of static friction (*friction of rest*); *in the case of viscous friction the force of friction of rest cannot appear.* In a motion of bodies touching each other along lubricated surfaces and also in a motion of a body in a viscous fluid or gas there arise forces of friction and it should be stressed that they only appear when the body is in the state of motion. The force of friction of rest is equal to zero in such cases. For instance, a body floating on the surface of a fluid (Fig. 95) starts moving under the action of any horizontal force, however small; this can readily be checked experimentally.

The observation of the motion of the floating body under the action of a constant horizontal force (see Fig. 95) shows that in some time the motion becomes almost uniform. This means that the motion is accompanied by the appearance of a force of friction (force of resistance) which increases together with the increase of the velocity until its magnitude achieves that of the external force, after which the two forces balance each other.

The forces of viscous friction (the resistance forces of fluids and gases) only appear in the process of motion and their action is always connected with the transformation of the mechanical energy into heat.

39. Viscous Friction

The forces of viscous friction (or, which is the same, the forces of resistance of a fluid or a gas medium acting on a body moving in a fluid or gas depend on the shape of the body, on the velocity of motion, and on some physical properties of the medium, namely, on its density and viscosity. The greater the viscosity of the medium, the greater is the magnitude of the friction force (provided all the other conditions remain invariable).

The viscosity of a medium is usually investigated by experiments in which, under certain conditions, the forces of friction acting on some bodies are measured. It was Newton who established experimentally the basic law of friction in a fluid or gas medium in which two parallel surfaces slide relative to each other (see Fig. 96*a*). It is supposed that the distance h between the parallel surfaces is small, the space between them being filled with a definite fluid or gas. When surface *1* with area S is in a uniform motion with velocity v relative to the parallel fixed surface *2* the force of friction f_{fr} acting on surface *1* is equal in its magnitude and opposite in its direction to the external force F producing the motion of surface *1*.

The measurements of the magnitudes of the velocity v and of the force F performed by Newton showed that the law connecting the quantities involved can be written in the form

$$f_{\text{fr}} = \mu S \frac{v}{h} \tag{39.1}$$

where μ is a constant coefficient called the *viscosity coefficient* (or, simply, the *viscosity*) dependent solely on the properties of the medium filling the space between the surfaces. This law proves valid when $h \ll \sqrt{S}$, that is when the distance between the sliding surfaces is very small in comparison with their

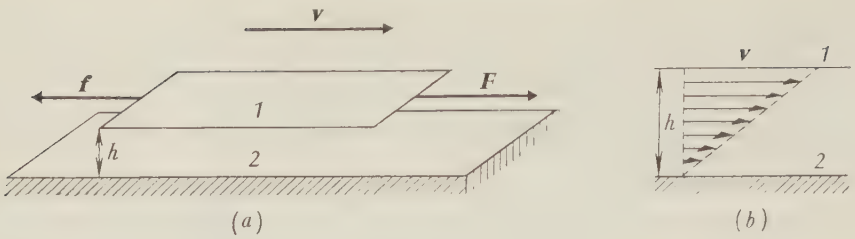


FIG. 96

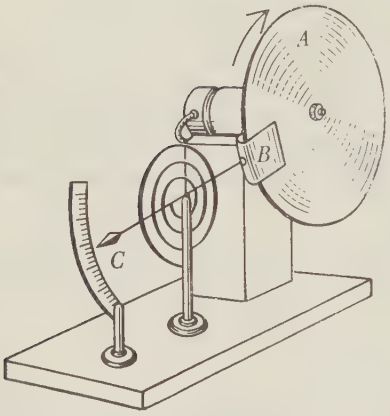


FIG. 97

dimensions. A more detailed investigation indicates that the particles of the medium (of a fluid or gas) adjoining surface *1* move with velocity v (they are "drawn" by the moving surface) while the particles adjoining surface *2* are at rest. The velocity of the particles of the medium increases directly proportionally to their distance from surface *2* (see Fig. 96*b*).

Let us divide (mentally) the fluid or the gas between the surfaces *1* and *2* into thin layers parallel to the surfaces. Each layer moves uniformly and pulls the lower layer adjoining it with a force of magnitude f_{tr} directed forwards (along v) while the latter layer pulls the former with a force of magnitude f_{tr} directed backwards. In this way the force of friction f_{tr} is transmitted from one layer to another and thus from surface *2* to surface *1*. Each layer is acted upon by two forces equal in their magnitudes and opposite in directions, which results in a uniform motion of the layer.

The viscosity coefficient μ of a medium is found experimentally; its dimension is kg/m·s in the SI system and g/cm·s in the cgs system.

The viscosity coefficient μ of the air can be determined experimentally with the aid of the apparatus depicted in Fig. 97. The disk A rotates with a known velocity between the two small plates B attached to the spring as shown in the figure. The force of friction acting on the plate is measured by the deviation of the arrow C. Knowing the area of the plates B, the distance between the disk and the plates, the angular velocity of the rotation of the disk and its radius we can determine the coefficient μ. Making the distance between the disk and the plates and the velocity of the disk vary we can check Newton's law of viscous friction for the air.

It should be noted that the viscosity coefficient of a medium can only be found in this way when it is known that the velocities of the particles of the medium are all the time parallel to the surfaces of the plates. In real circumstances this condition can be violated when the velocities are sufficiently large. For large velocities the motion of the particles is no longer *laminar* (which is the case for small velocities). Therefore Newton's formula (39.1) only holds

for velocities not exceeding some limiting value dependent on the properties of the medium and on other parameters of the motion.

Viscosity coefficients of fluids and gases can also be found from the velocity of their outflow from a pipe whose length and diameter are known. It was established that the amount Q of fluid or gas passing through the cross section of the pipe during a certain time under a definite difference of pressures at the ends of the pipe is inversely proportional to the viscosity coefficient (for more detail see Sec. 111).

Viscosity coefficient μ of some substances in g/cm·s

Air	0.00018	at 16 °C
Water	0.0114	at 15 °C
Glycerine	13.93	at 18 °C
Benzene	0.0053	at 18 °C
Mineral oil	0.833	at 50 °C

There are also devices with the aid of which the viscosity of a medium is found from the velocity of a small ball moving in the medium. The theoretical investigation shows that for a small velocity of motion of the ball the force of viscous friction is determined by the formula

$$f_{\text{fr}} = 6\pi\mu a v \tag{39.2}$$

where a is the radius of the ball. The validity of formula (39.2) is confirmed by experiment. However, practically it is more convenient to measure not the force of friction itself but the time of the fall of the ball in the viscous medium.

40. Fall of a Ball in a Viscous Medium

A ball falling in a viscous medium is acted upon by the following vertical forces (see Fig. 98): the downward gravity force $\rho V g$, the Archimedes upward hydrostatic supporting force $\rho_{\text{fl}} V g$ and the friction force $f_{\text{fr}} = 6\pi\mu a v$ (whose direction is opposite to that of the velocity of fall v). Here V is the volume of the ball, ρ and ρ_{fl} are, respectively, the densities of the material of the ball and of the fluid (or gas).

On setting the equation of motion in accordance with the second law of dynamics and projecting its left-hand and right-hand members on the vertical axis (directed downwards) we obtain

$$m \frac{dv}{dt} = (\rho - \rho_{\text{fl}}) V g - 6\pi\mu a v \tag{40.1}$$

Now taking into account that $V = 4/3\ \pi a^3$ and $m = \rho V$ we derive

$$\frac{dv}{dt} = \frac{\rho - \rho_{\text{fl}}}{\rho} g - \frac{9}{2a^2} \frac{\mu}{\rho} v \tag{40.2}$$

We see that the acceleration of the fall decreases as the velocity v increases. At the initial moment $t = 0$ we have $v = 0$, after which the velocity increases while the acceleration decreases and the rate of the increase of the velocity thus

decreases. The velocity continues to increase until it assumes the value

$$v_0 = \frac{\rho - \rho_{fl}}{\mu} \frac{2a^2 g}{9} \tag{40.3}$$

which, as follows from equation (40.2), cannot be exceeded (since the acceleration of the fall cannot be directed upwards). For $v = v_0$ we have $\frac{dv}{dt} = 0$; beginning with that moment the velocity no longer increases.

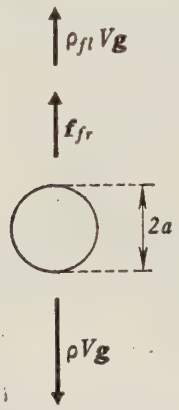


FIG. 98

Equation (40.2) can be rewritten as

$$\beta \frac{dv}{dt} = v_0 - v \quad \text{where} \quad \beta = \frac{2a^2\rho}{9\mu} \tag{40.4}$$

whence

$$\beta \frac{dv}{v - v_0} = -dt$$

The integration of the last equation gives

$$\beta \ln (v - v_0) = -t + C, \text{ that is}$$
$$v - v_0 = A_0 e^{-t/\beta}$$

where A_0 is a constant. Since $v = 0$ for $t = 0$, there must be $A_0 = -v_0$ and hence

$$v = v_0 (1 - e^{-t/\beta}) \tag{40.5}$$

The diagram representing the dependence of v on t/β is shown in Fig. 99. As $t \to \infty$ the magnitude v of the velocity approaches asymptotically the value v_0. The motion of the ball is rather intricate: at the initial moment and for $t \ll \beta$ the ball is in an almost uniformly accelerated motion $\left(\text{with acceleration approximately equal to } \frac{\rho - \rho_{fl}}{\rho} g\right)$, then the acceleration gradually decreases and, finally, for $t \gg \beta$ the motion becomes almost uniform. The time during which the motion becomes practically uniform and the distance travelled by the body during that transient period can be found from formula (40.5). The greater the value of μ and the smaller the radius a of the ball, the sooner does the motion become uniform. On the initial path (for $t \ll \beta$), when the velocity v is small, the motion, as has been said, is almost uniformly accelerated, the acceleration being practically equal to the acceleration of gravity g because usually $\rho_{fl} \ll \rho$. The diagram in Fig. 99 indicates that at the beginning of the motion the increase of the velocity is approximately proportional to time t.

Example. Let us consider the fall of a steel ball of radius 2.5 cm in the air and the fall of a steel ball of radius 0.1 cm in glycerine. In the first case $\beta \approx 6.08 \times 10^4$ s, that is the magnitude of β is of the order of 16 h, since the density of steel is $\rho = 7.88$ g/cm^3 and the viscosity of the air is $\mu = 0.00018$ g/cm·s. In the second case $\beta \approx 1.26 \times 10^{-3}$ s since the value of μ for glycerine is equal to 13.93 g/cm·s.

The fall of the steel ball in the air can be regarded, with high accuracy, as a uniformly accelerated motion during several first seconds (for $t \ll 6 \times 10^4$ s)

with acceleration g.* Hence, in these circumstances the fall of the ball in the air follows the same law as in the case of the fall in vacuum since the forces of friction can be neglected. That was the case in the well-known experiments of Galileo who proved experimentally that the free fall of bodies in vacuum has a constant acceleration; these experiments created the basis for the law of inertia.

The fall of a small ball in glycerine becomes a uniform motion in several milliseconds. By formula (40.3), the velocity of the motion is

$$v_0 = \frac{\rho - \rho_{\text{п}}}{9\mu} 2a^2 g = \frac{\rho - \rho_{\text{п}}}{\rho} \beta g = \frac{7.88 - 1}{7.88} 1.26 \times 10^{-3} \times 981 \text{ cm/s} \approx 1.06 \text{ cm/s}$$

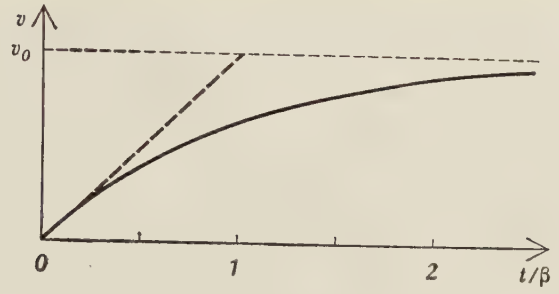


FIG. 99

In this uniform motion with constant velocity v_0 the force of gravity is completely balanced by the friction force and the ball is in an inertial motion. Observing this motion without taking into account the forces of viscous friction one may come to the incorrect conclusion that the velocity is proportional to the acting force; this coincides with Aristotle's idea which, as we know, is wrong.

Remark. When the resistance force of a medium is not proportional to the velocity of motion of a body and depends on it in a more complicated manner it also increases together with the increase of the velocity and therefore the fall of the body in such a medium will tend to a uniform motion. For instance, this is the case when a parachutist falls with his parachute closed or opened. The distinction between these cases is that the fall with the closed parachute has the velocity of the order of 60 m/s while the landing velocity with the opened parachute is about 5-6 m/s.

The dependence of the resistance force (friction force) on the velocity changes its character when the velocity becomes large. The nature of the flow around the moving body changes and the resistance force is no longer proportional to the velocity. In most cases, after a certain value of the velocity has been exceeded, the resistance force becomes proportional to the square of the velocity. Therefore the diagram showing the dependence of the resistance force f decelerating the ball on the velocity v is as the one in Fig. 100. For small velocities (region a in the figure) the resistance force f increases proportionally to the veloc-

* It should be noted that for the given ball the law we have used to describe the dependence of the force of friction on the velocity is not exactly valid when the velocity exceeds 10 m/s. However, more accurate calculations show that for the first three seconds the conclusion we have drawn remains applicable.

ity of motion as was indicated above, that is according to the law $f_{fr} = k_1 v$ ($k_1 = \text{const}$); for large values of the velocity (region c) the friction force increases proportionally to the square of the velocity according to the law $f_{fr} = kv^2$ ($k = \text{const}$). There is also a transition zone between the former and the latter laws (region b in the figure).

This principal change of the law describing the dependence of the friction force on the velocity is due to the change in the character of the flow-around: in region a the body is streamlined smoothly without a flow separation (more

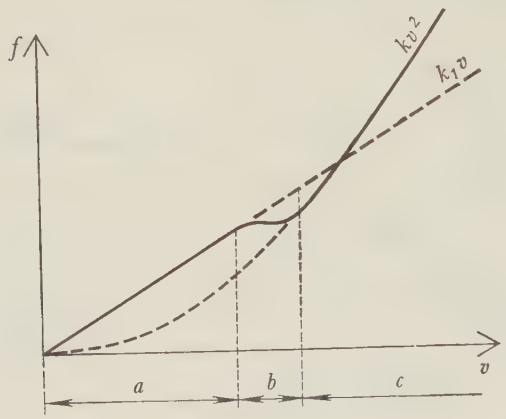


FIG. 100

precisely, with a slight separation) while in the flow-around corresponding to region c there is a considerable flow separation with the appearance of eddies which are carried away by the stream; the latter phenomenon plays an essential role in that zone (for more detail see Sec. 112).

41. Dry Friction

As has already been mentioned (see Sec. 38), a sufficiently small horizontal force applied to a body lying on an even horizontal surface does not produce its motion because there arises the force of static friction (friction of rest) $\boldsymbol{f}$ equal to $-\boldsymbol{F}$ where $\boldsymbol{F}$ is the external force (see Fig. 94). The force of friction of rest $\boldsymbol{f}$ is specified by the acting force $\boldsymbol{F}$: when we vary the tension of the thread the friction force $\boldsymbol{f} = -\boldsymbol{F}$ also varies. When the tension of the thread is increased the magnitude of the force of static friction also increases; when we change the direction of the force $\boldsymbol{F}$ (in the horizontal plane) we thus change the direction of the force of friction.

When the magnitude F of the external force is increased gradually the body starts to move beginning with some definite value f_0 of F. Simple experiments show that the body receives some acceleration when the force F exceeds that value f_0. Consequently, the force of static friction can be of any magnitude ranging from zero to f_0. Thus, the force of static friction has a maximum possible value f_0 (to which corresponds the *limiting friction* or friction of impending motion). When $F < f_0$ the acceleration of the body is equal to zero and the

body is at rest; when $F > f_0$ the body obtains some acceleration and starts moving.

Experiments show that the maximum magnitude of the force of static friction is specified by the physical properties of the bodies whose surfaces are in contact, by the state of the surfaces (the force of static friction is greater for rough surfaces and smaller for smooth ones) and by the magnitude of the normal force pressing one body to another.

Let a body lie on the table as shown in Fig. 94. In the state of equilibrium the magnitude N of the normal reaction of the table is equal to the force of gravity P of the body. Experiments show that the maximum value f_0 of the friction force is given by the formula

$$f_0 = \mu N \tag{41.1}$$

where μ is a dimensionless factor called the *coefficient of static friction* or, more precisely, the coefficient of static friction corresponding to the maximum (limiting) friction force dependent solely on the properties of the surfaces in contact. Formula (41.1) expresses *Amonton's law* established experimentally in 1699.

The value of the coefficient μ can be found from various experiments, for instance, from experiments with bodies sliding down inclined planes. A scheme of such an experiment is shown in Fig. 101. In this experiment we determine the minimum angle of inclination of the plane for which the body starts sliding down the inclined plane. When the body shown in the figure is at rest on the inclined plane the force of static friction f balancing the component of the gravity force along the plane is obviously expressed by the formula

$$f = P \sin \alpha \tag{41.2}$$

If the angle α is gradually increased the body starts sliding for some value α_0 of the angle. The resultant of the forces $\boldsymbol{P}_0$, $\boldsymbol{N}_0$ and $\boldsymbol{f}_0$ is equal to zero at that moment. The angle between $\boldsymbol{P}_0$ and $\boldsymbol{N}_0$ is equal to $180° - \alpha_0$ and consequently

$$\tan \alpha_0 = f_0/N_0 \tag{41.3}$$

As was said, $f_0 = \mu N_0$ and therefore $\tan \alpha_0$ is equal to the coefficient μ of static friction, that is the coefficient of (maximum) static friction is equal to the tangent of the angle for which the body starts sliding down the inclined plane.

Below are the values of the coefficient μ for the friction between dry smooth surfaces of some materials:

Steel on steel	0.15
Metal on oak (along grains)	0.62
Leather on oak	0.61
Brick on brick	0.5-0.73

No theory giving a thorough explanation of the laws of friction of dry surfaces has yet been created. A very rough scheme of the phenomena giving rise to the appearance of the friction forces is the following. Figure 102 shows an enlarged normal section of the contacting surfaces of two rigid bodies. In reality the surface of a body cannot be perfectly smooth; it always has some irregularities, asperities which are, to a certain extent, uniformly distributed over the surface and are, within some limits, of various shapes and dimensions. When two bodies are in contact and are pressed to each other these irregularities under-

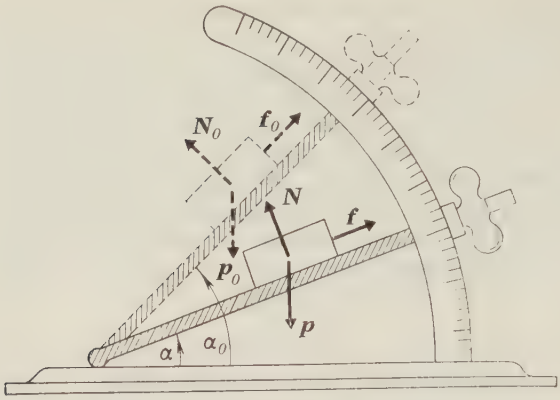


FIG. 101

go some deformation. The local deformation in the vicinity of a certain point of contact of the two surfaces depends on the local value of the pressure (and of course also on the mean pressure over the contacting areas) and it can be elastic or inelastic. The interlocking of the asperities of one body with those of the other and the squeezing of the protrusions of one surface into the hollows of the other obviously depend on the force pressing the bodies to each other.

In the state of rest, when the force of static friction arises, that is when $F < f_0$, the horizontal tangential components of the interaction forces between

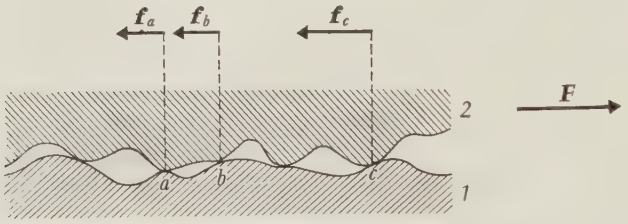


FIG. 102

the protrusions of the bodies "create" the friction force balancing the external force. Figure 102 shows schematically the appearance of the forces of static friction: when an external force is applied to body *2* there arise the tangential forces f_a, f_b and f_c in the regions adjoining the points a, b, and c, the resultant of these forces is in balance with the external force. (The vectors representing the forces f_a, f_b, and f_c are depicted above the bodies for the sake of visuality.)

When $F > f_0$ and the bodies are in relative motion there also arises the cohesion of the irregularities of the two bodies. In addition, these irregularities impact each other so that the resultant of the cohesion and impact forces gives rise to the friction force resisting the mutual sliding of the surfaces of the rigid bodies. The impact forces between the protrusions produce their oscillations which propagate in various directions in the contacting bodies. It is plausible

that the inelastic deformations accompanying the impact of the protrusions and irregularities of the contacting bodies also play an important role in this phenomenon.

The model we have discussed is very rough and it only describes approximately the real nature of the phenomenon of friction. We also have every reason to think that the chaotic molecular irregularities on the surfaces of the bodies may play the same role as the irregularities and protrusions of the contacting surfaces. The tangential forces arising on the contacting surfaces (they are produced by the molecular irregularities even when, in the macroscopic sense, the surfaces are perfectly smooth) may play the same role as the forces of interaction between the protrusions.

42. Force of Sliding Friction

Let a body lie on a horizontal surface. When the horizontal force acting on the body exceeds the (maximum) force of static friction, that is $F > \mu N$, the body obtains a non-zero acceleration and starts sliding, the velocity of the body increasing. To investigate this phenomenon we must consider the variation of the friction force between dry surfaces as the velocity of sliding increases.

Experiments show that in the general case the force of sliding friction first decreases and then increases as the velocity grows. In some cases the diagram of the dependence of the friction force on the velocity is of the type shown in Fig. 103*a*. (Along the v-axis in the figure we set off the scalar values of the velocity of sliding which is considered positive in a certain chosen direction and negative in the opposite direction; similarly, along the f-axis the scalar values of the friction force are set off.) When $v = 0$, that is in the state of rest, the friction force can assume any value from $-f_0$ to f_0. As the velocity becomes non-zero and starts increasing the friction force remains constant for some interval of variation of the velocity, then it decreases slowly and attains a minimum value after which it begins increasing. For different pairs of contacting surfaces the character of the dependence of the force of sliding friction on the velocity may be completely different.

When the velocity of relative sliding is sufficiently small the force of sliding friction between two dry metal surfaces can be considered constant, independent of the velocity and equal to the force of static friction. Experiments show that this assumption holds with a sufficiently high accuracy. In this approximation the graph of the dependence of the friction force on the velocity is of the type shown in Fig. 103*b*; for $v = 0$ the friction force can assume any value ranging from $-f_0$ to f_0.

This relationship between the force of sliding friction and the velocity is known as *Coulomb's law*. Coulomb's law also remains applicable when we consider the force of sliding friction between a metal surface and a homogeneous wooden surface, between a metal surface and leather, etc. For some limited ranges of variation of the velocities this law serves as an adequate approximation for the friction between various pairs of contacting surfaces sliding on each other.

Both the magnitude of the (maximum) force of static friction f_0 and the magnitude of the force of sliding friction depend on the force pressing the body to the surface it sildes on. As a rule, the force of sliding friction (like the force of friction of rest) is directly proportional to the force of normal pressure.

It should be noted that the known data concerning forces of sliding friction are to a large extent approximate and that there are many cases when the results of different measurements contradict each other. This can be accounted for by the fact that the force of sliding friction between two contacting surfaces depends on many factors including the polishing and the cleanliness of the surfaces. The contamination of the surfaces affects considerably the magnitude of the force of sliding friction, the kind of the contaminant also being very important. The character of the dependence of the forces of sliding friction on the

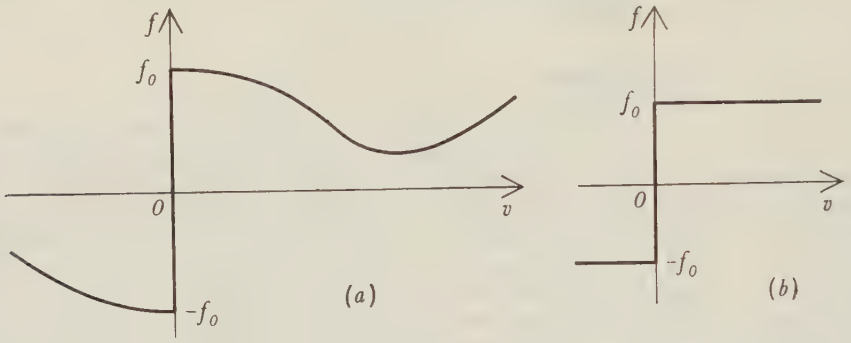


FIG. 103

velocity and also the values of f_0 are completely different for wet or contaminated surfaces with even very small spots of oil and for the same surfaces when they are purified.

The friction between two lubricated surfaces (with water, oil, and the like as a lubricant) in most cases has a character of a viscous friction. When there is a sufficient amount of a fluid lubricant between two sliding surfaces, that is a continuous fluid layer between them, the particles of the fluid adjoining the body stick to it and can be regarded as being at rest relative to the body. The velocity of motion of the particles of the fluid varies according to a linear law across the whole lubricant layer so that the force of friction is specified in this case by the value of the viscosity coefficient μ of the lubricant, by the areas of the contacting surfaces, and by the width of the lubricant layer. As to the width of the lubricant layer, it depends on both the kind of the lubricant and the force pressing the contacting and sliding bodies to each other.

The hydrodynamical theory of lubrication was developed in the classical theoretical and experimental studies of the Russian scientist N. P. Petrov*.

In engineering an important role is also played by the forces of rolling friction and the forces of friction arising in the adhesion of wheels (with sliding or without it). These phenomena will be discussed in Secs. 73 and 75 after the dynamics of rotating bodies has been studied.

It should be noted that the forces of static friction and sometimes also the forces of sliding friction play a fundamental role in the transmission of motion from one body to another. For example, when a man walks his motion relative to the ground is due to the friction forces between the soles of his shoes and

* N. P. Petrov, *Hydrodynamic Theory of Lubrication*, Izd. AN SSSR, 1948 (in Russian).

the ground. Similarly, a man walking in a passenger car or loads carried by a freight car obtain acceleration under the action of the forces of static friction arising between them and the car because of the cohesion between the corresponding contacting surfaces. The friction transmission of motion from one machine to another is widely used in engineering; for instance, this is the case in the transmission through a flexible connecter (for example, a flat belt) from one shaft to another. A belt transmission is only possible because of the forces of friction resulting from the cohesion between the belt and the pulleys. Another

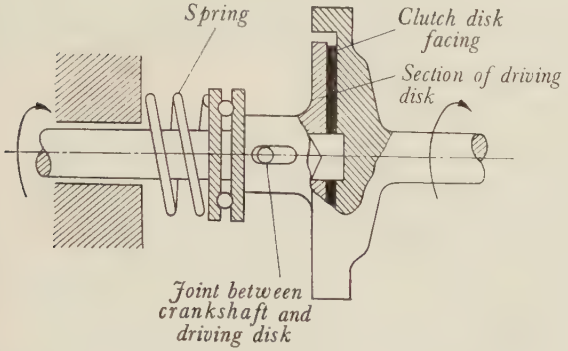


FIG. 104

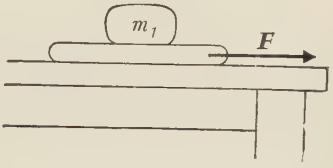


FIG. 105

example of this kind is a friction clutch joining the motor and the driving shaft of an automobile (see the scheme in Fig. 104).

Generally, without friction it would be very difficult to realize the transmission of motion and forces from one body to another. Most of the existing means of transmission and transportation would be impossible without using forces of friction.

Students often encounter difficulties when they start solving problems involving forces of static and sliding friction. It is therefore expedient to present here the solution of a typical problem of this kind.

Example. Let a load of mass m_1 be placed on a board lying on the horizontal surface of the table as shown in Fig. 105. The board is acted upon by a horizontal force F. Given the masses m_1 and m_2 of the load and of the board and the coefficient of friction μ_1 between the board and the load, and the coefficient of friction μ_2 between the board and the table, it is required to determine the accelerations of the board and of the load. We assume that Coulomb's law of friction is applicable here.

The answer to the problem depends of course on the magnitude F of the acting force and is different for different values of F. Let us begin with small magnitudes of the force F and then gradually pass to its larger values. It is quite obvious that for a very small force the accelerations of the load and of the board are equal to zero and both bodies are at rest. This situation takes place when

$$0 \leqslant F \leqslant \mu_2 (m_1 + m_2) g = F_0$$

that is when the acting force F does not exceed the limiting (maximum) force of static friction between the board with the load (regarded as one body) and the table.

When F slightly exceeds F_0 the board moves together with the load, their common acceleration being

$$a = (F - F_0)/(m_1 + m_2) \tag{42.1}$$

In this case the force of friction of rest between the board and the load is equal to $m_1 a$. As F increases (and, consequently, the acceleration a increases) the friction force between the load and the board also increases. This lasts as long as the condition $m_1 a < \mu_1 m_1 g$, that is $a < \mu_1 g$, is fulfilled. The situation changes

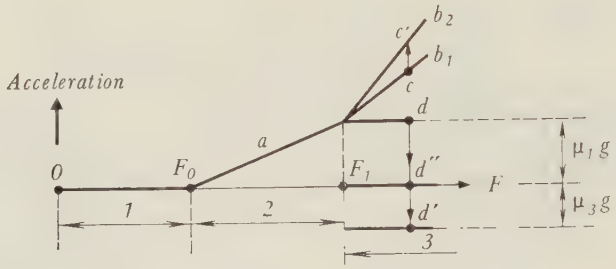


FIG. 106

when the friction force attains its maximum value, that is when $F = F_1$ where F_1 is that maximum value. For $F > F_1$ the load starts sliding relative to the board. The magnitude F_1 is determined by the condition $a = \mu_1 g$. On substituting this value into (42.1) we obtain

$$F_1 = F_0 + (m_1 + m_2)\,\mu_1 g = (\mu_1 + \mu_2)\,(m_1 + m_2)\,g$$

For $F > F_1$ the load leaves the board after some time because the acceleration of the board exceeds $\mu_1 g$ while the acceleration of the load remains equal to $\mu_1 g$. During that period the acceleration of the board is

$$b_1 = (F - F_0 - \mu_1 m_1 g)/m_2$$

and it can also be represented in the equivalent form

$$b_1 = (F - F_1)/m_2 + \mu_1 g$$

For a given force $F > F_1$ this period lasts as the load is sliding backwards relative to the board. The moment the load leaves the board the acceleration of the latter changes in a jump-like manner and instantaneously becomes equal to

$$b_2 = (F - \mu_2 m_2 g)/m_2 > b_1$$

because

$$b_1 = (F - \mu_2 m_2 g - \mu_1 m_1 g - \mu_2 m_1 g)/m_2$$

The conditions of the problem do not specify the acceleration of the load after it has left the board and is moving on the table. This unknown acceleration depends on the character of the motion, on whether the load slides on the table or moves in a more complicated manner (it can, for instance, roll). In case it slides, given the coefficient μ_3 of friction between the load and the table, the

acceleration of the load is equal to $-\mu_3 m_1 g$ and is directed backwards until the load stops sliding on the table.

We conclude the solution of the problem by representing the graph (Fig. 106) demonstrating the dependence of the accelerations on the magnitude of the force F. Region *1* corresponds to the state of rest; region *2* corresponds to the situation when the accelerations of the load and of the board are equal and are directly proportional to $(F - F_0)$. Finally, region *3* describes the motion in which the board and the load have different accelerations: the board first has the acceleration b_1 and then b_2 (the instantaneous change of the acceleration is depicted as the "jump" $c \to c'$ in Fig. 106). The load first has the acceleration equal to $\mu_1 g$ which then may become equal to $\mu_3 g$ and finally may turn into zero (see the corresponding transitions $d \to d' \to d''$ shown conditionally in the figure).

CHAPTER 6

Relative Motion

43. Inertial Reference Frames

Up till now we have assumed that the frames of reference relative to which we considered the motions of the bodies were rigidly connected with the Earth which itself was regarded as an immovable, fixed body. However, as we know, in reality the Earth is in permanent annual circular motion round the Sun (more precisely, it moves in an elliptic orbit) and, simultaneously, it rotates about its own axis. Hence, the assumtion that the Earth is immovable must lead to some errors; these errors will be examined in the present chapter.

The laws of dynamics can be stated the same way only for the *inertial* frames (systems) of reference which are in a uniform rectilinear motion relative to each other. Suppose that there are two frames of reference (see Fig. 107) one of which, denoted *1*, is regarded as being at rest (i.e. as being fixed) while the other, denoted *2*, moves relative to the former with a *constant velocity* $\boldsymbol{v}_0$. Then all the bodies which are in a state of rest with respect to the latter frame of reference will move with velocity $\boldsymbol{v}_0$ relative to the former and the bodies moving with velocity $\boldsymbol{v}_1$ relative to system *2* will obviously have the velocity $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_0$ with respect to system *1* (assumed to be fixed). The velocity $\boldsymbol{v}_0$ being constant, the acceleration of a body relative to the moving frame of reference coincides with that relative to the fixed frame of reference and vice versa. Thus, *in all systems of reference which are in uniform rectilinear motions relative to each other the acceleration of a moving body is the same.*

Experiments show that the forces acting on the bodies and the masses of the bodies are independent of the choice of any of these systems of reference relative to which the motions of the bodies are considered. For, the forces depend on the distances between the bodies, on their relative velocities, and on time, all these quantities not varying when we pass from one system of reference to another system of reference which is in a uniform rectilinear motion with respect to the former.

If we choose an arbitrary set of frames of reference which are all in uniform rectilinear motions relative to each other and if, in addition, it is known that the laws of dynamics hold for one of these frames then the first and the second laws of dynamics are stated in the same manner for all the frames of reference we have chosen. All such frames are referred to as *inertial* (or *Galilean*) frames of reference and the Galilean inertia law is valid only for such frames. This is the proposition we call *Galileo's relativity principle*; the transformation from one inertial system of reference to another is called a *Galilean transformation.*

A frame of reference which is in an accelerated motion with respect to an inertial frame of reference is spoken of as a *non-inertial frame of reference.* In

this connection there arises the following question: which of the systems of reference we deal with can be regarded as inertial one? It is extremely difficult to give a general answer to this question. However, the investigation of motions whose velocities are small in comparison with the velocity of light indicates that the coordinate system whose origin is connected with the centre of mass (see Sec. 55) of the bodies forming the Solar system and whose axes have invariable directions relative to the "fixed" stars* can be taken as an inertial frame

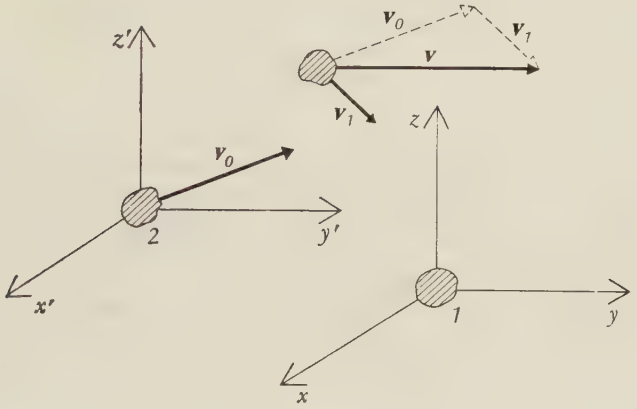


FIG. 107

of reference. The experimental data obtained both in the study of the motion on the Earth and from the astronomical observations confirm the validity of this assumption.

As to the frames of reference connected with the Earth, they can be considered inertial only approximately. This approximation involves some errors which will be analysed in Sec. 48.

In the theory of relativity this problem is treated in a different way. Firstly, it is assumed that all the inertial reference systems are equivalent in the sense that the laws of motion have the same form for all of them. This means that there is no inertial system which can be considered, in a certain sense, preferable or distinct from all the others. Secondly, not only the laws of dynamics but, generally, all the laws of nature are invariant relative to the choice of an inertial system of reference. Further, Galileo's relativity principle only means that Newton's laws are invariant with respect to the inertial system while Einstein's relativity principle extends this assertion to the laws of electrodynamics and all the other physical laws (see Chapter 17).

44. Motion of a Body in a Non-inertial System of Reference. Inertia Forces

When a motion of a body is considered in a system of reference which is in an accelerated motion the first and the second laws of dynamics are no longer applicable in their ordinary form. Indeed, it is evident that a body which is at

* This frame of reference is called the *primary inertial (coordinate) system.*

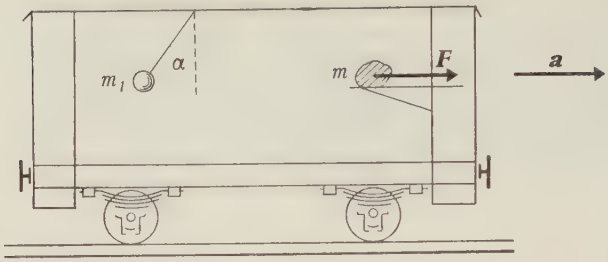


FIG. 108

rest relative to a non-inertial frame of reference must be subjected to the action of some external forces because the body is in an accelerated motion relative to the inertial frames of reference. Let us consider an example of a car on the rails (see Fig. 108) moving rectilinearly with an acceleration $\boldsymbol{a}$. A load of mass m lying on the horizontal table in the car has the acceleration $\boldsymbol{a}$ relative to the Earth and consequently it must be acted upon by a force giving it that acceleration, namely, by the force of friction which the surface of the table exerts on the load. Another load of mass m_1 (see Fig. 108) is suspended from a thread.

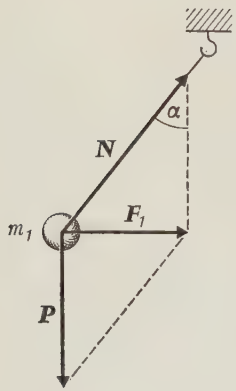
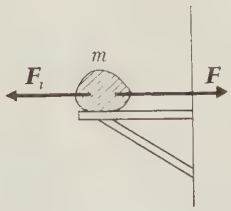


FIG. 109



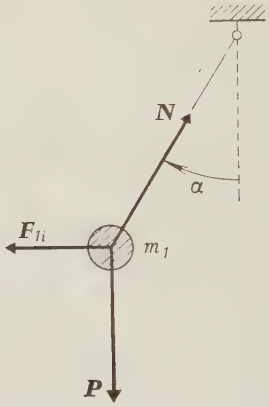


FIG. 110

The observation shows that the thread does not remain in the vertical position but departs from the vertical in the direction opposite to the direction of the acceleration. When the magnitude of the acceleration $\boldsymbol{a}$ is changed the angle α between the thread and the vertical also varies.

The latter load is subjected to the action of two forces, namely (see Fig. 109), the force of tension of the thread $\boldsymbol{N}$ (directed along the thread) and the force of gravity of the load $\boldsymbol{P}$ (directed towards the Earth). The sum $\boldsymbol{F}_1$ of the two forces, that is their resultant force, is in the direction opposite to that of the acceleration of the car:

$$\boldsymbol{F}_1 = m_1\boldsymbol{a} \text{ and } F_1 = P \tan \alpha \qquad (44.1)$$

When determining the forces acting on the bodies in the car we have taken into account the accelerations the bodies have relative to a system of reference which is connected with the Earth and considered inertial. Now there arises the following question: is it legitimate to express the laws of motion relative to the car (which is an accelerated system of reference) in the same way as in the case of an inertial system? The answer is obviously negative because we must take into account the acceleration of the non-inertial system of reference (the car).

In order to write correctly the laws of motion relative to a system having an acceleration *inertia forces* are often introduced. Their introduction makes it possible to retain the form which the first and the second laws of dynamics have in the inertial frames of reference for the bodies moving with respect to the non-inertial frames of reference. Usually this simplifies the analysis of the motion.

Let us suppose that every body which is in a motion relative to an accelerated system of reference is acted upon by an inertia force whose absolute value is equal to the mass of the body times the magnitude of the acceleration of the system of reference and whose direction is opposite to that of the acceleration. For instance, under this assumption, the load of mass m lying on the table in the car and the load of mass m_1 suspended from the thread are acted upon by the inertia forces $\boldsymbol{F}_{\mathrm{i}} = -m\boldsymbol{a}$ and $\boldsymbol{F}_{1\mathrm{i}} = -m_1\boldsymbol{a}$ as is shown in Fig. 110. Then we can assert that when the loads are at rest relative to the car the resultants of all the forces (including the inertia forces) applied to these bodies are equal to zero as in the case of a state of rest in an inertial system of reference. In fact, we can write

$$\boldsymbol{F}+\boldsymbol{F}_{\mathrm{i}}=0 \text{ and } \boldsymbol{F}_{1\mathrm{i}}+\boldsymbol{P}+\boldsymbol{N}=0 \tag{44.2}$$

If the table did not exert the force of friction on the load m the inertia force $\boldsymbol{F}_{\mathrm{i}}$ directed opposite to the acceleration of the car would move the load with acceleration $-\boldsymbol{a}$ relative to the car so that the load would slide away from the table. If that sliding motion of the load (in the absence of the friction force) is considered with respect to the rails we can say that the load is in a state of rest relative to the rails since there are no forces acting on it in the horizontal direction and that the car moving with acceleration $\boldsymbol{a}$ is being driven away from the load. Thus, the inertia forces should only be introduced when the motion in question is considered in an accelerated system of reference. When the same motion is considered in an inertial system of reference we do not need the introduction of any inertia forces.

If the load suspended from the thread in the car (see Figs. 109 and 110) were pushed it would start to oscillate like a pendulum. In case the acceleration of the car remains constant the oscillations of the pendulum can readily be investigated. Indeed, in such a situation the load is acted upon by its force of weight and, additionally, by the constant inertia force $\boldsymbol{F}_{1\mathrm{i}}$, the resultant of the two forces being inclined at an angle α to the vertical. Therefore, the pendulum must oscillate about the equilibrium position of the thread in which it forms the angle α with the vertical. In the state of equilibrium the force acting on the thread exceeds the force of gravity produced by the Earth: its absolute value is equal to the square root of the sum of the squares of the inertia force and the force of gravity and its direction is opposite to that of the force $\boldsymbol{N}$ (see Fig. 110). If the thread is cut the load starts falling in the car along the straight line forming the angle α with vertical, the absolute value of the acceleration of the load being $\sqrt{a^2+g^2}$. The trajectory of this motion of the load relative to the Earth

is a parabola whose shape is determined by the velocity of the car at the instant when the load starts falling and by the acceleration g.

For a body of mass m moving relative to a non-inertial system of reference having an acceleration $\boldsymbol{a}$ the second law of dynamics should be written in the form

$$\boldsymbol{F} + \boldsymbol{F}_1 = m\boldsymbol{w}_0 \tag{44.3}$$

where $\boldsymbol{w}_0$ is the acceleration of the body in the non-inertial system of reference, $\boldsymbol{F}_1 = -m\boldsymbol{a}$ is the inertia force, and $\boldsymbol{F}$ is the resultant of all the *external* forces acting on the body.

The presence of the inertia forces is connected with the accelerated motion of the system of reference relative to which the motion in question is considered. The inertia forces are introduced in order to describe the motion in the accelerated system of reference. In this sense there is no difference between them and the ordinary forces resulting from the interaction of the bodies. The principal distinction between the inertia forces and the other types of forces (resulting from the interaction of the bodies) lies in the fact that *there are no reaction forces corresponding to the inertia forces.* This means that there are no real bodies exerting the inertia forces. That is why an inertia force is sometimes referred to as a "fictitious" force. However, an inertia force is quite real because it is connected with the real situation when the coordinate system we deal with is in an accelerated motion. In this sense it is not at all fictitious, the only distinction from an ordinary force of interaction being the absence of a reaction force.

45. Inertia Forces Acting on a Body in a Rotating Frame of Reference

Let us consider the motion of several pendulums placed on the horizontal uniformly rotating disk (see Fig. 111). It turns out that the threads of the pendulums depart from the vertical position, the angle of the deviation increasing together with the distance from the pendulum to the centre of the disk.*

All the pendulums are at rest relative to the disk and are in a uniform circular motion relative to the Earth (the system of reference connected with the Earth is considered inertial). Since the radii of the circles in which the loads of the pendulum move are different the centripetal forces acting on them are also different because they are directly proportional to the distances from the loads to the centre of the disk. The centripetal force $\boldsymbol{F}$ (see Fig. 112) is due to the tensile force of the thread $\boldsymbol{N}$ and to the gravity force $\boldsymbol{P}$. Since $F = m\omega^2 R$ and $P = mg$ the angle of inclination α of the thread to the vertical must satisfy the relation

$$\tan\alpha = \frac{\omega^2 R}{g} \tag{45.1}$$

where R is the distance between the load and the centre of the disk, g is the acceleration of gravity, and ω is the angular velocity of the rotation of the disk.

---

* The pendulum suspended from the post placed near the axis of rotation of the disk will practically have no deviation from the vertical when the angular velocity ω of the rotation satisfies the condition $\omega \ll \sqrt{g/l}$.

Every pendulum is at rest relative to the disk, its thread being in an inclined position. Consequently, besides the force of weight, every pendulum is acted upon by a horizontal force directed from the centre, the absolute value of that force being different for different pendulums. This force is called the *centrifugal inertia force*. Its magnitude is equal to the mass of the load times the magnitude of the (centripetal) acceleration (relative to the Earth) of the point of the disk under which the load hangs; its direction is opposite to that of the acceleration. The centrifugal force is thus directed from the centre of the disk along the radius.

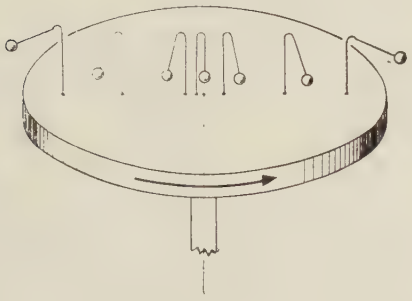


FIG. 111

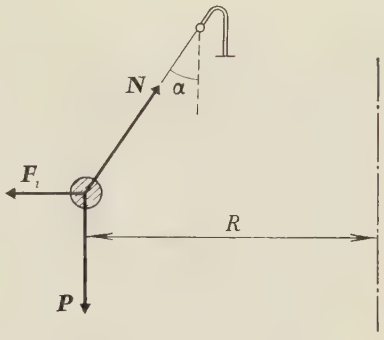


FIG. 112

Hence, when the pendulums are at rest relative to the disk each of them is acted upon by three forces, namely, the force of gravity $\boldsymbol{P}$, the tensile force $\boldsymbol{N}$, and the centrifugal inertia force

$$\boldsymbol{F}_i = m\omega^2 \boldsymbol{R} \tag{45.2}$$

where the vector $\boldsymbol{R}$ is directed from the axis of rotation. The resultant force of this system of three forces is equal to zero (see Fig. 112) and therefore the load is at rest relative to the (rotating) disk.

The inertia force acting on a body when it is in a state of rest in a rotating coordinate system depends on the position of the body relative to that system. When a body moves with respect to a rotating system of reference then, besides the centrifugal force, it is subjected to the action of some other inertia forces whose absolute values and directions will be determined in Sec. 48.

We conclude this section with the remark that in a system of reference which is in an accelerated rectilinear motion the inertia force is the same for all the points (this is not the case in a rotating system) and therefore the inertia force acting on a body in such a system does not depend on whether the body moves or is at rest.

46. Weightlessness

In the cabin of a spaceship, in the car of a falling lift, and in a flying airplane when it moves solely under the action of the gravity force *weightlessness* sets in. A body in the spaceship, in the lift, etc. "loses its weight" when it is acted upon *only* by the force of gravitational attraction produced by the Earth (or by other celestial bodies). The astronaut can freely "lie in the air" within the cabin without any support, an amount of fluid which does not wet the walls of the vessel

takes a spherical form, and so on. Before proceeding to the study of the phenomenon of weightlessness we must note that all the apparatuses in which the state of weightlessness is observed are in an accelerated motion under the action of the force of gravity alone, that is they are in a state of *free fall*.

Let us investigate how the weight of a body changes in a lift moving with an acceleration $\boldsymbol{a}$ directed downwards (Fig. 113). Here, like in Sec. 14, by weight we mean not the force of gravity $\boldsymbol{P}$ exerted on the body A by the Earth but the force of pressure $\boldsymbol{Q}$ with which the body acts on the floor of the car. The reaction force with which the floor acts on the body is denoted $\boldsymbol{Q'}$. By the laws of dynamics, we have

$$P - Q' = ma \text{ and } Q = Q'$$

where m is the mass of the body whence

$$Q = P - ma \qquad (46.1)$$

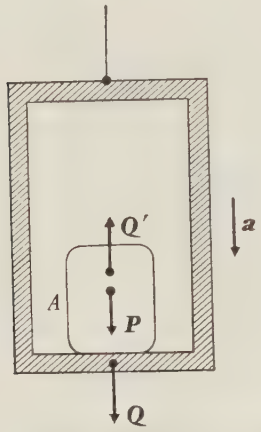


FIG. 113

(here the signs of the projections of the forces and of the acceleration are chosen in accordance with the arrows indicating the directions of the corresponding vectors in Fig. 113). The force P does not vary (because the distance from the lift to the Earth's surface is of course assumed to be very small in comparison with the radius of the Earth) while the force Q (the force of pressure exerted by the body on the floor) depends on the acceleration a.

If we associate with the lift an accelerated reference system moving with acceleration $\boldsymbol{a}$, all the bodies in the car must be regarded as being acted upon by the corresponding inertia forces. The projection of the inertia force applied to the body of mass m being equal to $-ma$, equation (46.1) can be interpreted as follows: the force of pressure Q with which the body acts on the floor is equal to the resultant of the force of gravity P and the inertia force $-ma$ applied to the body.

Formula (46.1) can also be written as

$$Q = m(g - a) \qquad (46.2)$$

whence it follows that for $a = g$ we have $Q = 0$, that is the force of pressure turns into zero and the body thus loses its weight. Hence, any body placed in a lift moving with acceleration of magnitude g directed downwards is in the state of weightlessness. The magnitude of the inertia force applied to the body is then equal to that of the force of gravity while the directions of the forces are opposite and the sum (the resultant) of the two forces is equal to zero. It should be noted that what has been said does not depend on whether the lift moves upwards or downwards; it is only required that its acceleration should be equal to g and directed downwards.

An analogous situation can be observed in the cabin of a spaceship when it moves solely under the action of the gravitational force because in this case all the bodies in the cabin have an acceleration of magnitude g (corresponding

to the point at which they are at the moment) produced by that force. When moving in a circular orbit round the Earth the spaceship is in a "permanent free fall", its acceleration being all the time perpendicular to its trajectory.

The state of weightlessness may appear in an airplane flying with a certain velocity when the pilot chooses the flying regime in such a way that the resultant of the forces with which the air acts on the airplane (these are the drag force and the lift force) is completely balanced by the thrust force. In these conditions the airplane "falls" under the action of the force of gravity with acceleration g.

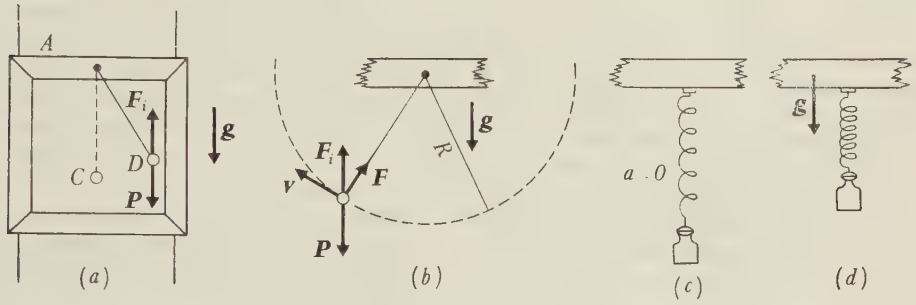


FIG. 114

For modern airplanes such regime may last about half a minute; during this time the people in the airplane observe the state of weightlessness.

Almost a hundred years ago Professor N. A. Lyubimov (the University of Moscow) suggested a very simple experiment with a pendulum on a falling frame to demonstrate the phenomenon of weightlessness. The scheme of the experiment is shown in Fig. 114*a*. The frame A carrying a pendulum can slide freely along two guides under the action of the gravity force. The frame is let to fall together with the pendulum as the latter oscillates. If this motion starts at a time when the load of the pendulum is in its uppermost position D (at this moment its instantaneous velocity relative to the frame turns into zero) then, as the frame is falling, the pendulum no longer oscillates; it stops moving relative to the frame and remains in the position D. If the frame is let to fall when the oscillating pendulum passes through the equilibrium point C the oscillatory motion of the pendulum goes into a uniform rotation about the point of suspension (see Fig. 114*b*).

We can also perform the following experiment with the falling frame. A load is suspended from a spring attached to the frame (see Fig. 114*c*). If the frame starts to fall when the load and the spring are in equilibrium and the spring is extended so that it exerts a force balancing the weight of the load then, after the fall has begun, the spring contracts as if there were no load suspended from it, that is as if the load "lost" its weight (see Fig. 114*d*).

All these phenomena can easily be explained as the results of the actions of the corresponding inertia forces on condition that the forces of friction (which are very small in the experiments) are neglected. When a body moves relative to the frame as the latter is itself in a uniformly accelerated motion (in a free fall) the body is acted upon by the force of gravity $\boldsymbol{P}$ and by the inertia force $\boldsymbol{F}_1$ (directed upwards) whose magnitude is equal to the mass of the body multi-

plied by the magnitude of the acceleration of the frame and whose direction is opposite to that of the acceleration. The acceleration of the frame being g, the inertia force balances the force of gravitation produced by the Earth. Consequently, the resultant of the forces acting on a body which moves or is at rest relative to the frame is equal to zero and therefore the body must be in a uniform rectilinear motion or retain its state of rest. That is why when the pendulum has a zero velocity at the beginning of the fall of the frame it remains in a fixed position relative to the frame. If the pendulum has a non-zero velocity relative to the frame at the moment the latter starts falling it begins to rotate uniformly about the point of suspension since the only force the pendulum is acted upon is the force F. This force is directed to the point of suspension and gives the pendulum the centripetal acceleration of the circular motion (it is of course meant that the mass of the frame is much larger than that of the pendulum). Similarly, the load extending the spring when the frame is fixed can no longer extend it after the fall has begun because in the fall the inertia force and the gravity force applied to the load are equal in their magnitudes and opposite in the directions and thus balance each other.

These phenomena can also be explained without the introduction of the inertia forces if we consider the motions of the frame and of the load relative to the Earth. After the frame is let to fall the accelerations of the frame and of the pendulum relative to the Earth become equal. Consequently, the velocities of the frame and of the pendulum are produced only by the action of the gravity forces and are therefore equal at any moment. Hence, the gravity forces cannot change the mutual positions of the frame and the pendulum during the free fall of the former. Thus, if the pendulum is at rest relative to the frame at the initial time of the fall, it remains in that state during the whole period of motion. Similarly, if the pendulum has a non-zero angular velocity at the initial time, it must be in a uniform rotation with the same angular velocity during the fall. (In these considerations it should also be taken into account that the mass of the frame is assumed to be considerably less than the mass of the pendulum).

Analogous arguments apply to the other cases of weightlessness.

The weightlessness which sets in when a spaceship moves in vacuum under the action of the gravitational forces can only be explained in the same manner. In this case the spaceship has an acceleration (created by the gravitational forces) relative to the reference system connected with the Sun. In the non-inertial reference system connected with the spaceship to all the bodies in the cabin of the spaceship are applied inertia forces whose direction is opposite to that of the acceleration of the system. Besides the inertia forces, the bodies are subjected to the action of the forces of gravitation, so that the resultant of the forces acting on each body is equal to zero. When a spaceship moves in a circular orbit round the Earth the role of the inertia forces is played by the centrifugal forces which balance the action of the gravitational forces on each body in the cabin.

47.* The Relationship Between the Vectors of Angular and Linear Velocities of a Particle

Before proceeding to the material of this section the reader should remember the definition of the vector product of two vectors (see Sec. 7). The formula expressing the vector product proves convenient for writing the formula connecting the angular velocity of a point moving in a circle (Fig. 115) with its linear velocity.

The *angular velocity* $\boldsymbol{\omega}$ (represented by a vector) of a point rotating about an axis is defined as a vector parallel to the axis of rotation whose length is equal to the number of units of length equal to the numerical value of the angular velocity ω (see (22.12)) and whose direction is such that the rotation of the point is seen counterclockwise from the terminus of the vector $\boldsymbol{\omega}$.

The vector $\boldsymbol{\omega}$ is perpendicular to the plane of the circle in which the point moves while the vector $\boldsymbol{v}$ (the *linear* velocity of the point) lies in that plane. Let us also consider the radius vector $\boldsymbol{R}$ (see Fig. 115) directed from the axis of rotation to the moving point. The vectors $\boldsymbol{R}$ and $\boldsymbol{v}$ are mutually perpendicular and lie in the plane perpendicular to the axis of rotation.

According to the definition of the vector product of two vectors, the velocity $\boldsymbol{v}$ is connected with the vectors $\boldsymbol{\omega}$ and $\boldsymbol{R}$ by the formula

$$\boldsymbol{v} = [\boldsymbol{\omega}\boldsymbol{R}] \tag{47.1}$$

In the case under consideration the angle α between the vectors $\boldsymbol{\omega}$ and $\boldsymbol{R}$ is equal to $90°$ ($\sin\alpha = \sin 90° = 1$). Thus, *the linear velocity $\boldsymbol{v}$ is equal to the vector product of the angular velocity $\boldsymbol{\omega}$ by the radius vector $\boldsymbol{R}$.*

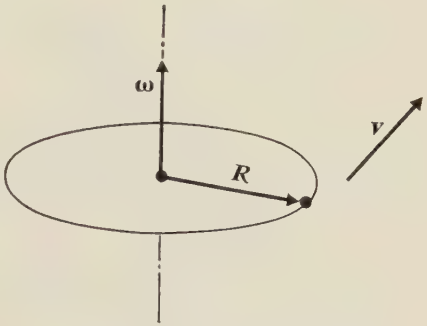


FIG. 115

In the simple situation we have considered the reason for writing formula (47.1) may not be quite clear to the reader but this representation of the velocity $\boldsymbol{v}$ in terms of $\boldsymbol{\omega}$ and $\boldsymbol{R}$ will be extremely convenient for our further analysis of more complicated motions.

48. Inertia Forces Acting on a Body Moving in a Rotating Frame of Reference

Let us consider Fig. 116 showing a body (a small ball) in a uniform rectilinear motion along a guide relative to a rotating disk. Let the velocity of the uniform rectilinear motion of the ball along the radius of the disk be $\boldsymbol{v}_0$. The motion of the ball relative to the fixed coordinate system x, y connected with the table (and regarded as an inertial reference system) is neither rectilinear nor uniform: the trajectory of the centre of the ball is a spiral, and the acceleration of the ball depends in a rather complicated manner on the motion along the trajectory. To determine the inertia forces acting on the body moving in a rotating frame of reference we must first of all find the acceleration of the body relative to the fixed coordinate system x, y.

Thus, let the centre of the ball move uniformly along the radius of the rotating disk. We shall find the acceleration of this point (the centre) relative to the fixed coordinate system. Let the point be at a distance R from the axis of rotation at time t. The position of the point is specified by the radius vector $\boldsymbol{R}$ in the xy-plane. The velocity of the point relative to the fixed coordinate system can be resolved into two components one of which is equal to the velocity $\boldsymbol{v}_0$ of the motion of the point relative to the disk and goes along the radius $\boldsymbol{R}$ and the other is perpendicular to the radius $\boldsymbol{R}$ and is equal to $[\boldsymbol{\omega}\boldsymbol{R}]$ where $\boldsymbol{\omega}$ is the angular velocity of the rotation of the disk (Fig. 117*a*).

Let us now determine the magnitudes and the directions of these components at time $t + dt$ where the time increment dt is very small. Firstly, during the time interval dt both components turn through a small angle

$$d\alpha = \omega\, dt \tag{48.1}$$

and, secondly, the absolute value of the component along the radius does not change and the absolute value of the tangential component (perpendicular to the radius) gains an increment equal to

$$\omega\, dR = \omega v_0\, dt \tag{48.2}$$

because during that time the distance from the point to the axis of rotation receives an increase of magnitude

$$dR = v_0\, dt \tag{48.3}$$

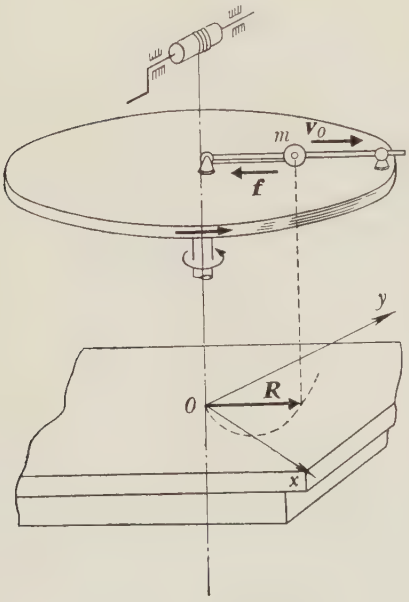


FIG. 116

For the sake of visuality, the components of the velocities at time moments t and $t + dt$ shown in Fig. 117*b* are drawn from a common origin. As is readily seen, the increment of the velocity of the point during time dt can be represented as the sum of three vectors $(d\boldsymbol{v})_1$, $(d\boldsymbol{v})_2$, and $(d\boldsymbol{v})_3$. The vectors $(d\boldsymbol{v})_1$ and $(d\boldsymbol{v})_2$ are perpendicular to the radius and go in one direction parallel to the tangential component $[\boldsymbol{\omega R}]$ of the velocity and the vector $(d\boldsymbol{v})_3$ is directed towards the axis of rotation. It should be noted that when determining both the directions and the magnitudes of these vectors we must take into account the fact that since the time interval dt is infinitesimal so is the angle $d\alpha$ and therefore the magnitudes of the vectors $(d\boldsymbol{v})_1$, $(d\boldsymbol{v})_2$, and $(d\boldsymbol{v})_3$ are very small relative to the magnitudes of the components of the velocity.

Now using Fig. 117*b* we can determine the magnitudes of the infinitesimal vectors $(d\boldsymbol{v})_1$, $(d\boldsymbol{v})_2$, and $(d\boldsymbol{v})_3$.

(1) The vector $(d\boldsymbol{v})_1$ is due to the rotation of the vector representing the velocity of the relative motion along the radius $\boldsymbol{R}$ (because the latter rotates). By formula (48.1) the magnitude of $(d\boldsymbol{v})_1$ is

$$(dv)_1 = v_0\, d\alpha = v_0 \omega\, dt \tag{48.4}$$

(2) The vector $(d\boldsymbol{v}_2)$ is due to the increase of the distance from the moving point to the axis of rotation, the increase of that distance results in the corresponding increase of the circular velocity. The magnitude of the vector $d\boldsymbol{v}_2$ is

$$(dv)_2 = \omega\,(R + dR) - \omega R = \omega v_0\, dt \tag{48.5}$$

(in these calculations we have taken into account formula (48.2)).

(3) The vector $(d\boldsymbol{v})_3$ appears because the rotation of the disk results in the motion of the point in a circle and therefore the component $[\boldsymbol{\omega R}]$ of the velocity

(perpendicular to the radius) changes its direction. The magnitude of the vector $(dv)_3$ is

$$(dv)_3 = \omega R\, d\alpha = \omega R\omega\, dt = \omega^2 R\, dt \tag{48.6}$$

To find the acceleration of the moving point we must divide the increment of the velocity by the infinitesimal time dt. The tangential component of the acceleration (perpendicular to the radius) is found by adding together (48.5)

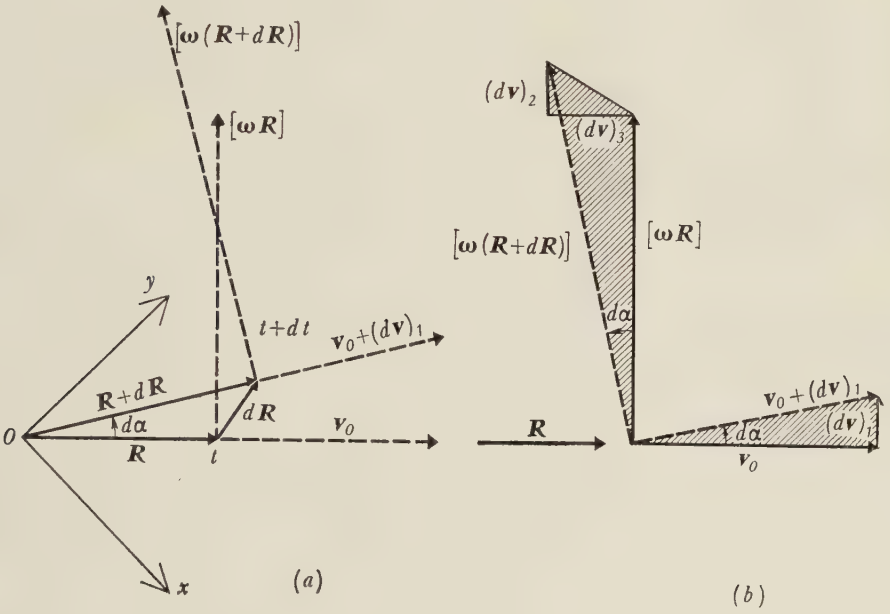


FIG. 117

and (48.4) and dividing the result by dt, this component is called the *Coriolis acceleration*. Denoting it by $\boldsymbol{w}_{cor}$ we can write for its absolute value the formula

$$w_{cor} = \frac{(dv)_1 + (dv)_2}{dt} = \frac{2\omega v_0\, dt}{dt} = 2\omega v_0 \tag{48.7}$$

The component of the acceleration along the radius is calculated in the same manner. It readily follows that this component is nothing but the *centripetal acceleration* $\boldsymbol{w}_c$ (Sec. 9), its magnitude being

$$w_c = \frac{(dv)_3}{dt} = \frac{\omega^2 R\, dt}{dt} = \omega^2 R \tag{48.8}$$

Thus, the magnitude of the Coriolis acceleration is twice the magnitude of the angular velocity times that of the velocity of the point relative to the disk.

Since the point has an acceleration relative to the reference system x, y it is acted upon by a force. The magnitude f of the component $\boldsymbol{f}$ of this force along the radius (the vector $\boldsymbol{f}$ is directed to the centre of the disk and corresponds to the centripetal acceleration) is $f = m\omega^2 R$. The magnitude F of the component $\boldsymbol{F}$ of the force perpendicular to the radius is $F = 2m\omega v_0$. The former component is the centripetal force and the latter is the force producing the Coriolis accele-

ration. This means that when the ball moves along the guide attached to the rotating disk the guide acts upon the ball with the force F perpendicular to the guide.

The Coriolis acceleration is perpendicular to the direction of the velocity v_0 of the relative motion and can be written in the form of the vector product of the vectors ω and v_0:

$$w_{cor} = 2[\omega v_0] \tag{48.9}$$

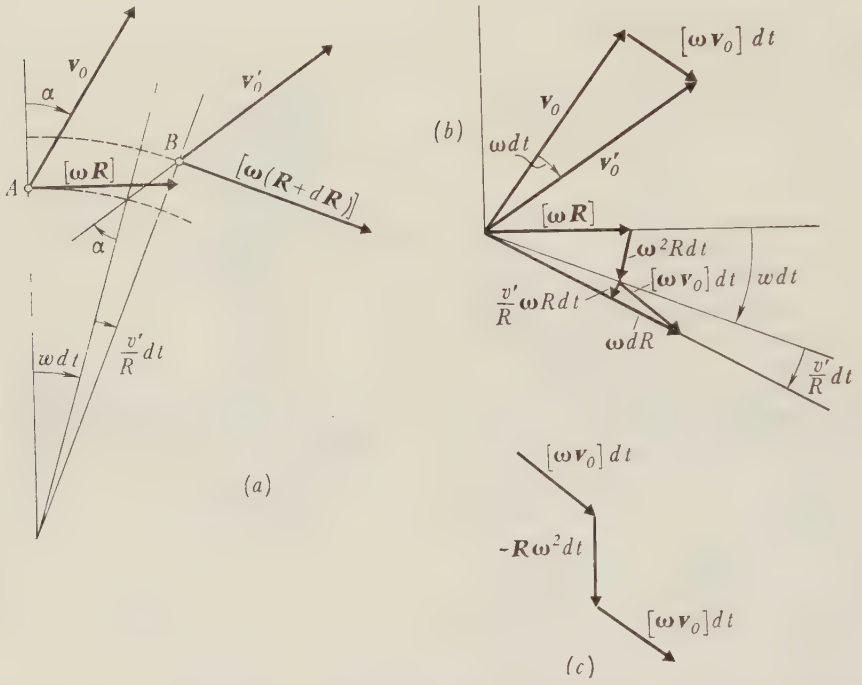


FIG. 118

Formula (48.9) is valid for any direction of the velocity v_0 relative to the rotating disk. In the general case this formula can be obtained with the help of Fig. 118*a* showing the components of the velocity of the point moving relative to the disk with an arbitrary constant velocity v_0. The velocity of the point relative to the fixed reference frame at time t is resolved into the two components v_0 and $[\omega R]$ the ball occupying the position A at that moment. At time $t + dt$ the ball is at the point B, its velocity being resolved into the two components v_0' and $[\omega(R + dR)]$. The latter component is turned through the angle $(\omega + v'/R)\,dt$ relative to the initial direction of the component at time t (v' is the component of the velocity v_0 perpendicular to the radius).

For the sake of visuality, in Fig. 118*b* the vectors representing the components of the velocity of the point relative to the fixed reference system at the instants t and $t + dt$ are drawn from a common origin. The increments of the velocity during time dt are also shown there. In the derivation of formula (48.9) it should be taken into account that $\omega\,dt$ and $\frac{v'}{R}dt$ are infinitesimal angles.

The three vectors whose sum is equal to the increment of the velocity are shown in Fig. 118*c* as forming a chain (for the sake of visuality). The first and the third vectors in the chain are perpendicular to $\boldsymbol{v}_0$ and have equal magnitudes while the second one is directed towards the centre. On dividing the sum of the first and the third vectors by dt we obtain the Coriolis acceleration $2\,[\boldsymbol{\omega v}_0]$. Similarly, the division of the second vector by dt results in the centripetal acceleration $-\omega^2\boldsymbol{R}$.

It should be noted that a motion in a rotating reference system in the direction parallel to the axis of rotation (i.e. when $\boldsymbol{v}_0 \parallel \boldsymbol{\omega}$) does not produce the Coriolis acceleration because in this case the vector $\boldsymbol{v}_0$ does not change its direction in space. However, we can say that the Coriolis acceleration is always equal to $2\,[\boldsymbol{\omega v}_0]$ (for any direction of $\boldsymbol{v}_0$) because if we write $\boldsymbol{v}_0 = \boldsymbol{v}_{\parallel} + \boldsymbol{v}_{\perp}$, where $\boldsymbol{v}_{\parallel}$ is parallel to the axis of rotation and $\boldsymbol{v}_{\perp}$ is perpendicular to it, then $[\boldsymbol{\omega v}_0] = [\boldsymbol{\omega v}_{\perp}]$ since $[\boldsymbol{\omega v}_{\parallel}] \equiv 0$.

Let us come back to the example of the ball moving along the radius of the rotating disk (Fig. 116). So far we have considered this motion relative to an inertial reference frame (i.e. relative to the fixed coordinate system). We shall now analyse the same motion in the reference system connected with the rotating disk.

Since the ball is in a uniform rectilinear motion (along the radius) relative to the disk no forces would act on the ball if the ordinary laws of dynamics were applicable to this motion. However, as has been established, there are two forces acting on the ball: the force $\boldsymbol{f} = -m\omega^2\boldsymbol{R}$ directed to the centre of the disk along its radius and the force $\boldsymbol{F} = 2m\,[\boldsymbol{\omega v}_0]$ perpendicular to the radius. The force $\boldsymbol{f}$ is exerted on the ball by the thread (see Fig. 116) attached to it which goes to the axis of rotation, passes around the pulley placed near the centre and then goes upwards to the reel it is wound onto. The force $\boldsymbol{F}$ is produced by the guide which bends slightly and acts on the ball in the direction of the rotation of the disk. These forces are permanently acting on the ball. The magnitude of $\boldsymbol{f}$ increases proportionally to the distance between the ball and the centre of the disk while the other force $\boldsymbol{F}$ is constant in its magnitude since v_0 and ω are constant. However, the ball is in a uniform rectilinear motion relative to the disk.

The forces $\boldsymbol{F}$ and $\boldsymbol{f}$ are quite real and do in fact act on the ball; their magnitudes and directions are independent of whether the motion is considered relative to the fixed reference frame or relative to the rotating one. Therefore, as in the case of an accelerated rectilinear motion, we can introduce inertia forces in order to retain the ordinary form of the first and the second laws of dynamics for the rotating reference system.

The introduction of the inertia forces makes it possible to describe the uniform motion of the ball along the rectilinear guide attached to the rotating disk in the following way. In the direction perpendicular to the guide the ball is acted upon by the force $\boldsymbol{F}$ exerted by the (slightly bent) guide. This force is balanced by the (*Coriolis*) *inertia force* $\boldsymbol{F}_1 = -2m\,[\boldsymbol{\omega v}_0]$ whose direction is opposite to that of the force exerted by the guide (see Fig. 119). Along the radius the ball is acted upon by the force $\boldsymbol{f}$ produced by the tension of the thread. The force $\boldsymbol{f}$ gives the ball the centripetal acceleration; it is balanced by the *centrifugal inertia force* $\boldsymbol{f}_1 = m\omega^2\boldsymbol{R}$. From this point of view, the resultant of the force of pressure of the guide, the force of tension of the thread, and the two inertia forces balancing them is equal to zero and the ball is therefore in a uniform rectilinear motion along the guide in the rotating reference system.

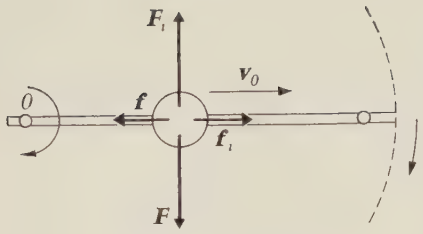


FIG. 119

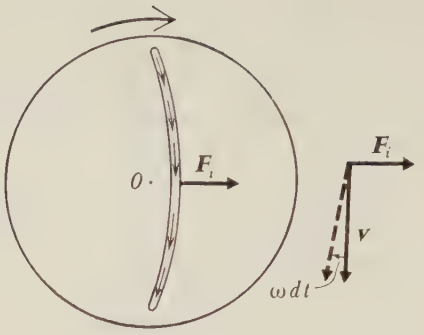


FIG. 120

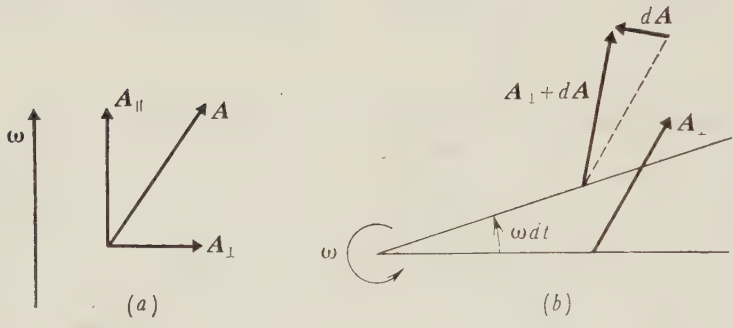


FIG. 121

Like all inertia forces, the Coriolis inertia force appearing in a rotating reference system is applied to the body moving relative to that system. When the relative motion is uniform and rectilinear this force is in balance with the forces exerted on the body by the other bodies. We once again stress that no inertia forces must be introduced when the same motion of the body is considered in an inertial reference system.

In the example we have considered the ball is permanently acted upon by the centrifugal inertia force which is directly proportional to the radius. If we cut the thread and let the ball move freely along the radius without the action of the external force $\boldsymbol{f}$ the motion along the guide is no longer uniform and becomes accelerated. The magnitude of this acceleration is specified by the magnitude $f_1 = m\omega^2 R$ of the inertia force and it increases together with R. The Coriolis inertia force also increases as the velocity of the motion of the ball relative to the disk grows. It can be shown that in this case the distance R between the ball and the centre of the disk increases according to the formula

$$R = a \cosh \omega t$$

where a is the distance between the ball and the centre at time $t = 0$ when its velocity is equal to zero, and $\cosh x$ is a function of the variable x which is called the hyperbolic cosine and is defined by the formula $\cosh x = 1/2\ (e^x + e^{-x})$.

The action of the Coriolis inertia force can be demonstrated with the aid of a visual experiment suggested by Professor S. E. Khaikin. The scheme of the experiment is shown in Fig. 120. Along the diameter of the rotating disk is placed a rubber pipe through which water is made to run as the disk rotates. The motion of the water is uniform relative to the disk and the water is acted upon by the Coriolis inertia force which bends the pipe in the direction opposite to the direction of the rotation of the velocity vector of the water particles. In this experiment the curvature of the pipe increases as the angular velocity of the rotation is increased. The same occurs when the velocity of the water flow in the pipe is increased. Both phenomena are more easily observed when the disk is lit by a stroboscopic tube.

Below we demonstrate another procedure for the derivation of the formulas describing the relationship between the accelerations of a moving body in a fixed and in a rotating coordinate system.

We begin with establishing the general rule for the differentiation of a vector function.

Let us consider a vector $\boldsymbol{A}$ simultaneously in a rotating reference system and in a fixed one.

If the vector $\boldsymbol{A}$ is invariable relative to the rotating coordinate system in which its components along the coordinate axes are constant then its components in the fixed system vary. Hence, the vector $\boldsymbol{A}$ itself varies in the fixed system. During time interval dt it receives the increment

$$d\boldsymbol{A} = [\boldsymbol{\omega}\, dt\, \boldsymbol{A}] \text{ whence } \frac{d\boldsymbol{A}}{dt} = [\boldsymbol{\omega}\boldsymbol{A}]$$

Indeed, let us resolve the vector $\boldsymbol{A}$ into the two components $\boldsymbol{A}_{\parallel}$ and $\boldsymbol{A}_{\perp}$ along the directions parallel and perpendicular to $\boldsymbol{\omega}$, respectively (see Fig. 121*a*). Then we can write

$$\boldsymbol{A} = \boldsymbol{A}_{\parallel} + \boldsymbol{A}_{\perp},\ [\boldsymbol{\omega}\boldsymbol{A}] = [\boldsymbol{\omega}\boldsymbol{A}_{\perp}] \text{ and } [\boldsymbol{\omega}\boldsymbol{A}_{\parallel}] = 0$$

The increment $d\boldsymbol{A}$ is shown in Fig. 121*b*, the plane of the figure being perpendicular to $\boldsymbol{\omega}$. As is seen,

$$d\boldsymbol{A} = [\boldsymbol{\omega}\, dt\, \boldsymbol{A}_{\perp}], \text{ that is } \frac{d\boldsymbol{A}}{dt} = [\boldsymbol{\omega}\boldsymbol{A}_{\perp}] = [\boldsymbol{\omega}\boldsymbol{A}]$$

If the vector $\boldsymbol{A}$ varies in the rotating system and receives an increment $(d\boldsymbol{A})_0$ during time dt then the derivative of $\boldsymbol{A}$ in the fixed coordinate system can be written as

$$\frac{d\boldsymbol{A}}{dt} = \frac{(d\boldsymbol{A})_0}{dt} + [\boldsymbol{\omega}\boldsymbol{A}] \tag{48.10}$$

The vector $(d\boldsymbol{A})_0/dt$ can be called the time derivative of the vector $\boldsymbol{A}$ in the rotating system. Formula (48.10) is valid for any vector function $\boldsymbol{A}$.

Let us apply formula (48.10) to the vector $\boldsymbol{R}$ joining a fixed point O on the axis of rotation of a moving particle with that particle (see Fig. 122). Then for the fixed system we have

$$\frac{d\boldsymbol{R}}{dt} = \frac{(d\boldsymbol{R})_0}{dt} + [\boldsymbol{\omega}\boldsymbol{R}] \tag{48.11}$$

where $\frac{d\boldsymbol{R}}{dt} = \boldsymbol{v}$ is the velocity of the particle in the fixed system and $\frac{(d\boldsymbol{R})_0}{dt} = \boldsymbol{v}_0$ is the velocity relative to the rotating system. The velocity $\boldsymbol{v}_0$ can have an

arbitrary direction. The derivative of the velocity vector

$$\boldsymbol{v} = \boldsymbol{v}_0 + [\boldsymbol{\omega}\boldsymbol{R}]$$

in the fixed system which is equal to the acceleration $\boldsymbol{w}$ of the particle relative to that system can be expressed as

$$\boldsymbol{w} = \frac{d\boldsymbol{v}}{dt} = \frac{d\boldsymbol{v}_0}{dt} + \left[\boldsymbol{\omega}\frac{d\boldsymbol{R}}{dt}\right] \tag{48.12}$$

provided that $\boldsymbol{\omega} = \text{const}$. According to (48.10), the derivatives $\frac{d\boldsymbol{v}_0}{dt}$ and $\frac{d\boldsymbol{R}}{dt}$ can be written in the form

$$\frac{d\boldsymbol{v}_0}{dt} = \frac{(d\boldsymbol{v}_0)_0}{dt} + [\boldsymbol{\omega}\boldsymbol{v}_0] \quad \text{and}$$

$$\frac{d\boldsymbol{R}}{dt} = \boldsymbol{v} = \boldsymbol{v}_0 + [\boldsymbol{\omega}\boldsymbol{R}]$$

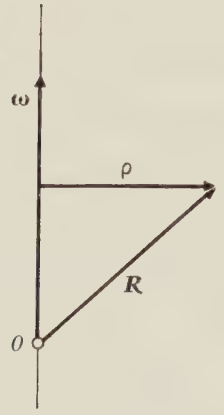


FIG. 122

On substituting these expressions into (48.12) we obtain

$$\boldsymbol{w} = \frac{(d\boldsymbol{v}_0)_0}{dt} + [\boldsymbol{\omega}\boldsymbol{v}_0] + [\boldsymbol{\omega}\,[\boldsymbol{v}_0 + [\boldsymbol{\omega}\boldsymbol{R}]]] =$$

$$= \frac{(d\boldsymbol{v}_0)_0}{dt} + 2\,[\boldsymbol{\omega}\boldsymbol{v}_0] + [\boldsymbol{\omega}\,[\boldsymbol{\omega}\boldsymbol{R}]] \tag{48.13}$$

which is the sought-for relation. The first term $\frac{(d\boldsymbol{v}_0)_0}{dt} = \boldsymbol{w}_0$ is the acceleration of the particle relative to the rotating system, the term $2\,[\boldsymbol{\omega}\boldsymbol{v}_0]$ is the Coriolis acceleration and $[\boldsymbol{\omega}\,[\boldsymbol{\omega}\boldsymbol{R}]] = -\omega^2\boldsymbol{\rho}$ is the centripetal acceleration (see Fig. 122). Thus, for the case $\boldsymbol{\omega} = \text{const}$ the general expression of the acceleration is

$$\boldsymbol{w} = \boldsymbol{w}_0 + 2\,[\boldsymbol{\omega}\boldsymbol{v}_0] - \omega^2\boldsymbol{\rho} \tag{48.14}$$

The introduction of the inertia forces can be interpreted from different points of view. To avoid misunderstanding we shall discuss the two possible approaches.

(1) *Without using the motion of inertia forces.*

In this approach the first and the second laws of dynamics are inapplicable to a non-inertial reference system and the forces of interaction between the bodies do not specify completely the acceleration of a moving body. Therefore, it is necessary to carry out the dynamical analysis of the motion of the given body in an inertial reference system and then determine its motion relative to the non-inertial system using the known rules of kinematics.

(2) *Using the motion of inertia forces.*

In this approach the addition of the inertia forces to the interaction forces makes it possible to apply (formally) the first and the second laws of dynamics to a non-inertial reference system as well. When the non-inertial system is in a translatory motion the inertia forces are the same for all the points and are independent of the velocity of the motion of the body relative to that system. In case the non-inertial system rotates the inertia forces are different for different points: the centrifugal inertia forces depend on the positions of the points at which they are applied and the Coriolis inertia forces depend on the relative velocity of the motion.

The physical significance of the inertia forces is that they make it possible to take into account the acceleration of a body which is in a uniform rectilinear motion relative to a non-inertial reference system, that is the acceleration resulting from the non-uniform motion of the system of reference. The addition of the inertia forces to the external forces is equivalent to the subtraction of a part of the external forces from their resultant, namely, the part of these forces which specifies the centripetal and the Coriolis accelerations of the body (in case the non-inertial reference system rotates) or the acceleration of the non-

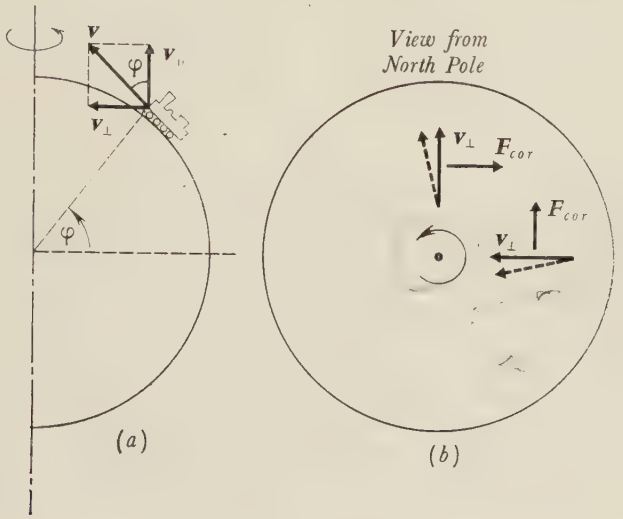


FIG. 123

inertial system (when it is in a translatory motion). The remaining part of the external forces determines the acceleration of the body relative to the non-inertial system of reference.

49. The Influence of the Earth's Rotation on the Motion of Bodies. The Foucault Pendulum

The action of the Coriolis inertia force accounts for the fact that rivers flowing in the northern hemisphere wash out their right banks (this is known as *Ber's law*). The action of this force also explains why in the northern hemisphere the right-hand rails of the railways are worn out faster than the left-hand ones.

Let us consider a train moving northward along the meridian in the northern hemisphere (see Fig. 123*a*). The velocity v of the motion along the meridian can be resolved into the two components $v_{\parallel}$ and $v_{\perp}$ parallel to the Earth's axis and perpendicular to it, respectively. The direction and the magnitude of the component $v_{\parallel}$ is not affected by the Earth's rotation and consequently this component is not connected with the inertia forces. As to the other component, it undergoes a variation similar to that of the velocity of a body moving along the radius of a rotating disk (see the example in Sec. 48, Fig. 116). Con-

sequently, the train is acted upon by the inertia force of magnitude

$$F_{\text{cor}} = 2m\omega v_{\perp} = 2m\omega v \sin \varphi \tag{49.1}$$

where m is the mass of the train and φ is the latitude. From Fig. 123*b* where the direction of the component $v_{\perp}$ at time $t + dt$ is shown in dotted line it is readily seen that the inertia force is always directed to the right from the direction of the motion of the train. It is therefore quite obvious that the fact

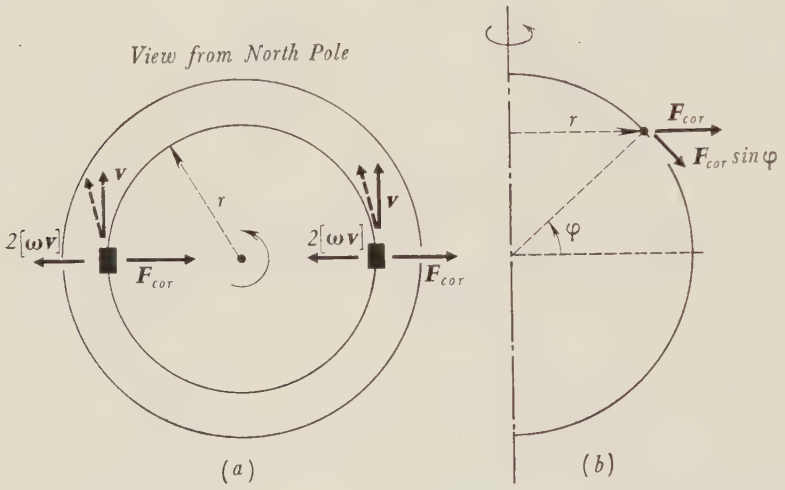


FIG. 124

that the right-hand rail* is worn faster than the left-hand one can only be observed for the double-track railways along whose each track the train always goes in one direction.

The Coriolis inertia force also acts in the case when the train moves not along the meridian but in an arbitrary direction. If, for instance, the train moves along a parallel of latitude (see Fig. 124) it has the Coriolis acceleration of magnitude $2\omega v$ directed to the axis of the Earth's rotation when the train moves eastward and directed from that axis when the train moves westward. Hence, in these cases there exists the inertia force

$$F_{\text{cor}} = 2m\omega v \tag{49.2}$$

directed from the axis of the Earth or to its axis. The projection of this force on the horizontal plane is equal to

$$F_{\text{cor}} \sin \varphi = 2mv\omega \sin \varphi \tag{49.3}$$

Thus, the magnitude of the projection is the same as in the case of the motion along the meridian and it again acts to the right from the direction of motion.

The fact that the right banks of the rivers flowing in the northern hemisphere (and the left banks in the southern hemisphere) are washed away and become

* Accordingly, the left-hand rail in the southern hemisphere.

steeper is explained in the same manner; this effect is independent of the direction of the flow.

Let the reader analyse the following problem: does the Coriolis inertia force appear when a train moves near the equator and does this force produce a non-uniform wear-out of the rails? (Answer: the Coriolis force appears in this case but it does not produce a non-uniform wear-out.)

Let us consider the free fall of a body in the reference frame connected with the Earth. During the fall the body is acted upon by three forces: the force of gravity, the centrifugal inertia force, and the Coriolis inertia force. When the height of the body over the Earth's surface is small (in comparison with the Earth's radius) the magnitudes of the inertia forces are small. The magnitude of the centrifugal acceleration is equal to

$$\omega^2 R \cos \varphi \approx \frac{(2\pi)^2 \times 6400 \times 10^3}{24^2 \times 36^2 \times 10^4} \cos \varphi \text{ m/s}^2 \approx 0.034 \cos \varphi \text{ m/s}^2 \qquad (49.4)$$

where ω is the angular speed of the Earth, R is the radius of the Earth, and φ is the latitude. Near the equator the centrifugal acceleration is about 0.3 % of the acceleration of gravity and therefore the variation of the centrifugal force during the fall can be neglected in approximate calculations. The influence of the Coriolis force is much more significant and in the northern hemisphere it results in an eastward departure of a falling body from the vertical.

This eastward deviation of a falling body can readily be computed approximately if we assume that, in the first approximation, the velocity $\boldsymbol{v}$ of the fall is directed downwards and that its magnitude is equal to gt (where t is the time of the fall) as if the Earth did not rotate. The Coriolis inertia force is equal to $-2m\,[\boldsymbol{\omega v}]$ and, to the approximation assumed, its magnitude is $2m\omega gt \cos \varphi$. Consequently, the magnitude of the eastward acceleration of the falling body is approximately equal to

$$a = 2\omega gt \cos \varphi \qquad (49.5)$$

On integrating twice the expression of the acceleration we find that the eastward deviation of the falling body is approximately equal to *

$$s = \frac{1}{3} \omega g t^3 \cos \varphi$$

In this approximate computation we assume that the Coriolis force is all the time directed eastward and neglect the variation of the direction of the velocity $\boldsymbol{v}$ and, consequently, the variation of the direction of the Coriolis inertia force. The substitution of concrete numerical data into the relation thus obtained shows that in the fall at the latitude of 45° the eastward deviation of the body from the vertical is equal to 3 cm for the time of the fall of 4 s (in this example the body falls approximately from the height of 80 m). Thorough experiments in which the eastward deviations were checked completely confirm the results of the calculations.

The facts we have discussed prove mechanically the Earth's rotation. They show that a reference system connected with the Earth is non-inertial. This means that the system connected with the Earth can be approximately con-

* We have $s = \int_0^t v\,dt$ where $v = \int_0^t a\,dt = 2\omega g \cos \varphi \int_0^t t\,dt = \omega g \cos \varphi \cdot t^2$.

sidered inertial only when the forces acting on the body are much greater than the Coriolis and centrifugal inertia forces.

The centrifugal inertia force has a definite direction and a definite magnitude at a given place irrespective of the motion of the body.

It acts on the body together with the gravitation force $\boldsymbol{P}$ as an equivalent force of weight $\boldsymbol{P}' = \boldsymbol{P} + \boldsymbol{f}$ (Fig. 125*a*). The force $\boldsymbol{P}'$ determines the local

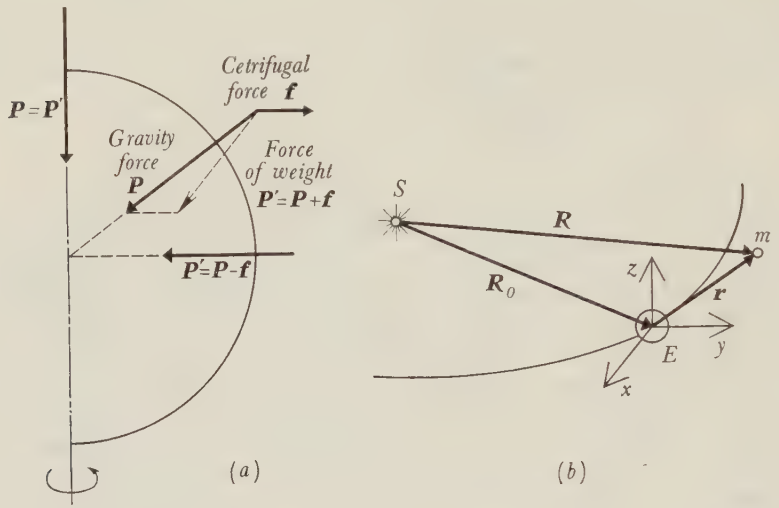


FIG. 125

vertical direction and the force of weight of the body at each point in space surrounding the rotating Earth (see Secs. 14 and 46).

The above discussion concerns only the diurnal rotation of the Earth about its axis. It can easily be shown that the influence of the inertia forces appearing in the annual rotation of the Earth round the Sun is considerably smaller. It is apparent that the magnitude of the Coriolis force in the annual rotation is approximately 360 times as small as the magnitude of the Coriolis force in the diurnal rotation of the Earth. The magnitude of the centrifugal force resulting from the rotation of the Earth round the Sun is of the order of 0.2 that of the centrifugal force of the diurnal rotation on the equator.

When physical bodies move near the Earth's surface the inertia forces acting on them because of the rotation of the Earth round the Sun and the forces of gravitational attraction of the bodies to the Sun practically balance each other and therefore in most cases they can simply be neglected. To illustrate what has been said let us write down the exact equation of motion of a material point of mass m in a spatial region near the Earth. Choosing the centre of mass of the Earth as the origin of a non-inertial reference system (see Fig. 125*b*) we obtain the equation

$$m\ddot{\boldsymbol{r}} = -\gamma \frac{mM_e}{r^3}\,\boldsymbol{r} - \gamma \frac{mM_S}{R^3}\,\boldsymbol{R} - m\boldsymbol{a}_0 + \boldsymbol{F}_{cor} + \boldsymbol{F}_c \qquad (49.6)$$

The terms on the right-hand side of (49.6) are, respectively, the force of gravitational attraction of the material point of mass m to the Earth, the force of gra-

vitational attraction of the point to the Sun, the inertia force appearing because of the Earth's rotation round the Sun in the elliptic orbit, the Coriolis inertia force, and the centrifugal inertia force appearing because of the Earth's diurnal rotation. The quantities $\boldsymbol{r}$, γ, M_e, M_s, $\boldsymbol{R}$ and $\boldsymbol{a}_0$ are, respectively, the radius vector of the material point, the constant of gravitation, the mass of the Earth, the mass of the Sun, the vector connecting the Earth to the Sun, and the acceleration of the Earth in its rotation round the Sun.

The acceleration $\boldsymbol{a}_0 = -\gamma \frac{M_s}{R^3} \boldsymbol{R}$ of the centre of mass of the Earth is due to the gravitational attraction of the Earth to the Sun. The distance $\boldsymbol{R}$ between the Earth and the Sun is approximately equal to $R_0 = 1.5 \times \times 10^8$ km.

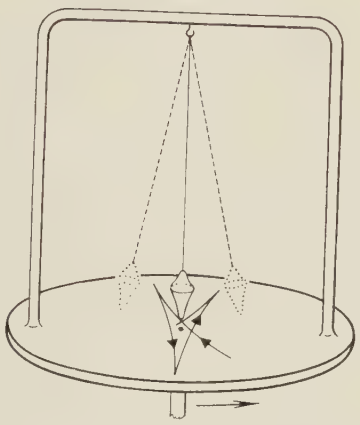


FIG. 126

The numerical comparison of the terms of equation (49.6) which represents the inertia force resulting from the non-uniform orbital motion of the reference system connected with the Earth and the force of gravitational attraction of the material point to the Sun shows that they balance each other to a high accuracy. Therefore their common input in equation (49.6) can be considered zero. Indeed, we have $r/R_0 \sim 10^4/10^8 = 10^{-4}$ and $\boldsymbol{R} = \boldsymbol{R}_0 + \boldsymbol{r} \approx \boldsymbol{R}_0$ whence it follows that

$$-\gamma \frac{mM_s}{R^3} \boldsymbol{R} - m\boldsymbol{a}_0 = -\gamma m M_s (\boldsymbol{R}/R^3 - \boldsymbol{R}_0/R_0^3) \approx 0$$

As was indicated above (see Fig. 125*a*), when introducing the notation $\boldsymbol{P}'$ we mean the sum of the force of attraction exerted on the body by the Earth and the centrifugal force. Equation (49.6) can therefore be written in the form

$$m\ddot{\boldsymbol{r}} = \boldsymbol{P}' + \boldsymbol{F}_{cor} = m\boldsymbol{g}_{eff} - 2m[\boldsymbol{\omega}\boldsymbol{v}_{rel}] \qquad (49.7)$$

where $\boldsymbol{g}_{eff} = \boldsymbol{P}'/m$ is the "effective" acceleration of gravity and $\boldsymbol{v}_{rel}$ is the velocity of the body relative to the Earth. Equation (49.7) describes the motion of a body of mass m in the near space relative to the frame of reference connected with the Earth.

Thus, a reference system connected with the Earth can be regarded as inertial only approximately. The error appearing in this approximation is specified by the ratios of the magnitudes of the inertia forces to the magnitude of the resultant of all the other forces acting on the body.

In 1852 J. B. L. Foucault, a French scientist, proved the Earth's rotation with the aid of his observation of the oscillations of a pendulum. Imagine that a pendulum is placed on the North pole. It is obvious that the plane of the oscillation of the pendulum rotates slowly (relative to the Earth) in the direction opposite to the rotation of the Earth. The rotation of the plane of the oscillation of the pendulum can be demonstrated by an experiment with a pendulum suspended over a rotating disk (Fig. 126). If the pendulum carries

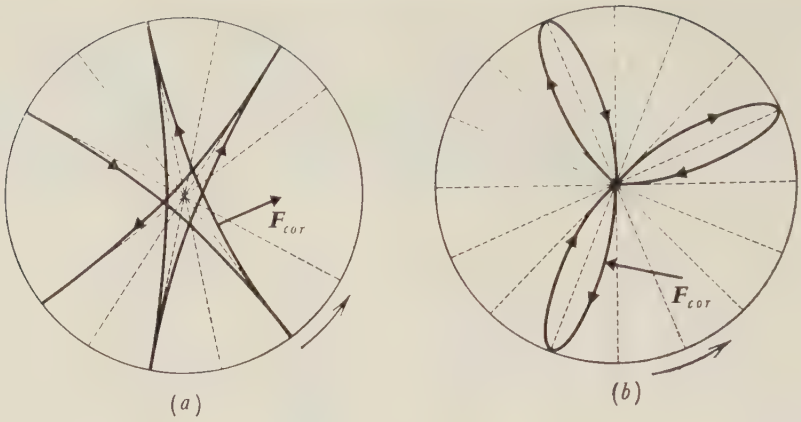


FIG. 127

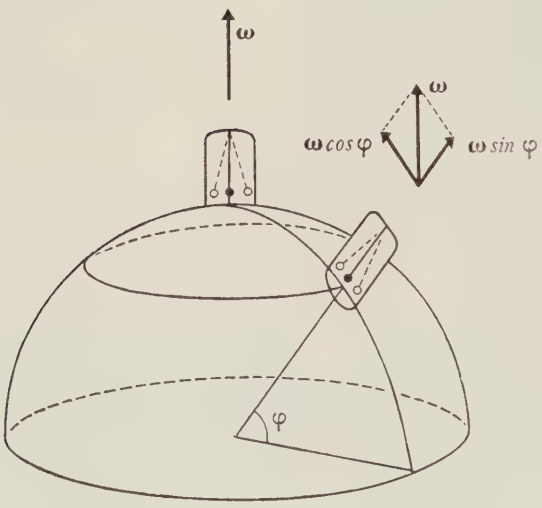


FIG. 128

a funnel from which the sand is pouring out then, after the pendulum has been made to oscillate and the disk has been made to rotate, the trace of the sand shows the rotation of the plane of the oscillation of the pendulum relative to the rotating disk.

In the fixed coordinate system there are no forces making the plane of the oscillation of the pendulum change its position and therefore the plane remains invariable in space while the disk (or the Earth) rotates relative to that plane. It is evident that on the North pole the plane of the oscillation of the pendulum rotates with the angular speed of the Earth (15° per hour). If the oscillation of the pendulum on the North pole is considered relative to a coordinate system connected with the Earth, the rotation of that plane can be accounted for by the action of the Coriolis force. Indeed, this force is perpendicular to the angu-

lar velocity vector and lies in the horizontal plane all the time. The magnitude of the force is proportional to the velocity of motion of the load of the pendulum and to the angular speed of the Earth's rotation, and the direction of the force is such that the trajectory of motion is curved in the corresponding direction.

The shape of the trajectory of the pendulum traced on the Earth (on the disk) depends on the way in which the pendulum is made to oscillate. Let us consider the sand trace on the rotating disk (Fig. 126) for the following two cases. Let us first give the pendulum an initial deviation from the vertical and simultaneously make the disk start to rotate so that when the pendulum begins to oscillate the funnel receives the same velocity as the point of the disk over which the funnel is placed. Then the trajectory will have a "star-like" shape (see Fig. 127*a*). The trajectory of the pendulum on the North pole of the Earth will be of the same shape when the pendulum starts oscillating from a position deviated from the vertical. Now let us consider the case in which the pendulum is made to oscillate when the disk is fixed and starts to rotate after the pendulum has begun to oscillate. Then the trajectory will be "rose-shaped" as shown in Fig. 127*b*. For the pendulum oscillating on the North pole of the Earth a trajectory of the latter type can be observed when the oscillation of the pendulum is produced by a sharp jerk at the moment when the pendulum is in the equilibrium (vertical) position. In both cases the trajectories are curved in the same direction under the action of the Coriolis force.

Thus, when the pendulum oscillates on the North pole the trajectory traced by the pendulum is curved and thus the plane of the oscillation of the pendulum rotates slowly under the action of the Coriolis force

$$\boldsymbol{F}_{\text{cor}} = -2m\,[\boldsymbol{\omega}\boldsymbol{v}] \tag{49.8}$$

which all the time lies in the horizontal plane and is always directed to the right from the direction of the motion of the load.

To demonstrate Foucault's experiment it is necessary to equip the pendulum with a device measuring the rotation of the trajectory. To increase the period of the oscillations of the pendulum its length should be made as long as possible; then the oscillation process will last long until the oscillation is damped (under the action of the friction forces). The experiment shows that the angular speed of the rotation of the plane of the oscillation of the pendulum is equal to

$$\omega \sin \varphi = 15 \sin \varphi \text{ deg/h}$$

where φ is the latitude (Fig. 128). The angular speed of the rotation round the vertical at the latitude φ is equal not to ω but to $\omega \sin \varphi$ which is the projection of the angular velocity $\boldsymbol{\omega}$ on the vertical.

The decrease of the angular speed of rotation at latitude φ (compared to that at the North pole) can also be explained by the fact that the projection of the Coriolis force on the horizontal plane drawn at the given latitude differs from the magnitude of that force on the North pole in the factor $\sin \varphi$ (since the rotation of the plane of the oscillation of the pendulum is only due to that projection). The Coriolis force acting on the load of the pendulum located at a given place with latitude φ lies in the plane perpendicular both to $\boldsymbol{\omega}$ and $\boldsymbol{v}$ and its magnitude is proportional to the sine of the angle between these vectors. The only case when the Coriolis force is directed horizontally is when the vector $\boldsymbol{v}$ lies in the meridional plane; in all the other cases this force does not lie in the horizontal plane.

CHAPTER 7

Motion of a Rigid Body

50. Translatory and Rotational Motions of a Rigid Body

Under certain conditions any physical body undergoes a deformation, that is it changes its shape in some way. By a *rigid body* (an *absolutely rigid body*) is meant a body which can never change its shape. This means that in all the circumstances the distance between its any two points (or, more precisely, between any two particles of the body) remains constant. It is obvious that the notion of a rigid body is an abstraction. However, there are many cases when the shape of a moving body does not change or changes so slightly that the change can be neglected; then it is expedient to state the laws of motion for such a body as if it were perfectly rigid.

Some physical bodies such as pieces of metal and the like consist of very small crystals. The atoms and molecules forming a crystal have a definite regular spatial configuration. In Fig. 129 we see a schematized model of a crystal where the "balls" are the atoms (or molecules) and the "rods" connecting them represent (conditionally) the forces keeping these "balls" in fixed positions

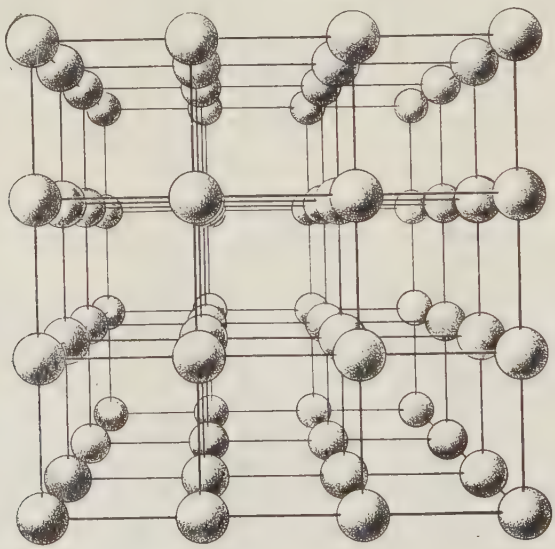


FIG. 129

relative to each other. Any rigid body can be thought of as consisting of crystals representable as certain configurations of such balls connected with weightless hard rods. Every particle of a rigid body can be represented as a set of a sufficiently large number of atoms and molecules constantly connected with each other. When such a body moves its every particle describes a certain trajectory but this trajectory is connected in a certain way with the trajectories of the other particles. Each particle is acted upon by some forces exerted by the neighbouring particles. It is these forces that are symbolized by the "rods" in Fig. 126. We speak of them as *internal forces* in the rigid body. In addition, every particle can also be acted upon by forces exerted by some other bodies. Such forces are referred to as *external forces*. An example of an external force is the force of gravitation with which the Earth acts on the particles of a body.

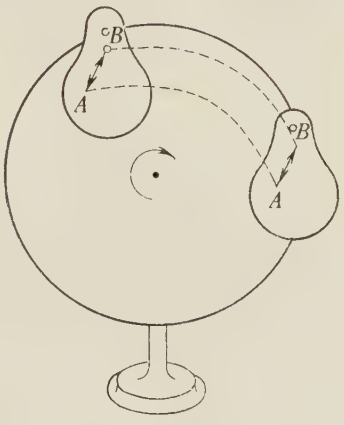


FIG. 130

Every compound motion of a rigid body can be regarded as a combination of *translatory* and *rotational* motions which are the simplest types of a motion. Therefore, we begin our investigation of a motion of a rigid body with considering separately these simplest motions.

A body is said to be in a *translatory* motion if the *line connecting any two particles of the body retains its direction in space*. A translatory motion is not necessarily rectilinear. For instance, in the model shown in Fig. 130 a heavy body is suspended from a rotating disk at a point B about which it can freely rotate. When the disk rotates the body is in a translatory motion. As is seen in the figure, the line AB remains parallel to itself during the motion although every point of the body moves in a circular trajectory.

A rigid body in a translatory motion moves without rotation and the line connecting any two particles of the body remains parallel to itself. Therefore, the displacement of all the points of the body are the same. It follows that in a translatory motion of a body all its points have the same velocity at any time and hence their accelerations are also the same. The simplicity of such a motion is characterized by the fact that to determine the motion completely it suffices to know the law of motion of any chosen point of the body since all the other points move in the same manner. For instance, a motorcyclist moving rectilinearly is in a translatory motion. The wheels of the motorcycle are in a compound motion representable as a combination of translatory and rotational motions while the piston of its engine is in a non-rectilinear translatory motion.

A motion of a rigid body is said to be *plane* if during the motion every point (particle) of the body remains in a fixed plane and all such planes are parallel to each other. In plane motion the trajectory of any point of the body lies in a fixed plane and the planes containing the trajectories either coincide or are parallel to each other. For example, in a rectilinear motion of a motorcycle its wheels and piston are in plane motions. A cylinder rolling on a table without sliding (Fig. 131*a*) is in a plane motion while a rolling cone (Fig. 131*b*) is not.

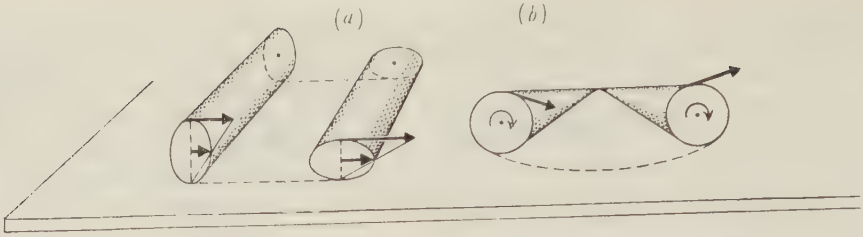


FIG. 131

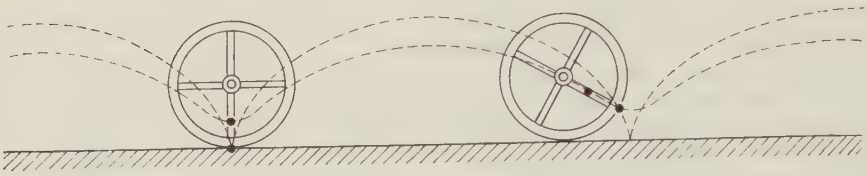


FIG. 132

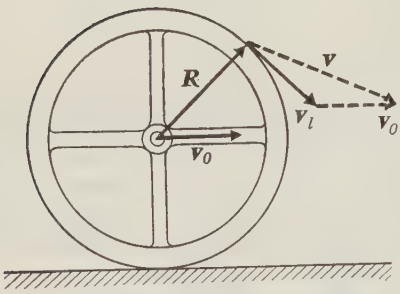


FIG. 133

A motion of a rigid body is said to be *rotational* if the *trajectories* of all the points of the body are *circles* whose centres lie in a common st raight line called the *axis of rotation*. For example, when the engine of a car works while the car does not move the driving shaft of the engine is in a rotational motion. The fixed axis of rotation passes through points which are invariably connected with the rotating body and remain fixed during the motion. The axis of rotation may lie outside the body or pass through it. A rotational motion about a fixed axis is always a plane motion.

The motion of a wheel of a vehicle in a rectilinear motion (of an automobile, of a car, of a train, etc.) is a *compound motion*. It consists of the rotation of the wheel about its axis and of the motion of the wheel and its axis together with the vehicle. In a fixed reference system the trajectories of different points of the wheel are complex curves called cycloids (Fig. 132). In a reference system connected with the vehicle the trajectories of the points of the wheel are circles. The displacement of any point of the wheel is representable as consisting of two parts: one part is determined by the displacement of the axis of the wheel and the other by the rotation of the wheel about the axis. Therefore, the velocity $\boldsymbol{v}$ of any point of the wheel relative to the ground is the sum of the velocity $\boldsymbol{v}_0$ of the axis (see Fig. 133) and the linear velocity $\boldsymbol{v}_l = [\boldsymbol{\omega R}]$ of the rotational motion about the axis.

A rotational motion of a body is determined by the magnitude of the angular velocity. Let us consider a line segment AB passing through two points A

and B of a rotating body and lying in a plane perpendicular to the axis of rotation. Let the segment occupy a definite position AB at time t and some other position $A'B'$ at time $t + dt$, the angle between the latter and the former positions being $d\alpha$. The quantity $\frac{d\alpha}{dt} = \omega$ is called the *angular velocity* (*the angular speed*) of the body. It is evident that the angular velocity is independent of the choice of the points A and B in any plane perpendicular to the axis of rotation; thus the angular velocity specifies the rotational motion of the body as a whole.

As was mentioned (Sec. 47), the angular velocity is represented by a vector directed along the axis of rotation.

51. Equilibrium Conditions for a Rigid Body with a Fixed Axis of Rotation

As was said, the relative positions of the points of a rigid body remain invariable, that is such a body is considered non-deformable. If two forces of equal magnitudes and opposite directions are applied to two points A and B of the body and act along the line AB connecting the points, their resultant force is equal to zero and their action does not affect the motion of the body. This means that any force acting on a rigid body can be transferred to any point along its line of action (see Fig. 134*a*).

Indeed, to any point B, lying on the line of action of a force $\boldsymbol{F}$ applied to a rigid body at a point A, it is allowable to apply two forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ of equal magnitudes and opposite directions. Let one of these forces, say $\boldsymbol{F}_2$, be equal to $\boldsymbol{F}$. The sum of $\boldsymbol{F}_1$ and $\boldsymbol{F}$ is equal to zero: $\boldsymbol{F}_1 + \boldsymbol{F} = 0$. Consequently the body can be thought of as being acted upon only by the force $\boldsymbol{F}_2$ applied at the point B. In other words, the action of the force $\boldsymbol{F}$ can be replaced by that of $\boldsymbol{F}_2$.

The indicated property can be stated mathematically in the following way: the action of a force on a rigid body can be represented by a *sliding* (or *free*) vector whose magnitude and direction remain invariable as it is transferred along its (fixed) line of action. A vector which cannot be transferred along its line and is connected with a definite point of application is said to be *localized* (or *bound* or *fixed*).

As is known, a body which can rotate about an axis is in an equilibrium state only in the case when the sum of the moments of all the forces acting on it is equal to zero.

Let us discuss a simple experiment with equilibrium of a body rotating about its axis under the action of two forces in order to demonstrate the expedience of the introduction of the notion of the moment of a force. Let a shaft carrying a disk be acted upon by two forces $\boldsymbol{f}$ and $\boldsymbol{F}$ (see Fig. 134*b*). The experiment shows that if the friction in the bearings supporting the shaft is negligibly small the equilibrium can only take place when the condition $fR = Fr$ is fulfilled, that is when the moments of the forces are of equal magnitudes and opposite directions.

The moment of a force about an axis is a physical quantity whose numerical value is equal to the product of the magnitude of the force by the arm of the force; the moments of the forces rotating the body in one chosen direction are considered positive and those rotating the body in the other directions are considered negative. The *arm* of a force is the shortest distance between the axis and the line of action of the force.

Consequently, in the study of the equilibrium of a body which can freely rotate about a fixed axis, it is not the balance of the forces but the balance

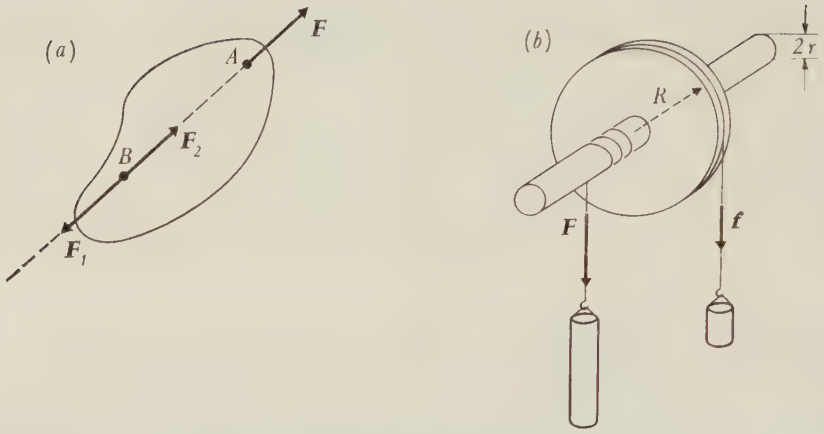


FIG. 134

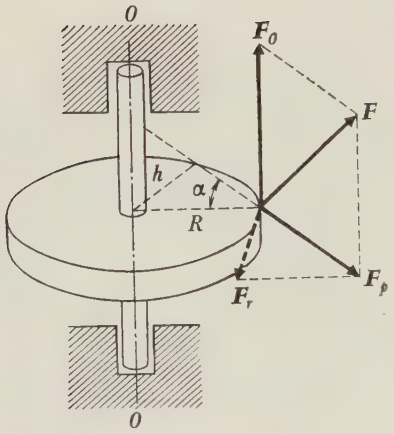


FIG. 135

of the moments of the forces about the axis of rotation that specifies the equilibrium. In a rotational motion of a body the moment of a force plays a role analogous to that of the force in the theory of motion of a material point.

Let us discuss in more detail the notion of the moment of a force about an axis. In the general case when the acting force is directed arbitrarily (see Fig. 135) it can be resolved into two components one of which ($\boldsymbol{F}_p$) is perpendicular to the axis of rotation while the other ($\boldsymbol{F}_0$) is parallel to the axis. The force $\boldsymbol{F}_0$ parallel to the axis of rotation OO' cannot produce a rotation of the body about the axis; its action does not affect the equilibrium of the body rotating about the fixed axis. The force $\boldsymbol{F}_0$ can bend the axis of rotation and deform the body; therefore, since we confine ourselves to the model of a rigid (non-deformable) body the component $\boldsymbol{F}_0$ can be neglected. Hence, we must only take into account the component $\boldsymbol{F}_p$ lying in the plane perpendicular to the axis of rotation. As is readily seen from Fig. 135, the moment of the force $\boldsymbol{F}$ about the axis OO' is equal to the product of the magnitude F_p of the component $\boldsymbol{F}_p$ of the force $\boldsymbol{F}$ by the arm h or, equivalently, it is equal to the product of the magnitude of the rotational component $\boldsymbol{F}_r$ by the distance R from the point of application of the force to the axis.

The rotational component $\boldsymbol{F}_r$ is perpendicular to the plane passing through the line R and the axis. Thus, the magnitude M of the moment of the force $\boldsymbol{F}$ about the axis OO' is expressed by the formula

$$M = F_p h = F_r R \qquad (51.1)$$

Now the equilibrium condition for a rigid body having a fixed axis of rotation can be stated in the following way: *the equilibrium takes place when the sum of the moments of the forces acting on the body about the axis of rotation is equal to zero.*

52. Law of Dynamics for a Body Rotating About a Fixed Axis

When a body rotates about a fixed axis the trajectories of all its points are circles whose centres lie in a common line coinciding with the axis of rotation. All the particles of the body are in plane motion. Generally speaking, the velocities and the accelerations of different particles are different: the greater the distance from the particle to the axis, the greater is its velocity. On the contrary, the angular velocity is the same for all the particles of the body. The angular velocity completely determines the character of the motion of the whole rigid body in its rotation about the fixed axis.

The angular velocity may vary as function of time t (it can increase or decrease) like in the case of a linear velocity, the variation of the angular velocity is characterized by the *angular acceleration.* The angular acceleration is the rate of change of the angular velocity and is therefore equal to the derivative $\frac{d\omega}{dt}$. Since at any given time the angular velocity is the same for all the parts of the rigid body, so is the angular acceleration $\beta = \frac{d\omega}{dt}$.

The linear accelerations of different particles of a rotating body may be different. The vector of linear acceleration of any particle lies in the plane containing the trajectory of the particle. Let us find the relationship between the angular acceleration and the component of the linear acceleration of the particle tangent to the trajectory. In Fig. 136 is shown the trajectory of a particle whose velocity is $v = \omega r$ where r is the distance from the particle to the axis. The magnitude w_t of the tangential component of the acceleration is specified by the variation of the velocity v of the particle along the trajectory; it is given by the formula

$$w_t = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\beta \tag{52.1}$$

because r is constant. Consequently, the magnitude of the tangential component of the acceleration of a particle of the body is equal to the angular acceleration β times the distance r from the particle to the axis of rotation.

Now let us find the relationship between the angular acceleration and the sum of the moments of the forces acting upon a body rotating about a fixed axis.

To find the connection between the angular velocity of the body and the resultant moment of the acting forces we begin with the study of the motion of an arbitrary particle of the body. Let a particle of mass Δm_i be at a distance r_i from the axis (Fig. 137). In the general case this particle is acted upon by external and internal forces. The former are exerted on the particle by some other bodies and the latter by the particles of the body itself. Let us project these forces on the line AB (Fig. 137) perpendicular to r_i and lying in the plane containing the ith particle and perpendicular to the axis of rotation. We can write down the sum of all these projections as

$$(f_i)_{\text{int}} + (f_i)_{\text{ext}}$$

where $(f_i)_{\text{int}}$ is the resultant rotational component of the internal forces and $(f_i)_{\text{ext}}$ is the resultant rotational component of the external forces. (The projec-

tions of the forces rotating the body counterclockwise are considered positive.) Then the equation expressing the second law of dynamics for the ith particle can be written in the form

$$\Delta m_i \frac{dv_i}{dt} = \Delta m_i r_i \frac{d\omega}{dt} = (f_i)_{\text{int}} + (f_i)_{\text{ext}} \tag{52.2}$$

Next we find the moment of the forces, acting on the particle, about the axis

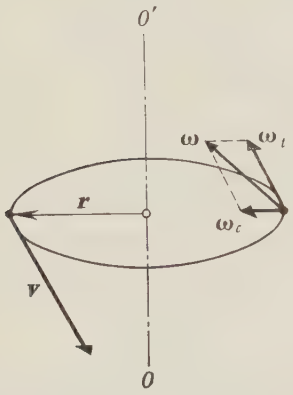


FIG. 136

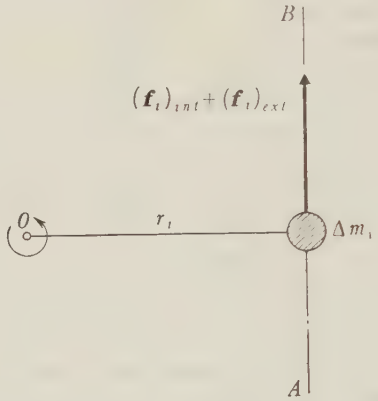


FIG. 137

of rotation. To this end, we multiply (52.2) by r_i and thus obtain

$$\Delta m_i r_i^2 \frac{d\omega}{dt} = r_i (f_i)_{\text{int}} + r_i (f_i)_{\text{ext}} \tag{52.3}$$

Let us write the analogous equalities for the other particles of the given rigid body and add together all of them. This results in

$$\frac{d\omega}{dt} \sum_i \Delta m_i r_i^2 = \sum_i r_i (f_i)_{\text{int}} + \sum_i r_i (f_i)_{\text{ext}} \tag{52.4}$$

It should be taken into account that $\sum_i r_i (f_i)_{\text{int}} = 0$, that is the resultant moment of all the internal forces is equal to zero. For, to each internal force there corresponds another internal force equal to the former in its magnitude and opposite in the direction, the arms of both forces being equal.

The sum $\sum_i r_i (f_i)_{\text{ext}} = M$ is the resultant rotational moment of all the external forces acting on the body. The quantity

$$I = \sum_i \Delta m_i r_i^2 \tag{52.5}$$

is called the *moment of inertia* of the body with respect to the given axis of rotation.

We can also rewrite equation (52.4) in the form

$$M = I \frac{d\omega}{dt} \tag{52.6}$$

Equation (52.6) reads: *the resultant moment of the forces rotating a body about a fixed axis is equal to the moment of inertia of the body with respect to that axis times the angular acceleration of the body.* This is the basic law of dynamics for a rigid body rotating about a fixed axis. It is analogous to the second law of dynamics for the motion of a material point, the only difference being that the force is replaced by the moment of the external forces about the axis, the linear acceleration is replaced by the angular one and the mass is replaced by the moment of inertia with respect to the axis of rotation.

The moment of inertia of a rigid body with respect to an axis is a physical quantity equal to the sum of the products of the masses of the particles forming the given body by the squares of the distances from the particles to the axis. The moment of inertia with respect to an axis depends not only on the mass of the body but also on the distribution of the mass with respect to the axis. If the distances of some particles of the body from the axis are increased the moment of inertia also increases.

The dimension of the moment of inertia is $kg \cdot m^2$ in the SI system and $g \cdot cm^2$ in the cgs system.

In order to find the moment of inertia of a body we must divide the body into sufficiently small parts, determine the distance from each part to the axis, multiply the mass of every part by the square of its distance from the axis and then add up all the results. For instance, consider the body shown in Fig. 138. It consists of eight identical balls placed at the vertices of a cube with edge a. Let us compute the moment of inertia of the body with respect to the axis passing through the centres of the balls *2* and *6*. It is of course supposed that the diameter of the balls is very small relative to the distance a. Denoting by m the mass of one ball we readily see that the sum of the moments of inertia of the four balls *1*, *3*, *5* and *7* is equal to $4ma^2$, the sum of the moments of inertia of the two balls *4* and *8* is also equal to $4ma^2$ while the moment of inertia of each of the balls *2* and *6* is equal to zero. Therefore, the moment of inertia of the whole system is

$$I = 4ma^2 + 4ma^2 = 8ma^2$$

The computation of the moments of inertia of some homogeneous solids will be discussed in Sec. 59.

The role of the moment of inertia in mechanics can be demonstrated by any example involving a body rotating about a fixed axis. As an example of this kind let us consider the system whose two projections are shown in Fig. 139. The system includes a small pulley rotating about a horizontal axis. The rod A is rigidly connected with the pulley; the rod carries two loads of masses m_1 and m_2 attached to it at equal distances from the axis of rotation. From the end of the thread attached to the pulley a load of mass m is suspended. The observation shows that the greater the moment of inertia of the rod with the loads m_1 and m_2, the faster does the load m move downwards. The moment of inertia can be varied by changing the distance from the loads m_1 and m_2 to the axis: as the distance is increased or decreased the moment of inertia also increases or decreases.

Let us discuss in more detail the motion of this system and find the law of motion. After the whole thread is wound off the pulley, the rod A continues to rotate by inertia and the load m starts to move upwards. When the rotation of the rod stops the load m again moves downwards and so on, that is the system is in an oscillatory regime. It is evident that the accelerations of the moving parts of the system (we mean the angular acceleration of the rotating pulley

together with the rod carrying the loads m_1 and m_2 and the linear acceleration of the load m) are constant and retain the same values as the load m goes upwards and downwards. Indeed, this clearly follows from the fact that the real acting force is the force of gravity $P = mg$ which is constant and is always directed downwards.

We now set the equations of dynamics for the rotating rod (with the loads m_1 and m_2) and for the load m. Denoting by $\beta = \frac{d\omega}{dt}$ the angular acceleration

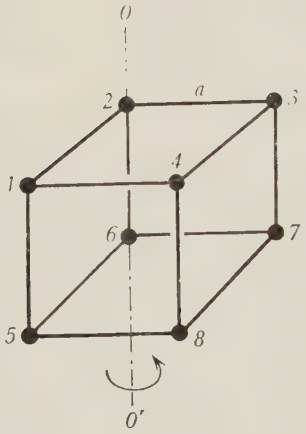


FIG. 138

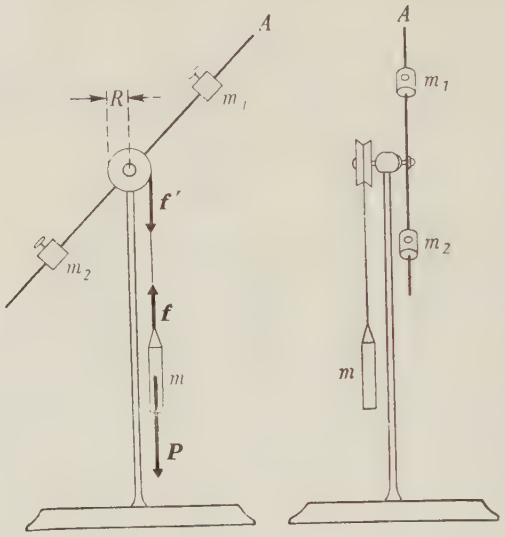


FIG. 139

we can write the equations

$$\left.\begin{aligned} f'R &= I\beta \\ P - f &= ma \end{aligned}\right\} \qquad (52.7)$$

for the rod and for the load m, respectively, where f and f' ($f = f'$) are the magnitudes of the forces with which the thread acts on the pulley and on the load m, I is the moment of inertia of the rod (with the loads), R is the radius of the pulley and a is the linear acceleration of the load m. The angular acceleration of the pulley and the linear acceleration of the load m are connected by the relation

$$a = R\beta$$

Indeed, when the pulley turns through a small angle α the load m receives a vertical displacement of magnitude $\Delta = R\alpha$. Indeed, on differentiating the equality $\Delta = R\alpha$ twice with respect to time we arrive at the indicated relations since $\frac{d^2\Delta}{dt^2} = a$ and $\frac{d^2\Delta}{dt^2} = \beta$.

On solving equations (52.7) we find the angular acceleration β of the pulley and the linear acceleration a of the load m:

$$\beta = \frac{g}{R}\,\frac{1}{1+\frac{I}{mR^2}} \quad \text{and} \quad a = \frac{g}{1+\frac{I}{mR^2}} \qquad (52.8)$$

From (52.8) it is seen that the acceleration a can be very small in comparison with g when the condition $I \gg mR^2$ holds.*

When the distance from the loads m_1 and m_2 to the axis of rotation is varied the moment of inertia I changes and this results in a change of the time during which the load m goes downwards or upwards. The measurement of the time in which the load m passes a definite distance and the computation of the acceleration a on the basis of this measurement make it possible to determine experimentally the moment of inertia I (provided that the mass m and the radius R are known).

53. Angular Momentum

In the analysis of a rotational motion of a body we deal with the moment of the acting force (instead of the force itself) and with the moment of inertia of the body (instead of its mass). Therefore, it is expedient to ask the following question: what is the corresponding analogue of the momentum of a particle? It turns out that the role of such an analogue is played by a new physical quantity called the angular momentum (or the moment of momentum).

The *angular momentum* (the *moment of momentum*) of a particle of a rotating body having a mass Δm_i is defined as the product of the distance r_i from that particle to the axis of rotation by the momentum of the particle. Thus, the numerical value of the angular momentum of the particle Δm_i is equal to

$$\Delta m_i v_i r_i \tag{53.1}$$

where v_i is the velocity of the particle.

By the angular momentum of a rigid body rotating about an axis is meant the sum of the angular momenta of the particles the body consists of: the numerical value N of the angular momentum of the body is given by the formula

$$N = \sum_i \Delta m_i r_i v_i \tag{53.2}$$

The expression for the angular momentum can be rewritten in the form

$$N = \sum_i \Delta m_i r_i r_i \omega = \omega \sum_i \Delta m_i r_i^2 = I\omega \tag{53.3}$$

Thus, the magnitude of the angular momentum of a rigid body rotating about an axis is equal to the product of the moment of inertia of the body with respect to that axis by the angular velocity.

The basic law of dynamics for a body rotating about an axis can now be written as

$$\frac{dN}{dt} = M \tag{53.4}$$

This law reads: *the time derivative of the angular momentum of a body rotating about an axis is equal to the moment of the forces, acting on the body, about that axis.*

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* The laws we have established do not apply to a short period when the load m is near its lowermost point and the thread stops winding off the pulley and starts winding onto it. During this period the load m gains a horizontal displacement and the moment of the forces rotating the pulley is not equal to fR.

If the resultant moment M of the external forces is equal to zero the angular momentum of the body retains a constant value:

$$\frac{dN}{dt} = 0, \quad N = \text{const} \text{ and } I\omega = \text{const} \tag{53.5}$$

Relation (53.5) expresses the *conservation law for the angular momentum.*

When a rigid body rotates about an axis and $M = 0$, that is when the angular momentum of the rotating body is constant, the motion of the body is analogous to an "inertial" motion for which $mv = \text{const}$. However, there is an important distinction between these analogous phenomena: the velocity of a point in an inertial motion remains constant while a rotating body with constant angular momentum N must not necessarily have a constant angular velocity ω because the moment of inertia I of the body can be varied during the motion. If the moment of inertia of a rotating body is changed the angular velocity ω can also change. If this is the case when the moment of the external forces is equal to zero the angular velocity ω varies inversely proportionally to the moment of inertia.

What has been said can be demonstrated by an experiment with a rotating platform suggested by N. E. Zhukovsky. Such a device (spoken of as Zhukovsky's platform) is shown schematically in Fig. 140. It consists of a horizontal disk which can freely rotate (with a negligibly small friction) about the vertical axis. A person with dumb-bells in his hands is sitting on the stool attached to the rotating platform. He stretches out his arms or presses his hands to his chest (see the figure). In the former case the moment of inertia I of the platform with the man attains its maximum value and in the latter case it attains a minimum value. Since the product $I\omega$ must remain invariable (because there are no external forces except the forces of friction whose moments are supposed to be negligibly small) the angular velocity ω takes a value ω_0 in the former case and a different value ω_1 in the latter case, the magnitudes of the angular velocities satisfying the condition $\omega_1 > \omega_0$. This variation of the angular velocity can easily be observed in the experiment.

Let us use the law of conservation of the angular momentum expressed by (53.4) to investigate the rotations of two bodies in the following experiment suggested by Professor S. E. Khaikin. The scheme of the experiment is shown in Fig. 141: the stator of a small electric motor is supported by two ball bearings so that it can rotate about the axis of the rotor of the motor. The electric supply of the winding of the motor is performed with the aid of two slip rings with which the stator is equipped. As is observed in the experiment, the stator and the rotor will simultaneously rotate in opposite directions. If the forces of friction in the bearings supporting the stator are negligibly small the angular momenta of the stator and of the rotor are of equal magnitudes and opposite directions. Indeed, the forces of magnetic interaction between the stator and the rotor and the forces of friction between them are internal forces. Therefore, the sum of the momenta of these forces acting on the stator and on the rotor is equal to zero. The only external forces in this experiment are the forces of friction in the bearings and in the slip rings of the stator. If the resultant moment of these friction forces is very small the total angular momentum of the system "stator and rotor" is equal to zero. This implies that the angular momenta of the stator and of the rotor are of equal magnitudes and opposite directions.

In reality the moment of the external forces of friction is always different from zero. This moment, however small, makes the angular momentum of the stator gradually decrease in comparison with the angular momentum of the

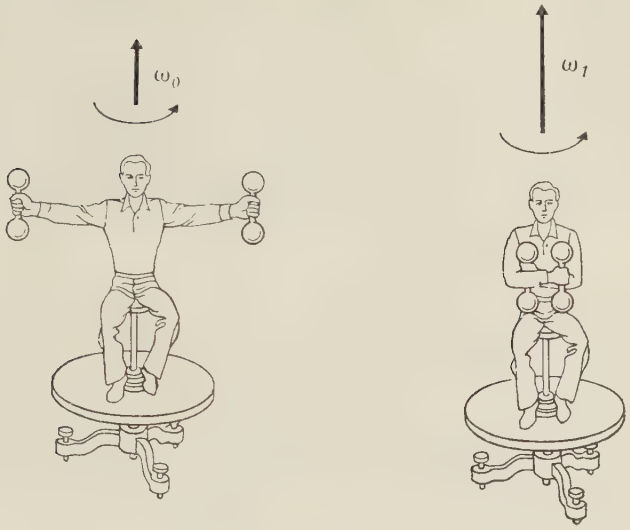

FIG. 140

rotor. The difference between the angular momentum of the stator and that of the rotor can be neglected when the electric motor is started because this difference is usually very small relative to the magnitudes of both momenta. However, this difference should be taken into account when we analyse the deceleration of the rotation of the stator and of the rotor which finally stops the motion some time after the electric current has been switched off. As can be observed, after the electric current has been switched off, the forces of friction first stop the rotation of the stator, then the stator is carried by the rotor and they

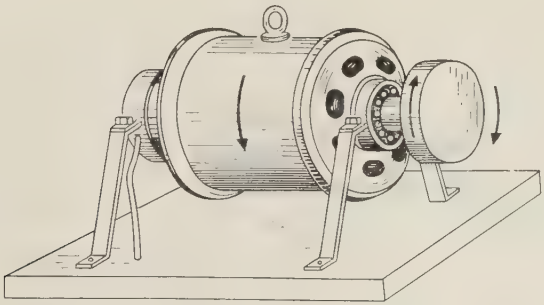

FIG. 141

both rotate in the direction of the rotation of the rotor until the whole system stops moving. It is evident that if there were no moments of the external forces of friction and the angular momenta of the stator and of the rotor were equal then the interaction forces between them (they are internal forces in the system in question) would stop the stator and the rotor simultaneously since the total

angular momentum of the system would be zero at every moment. But, as has been said, after the electric current has been switched off, the angular momentum of the rotor is in fact a little larger than that of the stator and, besides, the moment of the decelerative forces applied to the stator is greater than the moment of the forces acting on the rotor. Consequently, the difference between these moments will increase and the stator will stop earlier than the rotor, after which the friction between the moving rotor and the stator will make the latter rotate together with the former until both bodies stop.

54. Kinetic Energy of a Rotating Body

The kinetic energy of a rotating body can be found as the sum of the kinetic energies of the particles the body consists of. The kinetic energy of a particle of mass Δm_i which is at a distance r_i from the axis of rotation is equal to

$$\frac{\Delta m_i v_i^2}{2} = \omega^2 \frac{\Delta m_i r_i^2}{2} \tag{54.1}$$

because $v_i = \omega r_i$. The kinetic energy of the whole rotating body is therefore expressed as

$$E_k = \frac{\omega^2}{2} \sum_i \Delta m_i r_i^2 = \frac{I\omega^2}{2} \tag{54.2}$$

Formula (54.2) for the kinetic energy of a rotating body is analogous to the formula for the kinetic energy of a translatory motion, the difference being that in the case of rotation the *mass* is replaced by the *moment of inertia* I of the body and the *linear velocity* is replaced by the *angular velocity* ω.

The work of the forces acting on a body rotating about a fixed axis can be found in the ordinary way by computing the scalar products of the acting forces by the corresponding displacement vectors. Alternatively, this work can also be expressed in terms of the moments of the forces. Let $\boldsymbol{F}$ be the resultant of the external forces and let the point of application of this resultant force move in a circle of radius R. Denoting by F_τ the projection of $\boldsymbol{F}$ on the tangent line to the circle and taking into account that during time dt the point of application of the force receives a displacement of magnitude $R\,d\alpha$ where $d\alpha$ is the angle through which the body has turned during that time, we express the elementary work of the external forces during time dt:

$$F_\tau R\,d\alpha$$

The product $F_\tau R$ is equal to the moment M of the resultant force and consequently the elementary work can be written as

$$M\,d\alpha$$

Therefore, the work performed during time t is equal to $\int_0^t M\,d\alpha$. In other words,

the work of the forces acting on a rotating body is equal to the product of the moment of the forces by the angle of rotation when the moment is constant and to the integral of the moment of the forces with respect to the angle of rotation when the moment is variable.

Let us show that if the body is acted upon only by the moment of the forces M the work of the forces is equal to the increment of the kinetic energy. Indeed,

according to the law of dynamics expressed by (52.6), we have

$$M = I \frac{d\omega}{dt} \tag{54.3}$$

where I is the moment of inertia and ω is the angular velocity. On multiplying both members of equality (54.3) by $d\alpha = \omega\, dt$ (this is the angle through which the body turns during time dt) we obtain

$$M\, d\alpha = I \frac{d\omega}{dt} \omega\, dt = Id\left(\frac{\omega^2}{2}\right)$$

whence it follows that the work during time t is equal to

$$\int_0^t M\, d\alpha = I \int_0^t d\left(\frac{\omega^2}{2}\right) = \left(\frac{I\omega^2}{2}\right)_t - \left(\frac{I\omega^2}{2}\right)_0 \tag{54.4}$$

The proportion stated at the beginning of the paragraph has thus been proved.*

To compare the laws of motion for a moving particle and for a rotating rigid body it is expedient to compile the following table of analogous quantities:

Motion of a particle	Rotation of a rigid body about an axis
Mass m	Moment of inertia relative to the axis I
Resultant of external forces $\boldsymbol{F}$	Sum of the moments of the external forces about the axis $\boldsymbol{M}$
Displacement x	Angle of rotation α
Velocity v	Angular velocity ω
Acceleration a	Angular acceleration β
Momentum, $\boldsymbol{K} = m\boldsymbol{v}$	Angular momentum relative to the axis $\boldsymbol{N} = I\omega$
Kinetic energy $mv^2/2$	Kinetic energy $I\omega^2/2$
Work $F\, dx$	Work $M\, d\alpha$
Second law of dynamics	Law of dynamics
$\boldsymbol{F} = m\boldsymbol{a}$ or $\boldsymbol{F} = \frac{d\boldsymbol{K}}{dt}$	$\boldsymbol{M} = I\boldsymbol{\beta}$ or $\boldsymbol{M} = \frac{d\boldsymbol{N}}{dt}$

Let us discuss several experiments in which the kinetic energy of rotating bodies changes. We begin with studying the variation of the kinetic energy in the experiment with Zhukovsky's platform (see Fig. 140) described in the foregoing section. When the moment of inertia of the platform rotating together with the person holding the dumb-bells changes the kinetic energy also changes, namely, it increases when the moment of inertia decreases and vice versa. Indeed, the angular momentum $I\omega$ remains constant and therefore the angular velocity ω increases as I decreases. Consequently the kinetic energy which is equal to $I\omega^2/2$ increases. When the person changes the position of the dumb-bells and places them closer to his body he performs some work to overcome the action of the centrifugal inertia forces. This work is expended on the increase of the kinetic energy. When the dumb-bells are held at a greater distance from the axis of rotation the reverse transformation takes place: the kinetic energy decreases by the amount of work performed by the centrifugal inertia forces as the dumb-bells move away from the axis along the radius.

* Formulas (54.3) and (54.4) hold when I = const.

As an exercise, let the reader confirm by calculation that the work performed by the person on the platform when the dumb-bells are moved towards the axis of rotation is exactly equal to the change of the kinetic energy.

An analogous situation takes place in the experiment with the electric motor whose stator is supported by ball bearings allowing it to rotate (see Fig. 141). When the electric current is switched on and the stator and the rotor begin to rotate and their motion accelerates the kinetic energy of the system increases. If it is permissible to neglect the external friction forces then the body with

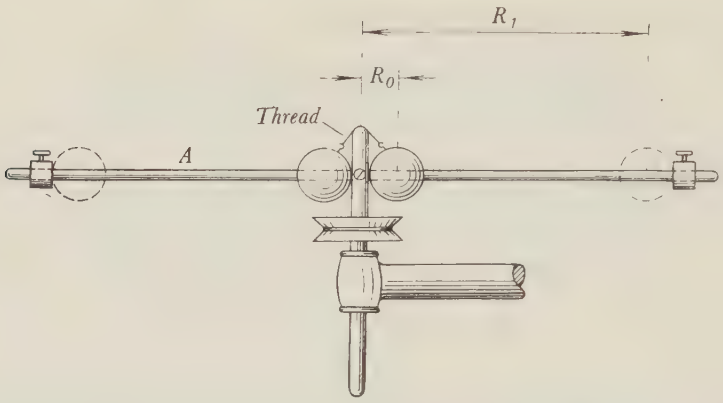

FIG. 142

a greater moment of inertia will at any instant possess a smaller amount of kinetic energy. For definiteness, let the moment of inertia of the stator be four times that of the rotor. Then the angular velocity of the rotor is four times that of the stator since the magnitudes of the angular momenta of the stator and of the rotor are equal. The kinetic energy of a rotating body being proportional to its moment of inertia and to the square of its angular velocity, the kinetic energy of the stator is four times as small as the kinetic energy of the rotor. Thus, when the two bodies (the stator and the rotor) rotate and are accelerated by the internal forces the total kinetic energy gained by the system is distributed so that the body with the smaller moment of inertia receives a greater portion of the energy. The kinetic energy of the system varies as the electromagnetic forces perform work and is distributed between the bodies so that the amount of the energy obtained by each of the two bodies is inversely proportional to its moment of inertia.

Let us also consider the following interesting problem. In Fig. 142 is shown a rod A rotating about the axis perpendicular to it. Two identical balls (shown in dotted line) may slide along the rod. Let the balls be first placed near the axis of rotation and joined together with a thread and let the system (the rod and the balls) be given an initial angular velocity ω_0. After the thread is cut the balls, naturally, slide away from the axis of rotation towards the ends of the rod (the friction forces are supposed to be negligibly small). On the rod there are two stops placed at equal distances from the axis which prevent the balls from leaving the rod. It is required to find the terminal angular velocity of the rotation of the rod on condition that the impact of the balls against the stops is perfectly plastic.

To solve the problem we assume that at the initial instant the balls are at a distance R_0 from the axis of rotation and that at time t they are at a distance R from it. Then the angular velocity ω at time t can be found from the condition of constancy of the angular momentum which is expressed by the equality

$$N = (I_0 + 2mR_0^2)\,\omega_0 = (I_0 + 2mR^2)\,\omega = \text{const} \tag{54.5}$$

where m is the mass of each ball and I_0 is the moment of inertia of the rotating system without the balls ($I_0 = \text{const}$). The kinetic energy of the rotational motion at time t is

$$E = \frac{1}{2}(I_0 + 2mR^2)\,\omega^2 = \frac{1}{2}N\omega \tag{54.6}$$

When the balls stop at a distance R_1 from the axis then, according to equality (54.5), the angular velocity is

$$\omega_1 = \frac{N}{I_0 + 2mR_1^2} \tag{54.7}$$

The angular velocity obviously decreases and, consequently, so does the kinetic energy, the decrease of the kinetic energy being equal to

$$\Delta E = \frac{1}{2}N(\omega_0 - \omega_1) \tag{54.8}$$

Now it is interesting to pose the following question: what is the amount ΔE of the kinetic energy expended on? (The friction forces are assumed to be negligibly small.) To answer this question let us consider the phenomenon in more detail.

First of all, as expression (54.6) shows, the kinetic energy E begins to decrease at the moment when the balls start moving away from the axis of rotation and their distance R from the axis becomes greater than R_0 because the angular momentum is constant while the angular velocity decreases. However, since the friction forces are neglected there must hold the energy conservation law. Therefore, formula (54.6) does not obviously express the total amount of kinetic energy of the moving bodies and represents only a part of that energy. Namely, formula (54.6) does not include the amount of kinetic energy corresponding to the rectilinear motion of the balls along the rod. The decrease of energy expressed by formula (54.8) is in fact equal to that amount.

The moment the balls stop at the ends of the rod the phenomenon of plastic impact takes place and, as usual in such a phenomenon, the whole kinetic energy is transformed into heat. The amount of mechanical energy which goes into heat is given by formula (54.8).

When the impact of the balls against the stops is entirely elastic and the mechanical energy is not transformed into heat the situation is quite different. It is evident that in this case the balls (having a velocity v_0 directed from the axis along the rod before the elastic impact) bounce off the stops after the impact with the same velocity v_0 directed towards the axis of rotation. The centrifugal inertia forces then decelerate the backward motion of the balls and the angular velocity of the whole system again increases. The motion of the balls towards the axis stops when R becomes equal to R_0; at that instant the angular velocity assumes its initial value ω_0. After this instant the balls again move away from the axis and the process goes on periodically. Hence, the balls are in an oscillatory motion along the rod, and the angular velocity also varies periodically. The situation we have described takes place when there are no friction forces in the rotation of the rod and in the sliding of the balls along the rod.

In reality the impact of the balls against the stops is neither perfectly plastic nor entirely elastic and, besides, the friction forces between the balls and the rod, though small, should also be taken into account. Therefore, after the balls bounce off the stops they do not return exactly to their original positions near the axis of rotation and start moving again towards the ends of the rod from a larger distance from the axis of rotation than before. The amplitude of the oscillation of the balls thus decreases and after several impacts the balls stop at the ends of the rod. The transition of the kinetic energy into heat takes place both when the balls strike the stops and when they move along the rod, the total amount of the dissipated mechanical energy being equal to the expression ΔE given by formula (54.8) (provided that the energy losses connected with the friction between the axis of rotation and the bearing supporting it are negligibly small).

55. Centre of Gravity and Centre of Mass of a Rigid Body

In the description of a mechanical motion of a rigid body an important role is played by a point called the *centre of mass* (or the *centre of inertia*) of the body. Practically the centre of mass of a body coincides with its *centre of gravity*, the latter notion being familiar to the reader from the elementary course of physics. Let us discuss how the centre of gravity of a rigid body can be found.

Each of the particles the given body consists of is acted upon by the force of gravity exerted by the Earth. If the dimensions of the body are small in comparison with the radius of the Earth* all the gravitational forces acting on the particles can be considered parallel to each other. Their resultant force is parallel to the constituent forces and the magnitude of the resultant is equal to the sum of the magnitudes of the forces of gravity of all the particles because these parallel forces go in one direction. It turns out that for any position of the body in space this resultant force always passes through a point occupying a fixed position relative to the body. This point is called the *centre of gravity* of the body.

When the centre of gravity of a body is fixed so that the body can freely turn about it the body is in an equilibrium state for all possible positions it can occupy. This implies that *the sum of the moments of the forces of gravity of all the particles of the body about any horizontal axis passing through the centre of gravity is equal to zero.* After any rotation about any axis passing through the centre of gravity the body with the fixed centre of gravity remains in a state of equilibrium because the resultant of the gravity forces passes through that point of support.

Let us take a rectangular coordinate system x', y', z' whose origin coincides with the centre of gravity and denote by x_i', y_i' and z_i' the coordinates of the ith particle having a mass Δm_i (Fig. 143). Let us turn the body in such a way that the plane $x'O'y'$ becomes horizontal. Since the sums of the moments of the forces of gravity (acting on all the particles of the body) about the horizontal axes $O'x'$ and $O'y'$ passing through the centre of gravity are equal to zero the conditions

$$\sum_i \Delta m_i g y_i' = 0 \quad \text{and} \quad \sum_i \Delta m_i g x_i' = 0$$

* This is in fact the only case when the notion of the centre of gravity of the body makes sense.

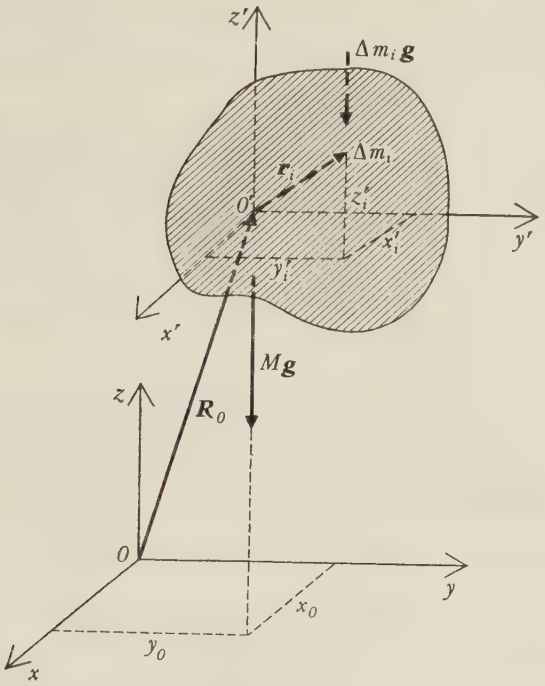

FIG. 143

must hold. After the body is turned through an angle of 90° about the y'-axis the axis $O'z'$ will occupy a horizontal position and the sum of the moments of the gravity forces with respect to that axis must become zero, whence

$$\sum_i \Delta m_i\, g z_i' = 0$$

On cancelling these equalities by the constant g we arrive at the conditions

$$\left.\begin{aligned} \sum_i \Delta m_i\, x_i' &= 0 \\ \sum_i \Delta m_i\, y_i' &= 0 \\ \sum_i \Delta m_i\, z_i' &= 0 \end{aligned}\right\} \qquad (55.1)$$

which imply

$$\sum_i \Delta m_i \boldsymbol{r}_i' = 0 \qquad (55.2)$$

where $\boldsymbol{r}_i'$ is the radius vector issued from the origin O' whose terminus is at the point where the ith particle is placed. It is evident that equalities (55.1) and (55.2) remain valid for any choice of the directions of the axes $O'x'$, $O'y'$, and $O'z'$ relative to the body.

The centre of gravity can be found experimentally as follows. The body is suspended from an arbitrary point. In the state of equilibrium the position occupied by the body must be such that the centre of gravity is on the vertical

line under the point of suspension. This line is marked on the body; as has been said, the centre of gravity is somewhere on that line. Then the body is suspended from another point and again the corresponding vertical line is marked on it. The centre of gravity must obviously be at the point of intersection of the two lines.

Experiments readily show that the centre of gravity is a uniquely determined point: irrespective of the choice of the point of suspension the corresponding vertical lines always pass through the same point which is the centre of gravity.

The coordinates of the centre of gravity of a body with respect to an arbitrary fixed coordinate system can be found provided that the coordinates of the particles forming the body are known. To this end it is convenient to use the condition that the moment of the force of gravity of the whole body about any axis must be equal to the sum of the moments of the forces of gravity (acting on all the particles of the body) about the same axis.

Let us consider a fixed coordinate system $Oxyz$ whose axes Ox and Oy are horizontal (see Fig. 143). Given the coordinates x_i, y_i, and z_i of each of the particle with mass Δm_i, it is required to determine the coordinates x_0, y_0, and z_0 of the centre of gravity in the system $Oxyz$. According to the condition stated above, we can write for the axes Ox, Oy, and Oz the equalities

$$mx_0g = \sum_i \Delta m_i x_i g, \quad my_0g = \sum_i \Delta m_i y_i g, \quad \text{and} \quad mz_0g = \sum_i \Delta m_i z_i g^*,$$

On cancelling them by g and dividing by m (the mass of the whole body) we find the sought-for coordinates of the centre of gravity:

$$x_0 = \frac{\sum_i \Delta m_i x_i}{m}, \quad y_0 = \frac{\sum_i \Delta m_i y_i}{m}, \quad \text{and} \quad z_0 = \frac{\sum_i \Delta m_i z_i}{m} \tag{55.3}$$

Denoting by $\boldsymbol{R}_0$ the radius vector of the point (x_0, y_0, z_0) and by $\boldsymbol{r}_i$ the radius vector of the point (x_i, y_i, z_i) we can rewrite formulas (55.3) as one vector formula:

$$\boldsymbol{R}_0 = \frac{\sum_i \Delta m_i \boldsymbol{r}_i}{m} \tag{55.4}$$

The coordinates of the centre of gravity are usually found theoretically by means of formulas (55.3) (or formula (55.4)). When the mass of the body is distributed continuously these formulas involve the corresponding integrals instead of the sums. Let ρ be the mass density of the body. Then an infinitesimal particle with volume $dx\,dy\,dz$ having coordinates x, y, and z has a mass equal to $\rho\,dx\,dy\,dz$ and the expression of the coordinate x_0 in formulas (55.3) must be replaced by

$$x_0 = \frac{\int\int\int \rho x\,dx\,dy\,dz}{m} \tag{55.5}$$

The formulas for y_0 and z_0 are of analogous structure.

The centre of gravity of an axially symmetric homogeneous body must lie on its axis of symmetry. It should be noted that the centre of gravity of a body may lie outside the body; for instance, this is the case for the centres of gravity of a ring, of a nut, of a washer, and the like. When computing the coordinates

* To establish the last equality we should rotate the coordinate system $Oxyz$ together with the body about the x-axis (or about the y-axis) as was done earlier.

of the centre of gravity of a body we can divide the body into a number of parts, find the coordinates of the centres of gravity of these parts, after which, considering the mass of each part as being concentrated at the corresponding centre of gravity, we can determine the position of the centre of gravity of the whole body.

The notion of the centre of gravity understood as the point of application of the resultant of the forces of gravity makes sense only for the bodies whose dimensions are not very large or, more precisely, are small relative to the

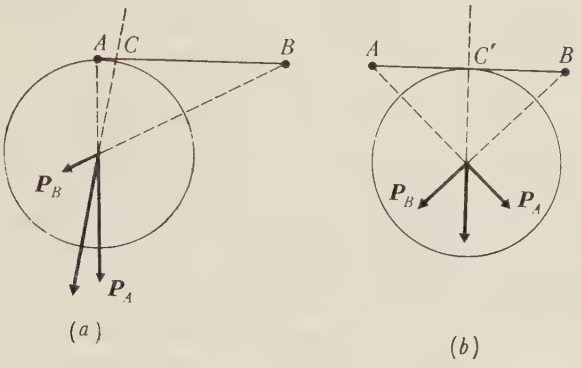

FIG. 144

Earth's radius. Only in this case is it allowable to regard the gravity forces (acting on the particles of the body) as parallel to each other. When the indicated condition is not fulfilled there is no (fixed) point invariably connected with the body through which the resultant of the gravitational forces always passes. In order to illustrate what has been said we shall discuss the following example.

Consider a body consisting of two identical material points A and B placed at the ends of an imaginable rigid weightless rod AB of length $2R$ where R is the radius of the Earth (see Fig. 144). Let us determine the positions of the lines of action of the resultant of the gravity forces for the two different cases shown in Fig. 144*a* and *b*. In the former case this line passes through a point C of the rod lying near the point A. Since the distance from the point B to the centre of the Earth is $\sqrt{5}$ times the distance from the point A to the centre of the Earth (this follows from Pythagoras' theorem) and since, as is known from Newton's law of universal gravitation (see Chapter 9), the force of gravitational attraction is inversely proportional to the square of the distance, we have $CB = 5\,AC$, which determines the position of the point C on the axis of the rod. In the latter case (Fig. 144*b*) it is evident that the line of action of the resultant gravity force passes through the midpoint C' of the rod. It is clear that varying the position of the rod in all possible ways we obtain different points on the axis of the rod through which the line of action of the resultant gravity force passes.

As will be seen later, there are problems in which it is necessary to determine the centre of gravity of a system of several bodies which are not rigidly connected with each other. In such a case the centre of gravity of the system is again determined by formulas (55.3) in which by m is meant the sum of the masses of all the bodies forming the system in question. As these bodies move the centre of

gravity may change its position both in space and relative to the bodies themselves.

Now let us consider the case when all the particles forming a body are acted upon by some parallel *mass forces* (this means that the magnitudes of the forces are proportional to the masses of the particles). As in the case of the parallel gravity forces, the line of action of the resultant of all these mass forces always passes through the same point invariably connected with the body irrespective of the position of the body in space and of the direction of the forces. This point is called the *centre of mass* or the *centre of inertia* of the body. It is evident that the coordinates of this point are determined by formulas (55.3) or (55.4) and that the centre of gravity and the centre of mass of a body coincide. That is why we usually do not distinguish between them.

56. The Law of Motion of the Centre of Mass of a Body

In this section we shall establish the law governing the motion of the centre of mass of a body. To begin with, let us discuss the following example. Suppose that we consider the motion of a box (see Fig. 145) which can slide on a smooth floor. If we pull the rope attached to the angle of the box the latter will be in a compound motion resulting from a translatory motion and a rotation and will possess both an angular velocity and an angular acceleration. However, it turns out that the following remarkable property always holds: under the action of a force $\boldsymbol{F}$ applied to a body, irrespective of the character of the motion produced by the force, *the centre of mass of the body moves with an acceleration* $\boldsymbol{a}$ *whose direction coincides with that of the force* $\boldsymbol{F}$ *and whose magnitude is equal to* F/m *where* m *is the mass of the body.*

In other words, the *centre of mass of a rigid body always moves as if all the external forces were applied at the centre and the mass of the whole body were concentrated at the centre.*

We now proceed to the proof of this proposition. As was found, the position of the centre of mass (coinciding with the centre of gravity) is determined by formulas (55.3) (or, which is the same, by formula (55.4)). Therefore, if $\boldsymbol{r}'_i$ denotes the radius vector of the ith particle drawn from the centre of mass, condition

$$\sum_i \Delta m_i \boldsymbol{r}'_i = 0 \tag{56.1}$$

must be fulfilled where Δm_i is the mass of the ith particle.

The centre of mass is defined in the same way in the more general case when a given body (or a system of bodies) undergoes deformations. The only difference is that the centre of mass of a rigid body occupies an invariable position relative to the body while in the case of a deformable body the centre of mass may move in a certain way relative to the particles the body consists of. However, if the positions of the particles of the body are known for any instant the location of the centre of mass can be found with the aid of the same formulas (56.1) or (55.4). Therefore, it is advisable to establish the law of motion of the centre of gravity for the most general case when the particles of the body can move relative to each other.

The vector sum of the momenta of all the particles of a body is called the momentum of that body. We will show that the momentum of any body is completely specified by the mass of the whole body and the velocity of its centre of mass. Let us denote by Δm_i the mass of the ith particle of the body and by $\boldsymbol{v}_i$ its velocity relative to a fixed coordinate system. We will also consider

a (moving) coordinate system with origin at the centre of mass of the body whose axes always retain their directions in space, that is always remain parallel to certain definite directions. Denoting by $\boldsymbol{v}_0$ the velocity of the centre of mass relative to the fixed coordinate system and by $\boldsymbol{v}_i'$ the velocity of the ith particle in the moving coordinate system we can write

$$\boldsymbol{v}_i = \boldsymbol{v}_0 + \boldsymbol{v}_i'$$

Thus, the velocity of the ith particle relative to the fixed reference system is equal to the sum of the velocity $\boldsymbol{v}_0$ of the centre of mass and the velocity $\boldsymbol{v}_i'$ of

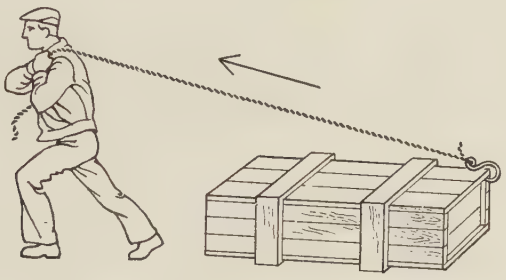

FIG. 145

the particle relative to the moving reference system. The momentum of the ith particle is obviously equal to

$$\Delta \boldsymbol{k}_i = \Delta m_i \boldsymbol{v}_i = \Delta m_i \boldsymbol{v}_0 + \Delta m_i \boldsymbol{v}_i'$$

The sum of the momenta of all the particles of the body, that is the momentum of the whole body, can therefore be written as

$$\boldsymbol{K} = \sum_i \Delta \boldsymbol{k}_i = \sum_i \Delta m_i \boldsymbol{v}_0 + \sum_i \Delta m_i \boldsymbol{v}_i' \tag{56.2}$$

The first sum on the right-hand side can be rewritten as

$$\sum_i \Delta m_i \boldsymbol{v}_0 = \boldsymbol{v}_0 \sum_i \Delta m_i = m\boldsymbol{v}_0$$

(where m is the mass of the whole body) and for the second sum we have

$$\sum_i \Delta m_i \boldsymbol{v}_i' = 0$$

The last relation follows from the condition that the origin of the moving coordinate system coincides with the centre of mass. Indeed, let $\boldsymbol{r}_i'$ be the radius vector of the ith particle in the moving reference system; then $\frac{d\boldsymbol{r}_i'}{dt} = \boldsymbol{v}_i'$, and equation (56.1) which holds for any time t implies that $\sum_i \Delta m_i \boldsymbol{v}_i' = 0$. Therefore, expression (56.2) can be brought to the form

$$\boldsymbol{K} = m\boldsymbol{v}_0 \tag{56.3}$$

Thus, *the momentum* $\boldsymbol{K}$ *of the whole body is equal to the product of its mass* m *by the velocity* $\boldsymbol{v}_0$ *of its centre of mass.* The total momentum $\boldsymbol{K}$ is independent of

the motions of the particles of the body relative to the moving reference system which is in a translatory motion together with the centre of mass of the body. In its motion the rigid body can only rotate relative to that moving reference system. Consequently, the momentum of a rigid body does not depend on the rotation of the body about any axis passing through its centre of mass.

We now proceed to the dependence of the momentum of a body on the forces acting on it.

Let us apply the second law of dynamics to the motion of each particle. Let the ith particle be under the action of the force

$$(\boldsymbol{f}_i)_{\text{int}} + (\boldsymbol{f}_i)_{\text{ext}}$$

where $(\boldsymbol{f}_i)_{\text{int}}$ is the resultant force exerted on the ith particle by the other particles of the same body (this is an internal force) and $(\boldsymbol{f}_i)_{\text{ext}}$ is the resultant of the forces exerted by the other bodies (this is an external force). The time derivative of the momentum of each of the particles is equal to the corresponding force, that is

$$\frac{d\,\Delta \boldsymbol{k}_i}{dt} = (\boldsymbol{f}_i)_{\text{int}} + (\boldsymbol{f}_i)_{\text{ext}} \tag{56.4}$$

On writing down these equations for all the particles and summing them together we obtain

$$\sum_i \frac{d\,\Delta \boldsymbol{k}_i}{dt} = \sum_i (\boldsymbol{f}_i)_{\text{int}} + \sum_i (\boldsymbol{f}_i)_{\text{ext}} \tag{56.5}$$

For the first sum on the right-hand side we have $\sum_i (\boldsymbol{f}_i)_{\text{int}} = 0$ because $(\boldsymbol{f}_i)_{\text{int}}$ are internal forces (each of these forces has a reaction force among the internal forces which has the same magnitude and opposite direction). For the second sum on the right-hand side of (56.5) we have $\sum_i (\boldsymbol{f}_i)_{\text{ext}} = \boldsymbol{F}$ where $\boldsymbol{F}$ is the resultant of all the external forces acting on the body. Therefore, equality (56.5) can be rewritten in the form

$$\frac{d}{dt} \sum_i \Delta \boldsymbol{k}_i = \frac{d\boldsymbol{K}}{dt} = \boldsymbol{F} \tag{56.6}$$

Taking into account relation (56.3) expressing the momentum of the body we can write the last equality as

$$m \frac{d\boldsymbol{v}_0}{dt} = \boldsymbol{F} \tag{56.7}$$

where $\frac{d\boldsymbol{v}_0}{dt}$ is the acceleration of the centre of mass of the body.

Consequently, *for any body, the time derivative of its momentum is equal to the sum of the external forces acting on the body* (formula (56.6)). Since the momentum of the whole body is equal to the product of its mass by the velocity of the centre of mass, it follows that *the acceleration of the centre of mass is equal to the ratio of the sum of all the external forces to the mass of the whole body* (formula (56.7)). This means that the acceleration of the centre of mass is independent of the positions of the points of application of the external forces acting on the body and is dependent solely on the magnitudes of the forces and their directions. Suppose that we have struck a match-box lying on the table so that the line of action of the force passes through the centre of mass. The box will then

be in a translatory motion and its momentum will be

$$m\boldsymbol{v}_0 = \int \boldsymbol{F}\, dt \tag{56.8}$$

Now imagine that we strike the box at its angle and that the force of action has the same magnitude. In this case the line of action of the force does not pass through the centre of mass and the box will be in a compound motion involving a rotation. However, if the acting force $\boldsymbol{F}$ is directed as in the former case and acts during the same period the momentum of the box will again be equal to $m\boldsymbol{v}_0$ and will be expressed by formula (56.8)*. If the magnitude and the direction of the acting force remain invariable the resulting motion of the centre of mass of the body does not depend on the point of application of the force: the centre will always move in the direction of the force and the momentum of the body will be expressed by formula (56.8). In this sense the second law of dynamics is applicable to any body; that is why the notion of the centre of mass plays such an important role in dynamics.

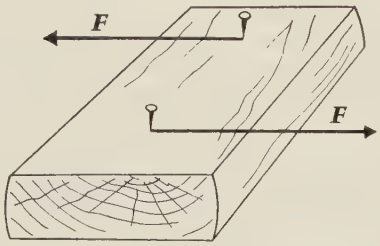

FIG. 146

It is expedient to make the following remarks concerning the basic notions introduced in this section.

(1) The system of the external forces can be replaced by their sum (by the resultant external force), that is by one force under which the motion of the body remains the same as under the action of the given forces, only when the lines of action of all the forces $(\boldsymbol{f}_i)_{\text{ext}}$ meet at one point or when the forces are parallel. At the same time, in all cases the resultant force completely specifies the motion of the centre of mass of the body.

(2) Neither the resultant force nor the momentum of the body gives any information about the rotation of the body. At the same time, the law of motion of the centre of mass we have established implies the following important conclusion: if the resultant of the external forces is equal to zero the momentum of the body does not vary during the time of motion or, in other words, the velocity of the centre of mass remains constant under the action of a system of external forces whose resultant is zero. When a body is first at rest relative to an inertial reference system and then starts moving under the action of a system of external forces with zero resultant force the centre of mass of the body remains at rest during the time of motion.

The simplest example of a system of forces with zero resultant is a couple of forces. *A couple of forces is a system of two parallel forces with equal magnitudes, opposite directions and different lines of action* (Fig. 146). The action of a couple does not produce any displacement of the centre of mass.

(3) The law of motion of the centre of mass describing the variation of the momentum and expressed by equations (56.6) and (56.7) which we have established for a single body remains valid for an arbitrary system of bodies. This general statement is proved in an analogous way. To this end, every body form-

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* Let the reader answer the following question: will the work performed by the force F be the same in both cases?

ing the system is divided into particles and the position of the centre of mass of the system is found for any time with the aid of formula (55.2) or (55.4). The mass of the system is understood as the sum of the masses of the bodies forming the system. By external forces are meant those forces which are exerted by the bodies not included in the system. The forces of interaction between the particles of one body or of different bodies forming the system are of course understood as internal forces, their sum being always equal to zero. The law expressed by formula (56.6) or (56.7) shows that when the resultant of the external forces is equal to zero the bodies included into the mechanical system in question can only move in such a way that the total momentum of the system remains invariable and the centre of mass is at rest or in a uniform rectilinear motion.

The resultant of the external forces applied to a mechanical system can be equal to zero in the following two cases: either the system of the external forces reduces to a couple of forces or the action of the external forces does not produce any change in the mechanical state of the system. In the latter case we say that the system in question is isolated or closed. In such a system only the action of the internal forces should be taken into account and these forces cannot change the momentum of the system.

The law of inertia stated earlier for a material point (particle) can now be generalized to any system of material bodies (particles): *the momentum of an isolated mechanical system remains invariable and its centre of mass is either at rest or in a uniform rectilinear motion.* This is the most complete and precise formulation of the law of conservation of momentum (expressing the inertia law) which holds for any isolated system of material bodies. Thus, the inertia law holds both for a single isolated particle and for any isolated system of particles.

The law describing the variation of the momentum of a body or of a system of bodies written in form (56.6) and (56.7) gives some information about the motion of the system under the action of the given forces, namely, it specifies the translatory part of the motion. But it gives no information about the motion of the particles of two systems. Therefore, equation (56.6) or (56.7) does not itself provide a complete description of the process of motion within the system. However, there are many important cases when a detailed description of the motion of the particles forming the system relative to the centre of mass is unnecessary and when it is quite sufficient to know the translatory part of the motion. Then the whole system can be replaced by a point (placed at the centre of mass of the system) at which the whole mass of the system is concentrated and to which the resultant of all the external forces acting on the system is applied. In such cases it is sufficient to use only equation (56.7) to obtain the necessary information about the motion.

57. Plane Motion of a Body

The investigation of the motion of a body when the point of application of the force applied to it is in an arbitrary position is a rather complicated task. We shall therefore begin with a plane motion of a body when all particles of the body move in planes parallel to a given fixed plane. For example, this is the case when a box is sliding on a smooth horizontal ice surface, a cylinder is rolling on a plane, and the like.

Let us first discuss the results of some experiments. Suppose that, as shown in Fig. 147, a smooth board can only move on the smooth surface of the table. If we drive a nail into the board, attach a thread to it, and pull the thread the

board starts to move in such a way that the acceleration of the centre of mass of the board coincides in its direction with the acting force. If the motion starts from a state of rest and the thread has an invariable direction the velocity of the centre of mass must be in the direction of the thread. However, besides that motion, the board may rotate about a vertical axis. Changing the position of the nail in the board we observe that the rotation becomes slower when the point of application of the force is closer to the centre of mass. When the nail is exactly at the centre of mass, the board is in a translatory motion without

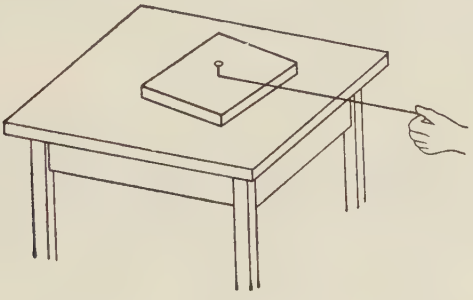

FIG. 147

rotation. In this case the moment of the external force about the vertical axis passing through the centre of mass is equal to zero. We can therefore draw the conclusion that the rotation of the board depends on the moment of the external force about that vertical axis.

We now proceed to the following question: what is the relationship between the rotation and the moment of the force? It turns out that, as in the case of a body rotating about a fixed axis passing through its centre of mass, *the moment of the force is equal to the moment of inertia of the body times the angular acceleration.*

Let us prove this assertion. To facilitate the computations we state a more general definition of the moment of a force about an axis. Let $\boldsymbol{R}$ be the vector joining a point O on an axis with the point of application of a force $\boldsymbol{F}$ acting in a plane perpendicular to that axis and passing through the point O (see Fig. 148*a* where the axis is meant to be perpendicular to the plane of the figure). By the moment of the force $\boldsymbol{F}$ about the axis we mean the vector product of the vectors $\boldsymbol{R}$ and $\boldsymbol{F}$:

$$\boldsymbol{M} = [\boldsymbol{RF}] \qquad (57.1)$$

According to the definition of the vector product, the magnitude of the moment is

$$M = FR \sin \alpha = Fh$$

where $h = R \sin \alpha$ is the arm of the force; this coincides with the former definition of the magnitude of the moment of a force. Formula (57.1) automatically indicates the sign and the direction of the moment which is a vector perpendicular to the plane of the motion.

We now consider a plane motion of a body with respect to a fixed reference system Oxy (see Fig. 148*b*) not connected with the moving body. Let us take a sufficiently small particle of the body with mass Δm_i which can be regarded

as a material point; the position of the particle is specified by the radius vector $\boldsymbol{R}_i$ drawn from the origin O. Suppose that the ith particle is acted upon by an external force $(\boldsymbol{f}_i)_{\text{ext}}$ and by internal force $(\boldsymbol{f}_i)_{\text{int}}$. Then the law of motion can be written for that particle in the form

$$\Delta m_i \frac{d\boldsymbol{v}_i}{dt} = (\boldsymbol{f}_i)_{\text{ext}} + (\boldsymbol{f}_i)_{\text{int}}. \tag{57.2}$$

where $\boldsymbol{v}_i = \frac{d\boldsymbol{R}_i}{dt}$ is the velocity of the particle relative to the fixed frame of

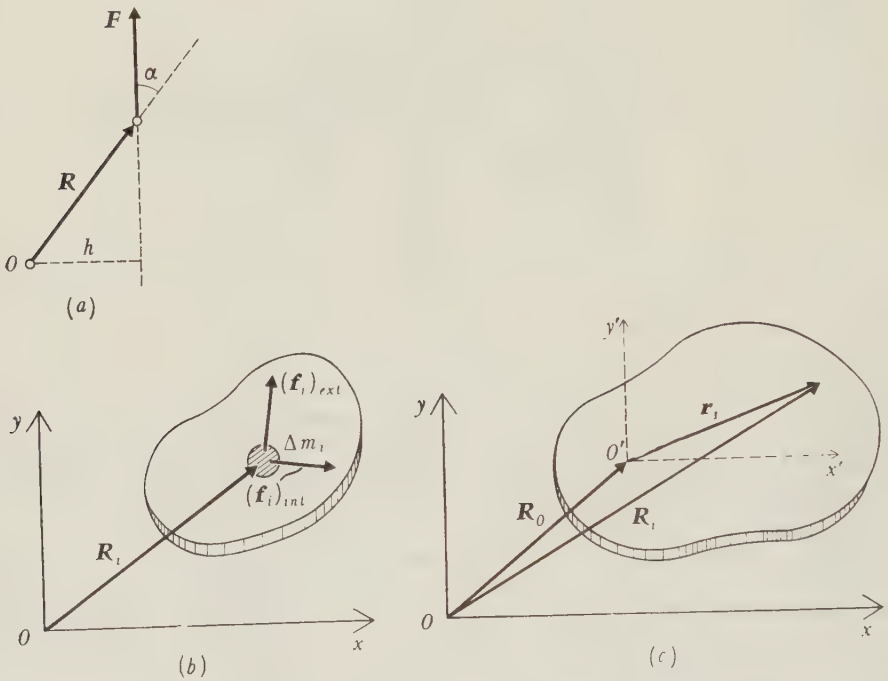

FIG. 148

reference Oxy. On performing the vector multiplication of both members of equation (57.2) by $\boldsymbol{R}_i$ we obtain

$$\Delta m_i \left[\boldsymbol{R}_i \frac{d\boldsymbol{v}_i}{dt}\right] = [\boldsymbol{R}_i (\boldsymbol{f}_i)_{\text{ext}}] + [\boldsymbol{R}_i (\boldsymbol{f}_i)_{\text{int}}] \tag{57.3}$$

The term involving the time derivative can be rewritten as $\frac{d}{dt}[\boldsymbol{R}_i \Delta m_i \boldsymbol{v}_i]$ because

$$\boldsymbol{v}_i = \frac{d\boldsymbol{R}_i}{dt}$$

and therefore

$$\left[\frac{d\boldsymbol{R}_i}{dt} \Delta m_i \boldsymbol{v}_i\right] \equiv 0$$

On adding together equations of type (57.3) written for all particles of the body we obtain

$$\frac{d}{dt}\sum_i [R_i\,\Delta m_i v_i] = \sum_i [R_i\,(f_i)_{\text{ext}}] + \sum_i [R_i\,(f_i)_{\text{int}}] \tag{57.4}$$

The first sum on the right-hand side is equal to zero since for every internal force there is a reaction force among the system of the internal forces which has the same magnitude and opposite direction and is applied to some other particle of the body. The sum of the moments of these two forces is equal to zero since their directions are opposite while the arms are equal.

The quantity $[R_i \Delta m_i v_i]$ is the *angular momentum* of the ith particle and the sum

$$N = \sum_i [R_i\,\Delta m_i v_i]$$

is the *angular momentum* (the *moment of momentum*) *of the whole body* (with respect to the axis passing through the point O). Now we can write the general law for a plane motion of a body regarded as a rotation of the body about a fixed axis passing through O:

$$\frac{dN}{dt} = M \tag{57.5}$$

where

$$M = \sum_i [R_i (f_i)_{\text{ext}}].$$

Formula (57.5) reads: *the time derivative of the angular momentum is equal to the sum of the moments of the external forces.* Equation (57.5) is similar to formula (53.4) written for the case of a rotation of a body about an axis rigidly connected with the body. The distinction between the formulas is that in formula (57.5) the velocities v_i of the particles are not necessarily perpendicular to the vectors R_i and that the magnitudes and the directions of the vectors R_i may vary in the course of the motion.

Equation (57.5) can be given a clearer physical interpretation if we take into account the law of motion of the centre of mass (56.7). Let

$$R_i = R_0 + r_i$$

where R_0 is the radius vector of the centre of mass O' and r_i is the radius vector of the ith particle in the coordinate system $O'x'y'$ which does not rotate and is connected with the centre of mass of the body (Fig. 148*c*). Then the velocity of the ith particle can be represented in the form

$$v_i = v_0 + u_i$$

where

$$v_0 = \frac{dR_0}{dt} \quad \text{and} \quad u_i = \frac{dr_i}{dt}$$

The radius vector of each particle can be written as the sum of the radius vector of the centre of mass and the radius vector of that point in the reference system connected with the centre of mass. The same relations hold for the velocities. On substituting the indicated expressions in formula (57.4) we arrive at

$$\frac{d}{dt}\sum_i [(R_0 + r_i)\,\Delta m_i\,(v_0 + u_i)] = \sum_i [(R_0 + r_i)\,(f_i)_{\text{ext}}]$$

that is

$$\frac{d}{dt}\left\{[\boldsymbol{R}_0\boldsymbol{v}_0]\sum_i \Delta m_i + \left[\boldsymbol{R}_0 \sum_i \Delta m_i \boldsymbol{u}_i\right] + \left[\sum_i \boldsymbol{r}_i \Delta m_i \boldsymbol{v}_0\right] + \sum_i [\boldsymbol{r}_i \Delta m_i \boldsymbol{u}_i]\right\} =$$

$$= \left[\boldsymbol{R}_0 \sum_i (\boldsymbol{f}_i)_{\text{ext}}\right] + \sum_i [\boldsymbol{r}_i (\boldsymbol{f}_i)_{\text{ext}}] \qquad (57.6)$$

According to the definition of the centre of mass the second and the third sums in the curly brackets on the left-hand side of (57.6) are equal to zero. Indeed, we have the equality

$$\sum_i \Delta m_i \boldsymbol{r}_i = 0$$

The differentiation of this relation leads to

$$\sum_i \Delta m_i \frac{d\boldsymbol{r}_i}{dt} = \sum_i \Delta m_i \boldsymbol{u}_i = 0$$

Consequently,

$$\frac{d}{dt}\left\{[\boldsymbol{R}_0 m \boldsymbol{v}_0] + \sum_i [\boldsymbol{r}_i \Delta m_i \boldsymbol{u}_i]\right\} = [\boldsymbol{R}_0 \boldsymbol{F}] + \sum_i [\boldsymbol{r}_i (\boldsymbol{f}_i)_{\text{ext}}] \qquad (57.7)$$

where $m = \sum_i \Delta m_i$ is the mass of the whole body and $\boldsymbol{F} = \sum_i (\boldsymbol{f}_i)_{\text{ext}}$ is the sum of all external forces acting on the body. The first terms on the right-hand and left-hand sides of (57.7) are equal to each other because

$$\frac{d}{dt}[\boldsymbol{R}_0 m \boldsymbol{v}_0] = \left[\frac{d\boldsymbol{R}_0}{dt} m \boldsymbol{v}_0\right] + \left[\boldsymbol{R}_0 m \frac{d\boldsymbol{v}_0}{dt}\right] = [\boldsymbol{R}_0 \boldsymbol{F}]$$

since $\frac{d\boldsymbol{R}_0}{dt} = \boldsymbol{v}_0$ and, according to (56.7), $m \frac{d\boldsymbol{v}_0}{dt} = \boldsymbol{F}$.

We thus arrive at a remarkable relation which can be written as

$$\frac{d\boldsymbol{N}_0}{dt} = \boldsymbol{M}_0 \qquad (57.8)$$

where $\boldsymbol{N}_0 = \sum_i [\boldsymbol{r}_i \Delta m_i \boldsymbol{u}_i]$ is the angular momentum with respect to the axis passing (at any given time t) through the centre of mass O' and $\boldsymbol{M}_0 = \sum_i [\boldsymbol{r}_i (\boldsymbol{f}_i)_{\text{ext}}]$ is the moment of the external forces with respect to that axis.

Formula (57.8) indicates that for any compound plane motion of a body *the time derivative of the angular momentum with respect to the axis passing through the centre of mass of the body is equal to the resultant moment of the external forces with respect to the same axis.* The rotational part of the motion is quite similar to a rotation about a fixed axis rigidly connected with the body and occupying an invariable position in space (see 53.4).

Since the centre of mass of a rigid body is in a fixed position relative to the body, the velocity $\boldsymbol{u}_i$ can be represented in the form $\boldsymbol{u}_i = [\boldsymbol{\omega} \boldsymbol{r}_i]$ where the direction of the vector $\boldsymbol{\omega}$ is always perpendicular to the plane of motion. Therefore,

the expression of the angular momentum can be transformed as follows:

$$\sum_i [\boldsymbol{r}_i \Delta m_i \boldsymbol{u}_i] = \sum_i [\boldsymbol{r}_i \Delta m_i [\boldsymbol{\omega} \boldsymbol{r}_i]] = \boldsymbol{\omega} \sum_i \Delta m_i \boldsymbol{r}_i^2 *$$

The quantity $I_0 = \sum_i \Delta m_i r_i^2$ is nothing but the moment of inertia of the body with respect to the axis passing through the centre of mass. Hence, as in the case of a rotation with respect to a fixed axis, equation (57.8) assumes the form

$$I_0 \frac{d\boldsymbol{\omega}}{dt} = \boldsymbol{M}_0 \tag{57.9}$$

The vectors $\boldsymbol{\omega}$, $\frac{d\boldsymbol{\omega}}{dt}$, and $\boldsymbol{M}_0$ are always perpendicular to the plane of motion.

When the resultant of the external forces meets the axis passing through the centre of mass we have $\boldsymbol{M}_0 = 0$ and $\frac{d\boldsymbol{\omega}}{dt} = 0$. This means that in case the line of action of the resultant external force intersects the axis passing through the centre of mass the angular acceleration is constant. If at an initial instant the body is at rest, that is $\boldsymbol{\omega}_0 = 0$ where $\boldsymbol{\omega}_0$ is the initial angular velocity, then $\boldsymbol{\omega}$ must all the time be equal to zero. In other words, in this case the body is in a translatory motion. If the initial angular velocity $\boldsymbol{\omega}_0$ is different from zero then, under the same conditions, the body continues to rotate with the constant angular velocity $\boldsymbol{\omega} = \boldsymbol{\omega}_0$ since $\frac{d\boldsymbol{\omega}}{dt} = 0$.

Let us discuss a special case when a plane motion is produced by a couple of forces. Since the parallel forces forming a couple are of equal magnitudes and opposite directions the resultant force of a couple is always equal to zero. In the case under consideration the centre of mass must therefore be at rest or in a uniform rectilinear motion. If the centre of mass is at rest before the couple has been applied to the body it remains at rest under the action of the couple. Thus, *a body acted upon by a couple of forces is in a rotational motion about the axis passing through its centre of mass irrespective of the positions of the points of application of the forces forming the couple, the rotation being completely specified by the direction and the moment of the couple.*

The moment of a couple of forces about any axis has the same magnitude equal to the product of the magnitude F of the forces forming the couple by the distance H between the lines of action of the forces (see Fig. 149; the distance H is the *arm of the couple*). Indeed, the sums of the moments of the forces about the axes passing through two arbitrary points O_1 and O_2 are

$$M = F(H + h_1) - Fh_1 = FH \quad \text{and} \quad M = Fh_2 + Fh_2' = FH$$

respectively.

Let us sum up the results we have established. A plane motion of a body under the action of external forces can be represented as a combination of a translatory motion and a rotational motion in which the body takes part simultaneously. The acceleration of every particle of the body is equal to the sum of the accelerations of the translatory and of the rotational motion of the particle about the axis passing through the centre of mass of the body. The acceleration of the

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* In these calculations we use the formula $[\boldsymbol{a}[\boldsymbol{bc}]] = \boldsymbol{b}(\boldsymbol{ac}) - \boldsymbol{c}(\boldsymbol{ab})$ known from vector algebra.

translatory motion is the same for all particles and is given by the formula

$$a = \frac{F}{m} \tag{57.10}$$

where $F = \sum_i (f_i)_{\text{ext}}$ is the resultant of all external forces and m is the mass of the body. The direction of the acceleration a coincides with that of the resultant force F. The magnitude of the angular acceleration of the rotational motion

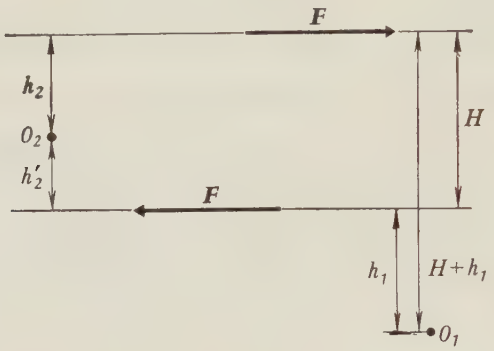

FIG. 149

about the axis passing through the centre of mass is expressed by the formula

$$\frac{d\omega}{dt} = \frac{M}{I} \tag{57.11}$$

where M is the magnitude of the resultant moment of all the external forces about the axis passing through the centre of mass and I is the moment of inertia of the body with respect to that axis.

58. Rolling Motion of a Cylinder on a Plane. Maxwell's Pendulum

As an example of a plane motion, we consider a rolling motion of a cylinder (or a wheel) on a plane. Such a motion is shown schematically in Fig. 150.

The axis of the cylinder moves with a velocity v_0 while the cylinder itself rotates with an angular velocity ω. If the cylinder rolls on the plane without sliding then v_0 and ω are connected by the relation

$$v_0 = \omega R_0 \tag{58.1}$$

where R_0 is the radius of the cylinder. This obviously follows from the fact that the displacement $v_0\,dt$ of the axis during time interval from t to $t + dt$ is equal to the length $\omega R_0\,dt$ of the corresponding circular arc of radius R_0 travelled by the point of contact of the cylinder and the plane during the same time.

When $v_0 > \omega R_0$ or $v_0 < \omega R_0$ the point of contact slides and the cylinder rolls with sliding. In the first case the velocity of the point of contact is directed forwards and in the second case backwards.

In both cases the velocity of a point of the cylinder is given by the formula

$$v_0 + [\omega r] \tag{58.2}$$

where r is the radius vector joining the axis of the cylinder to that point (r is perpendicular to the axis of the cylinder). The motion of the cylinder is the combination of the translatory motion with velocity v_0 and the rotational motion with angular velocity ω.

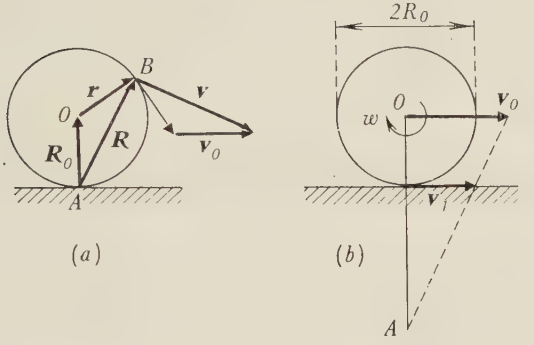

FIG. 150

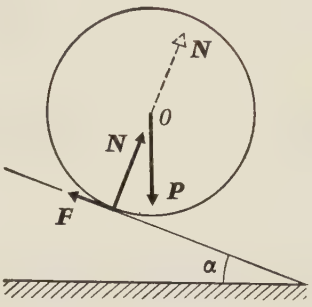

FIG. 151

An arbitrary plane motion of a body can be represented as a (pure) rotation about an *instantaneous axis*. By that axis is meant a straight line passing through those points fixed relative to the body which are at rest at the given instant. In the case of a cylinder rolling without sliding on a plane the instantaneous axis passes through the point of contact of the cylinder and the plane. This point of the cylinder has a zero velocity at any given instant. The velocities of all the other points of the body can be found if we take into account that during time dt the cylinder turns about the instantaneous axis (passing through the point A in Fig. 150*a*) with angular velocity ω. Let $R = R_0 + r$ be the radius vector of a point B (see Fig. 150*a*) issued from the instantaneous axis perpendicularly to it and let R_0 be the radius vector drawn to the axis of the cylinder from the point A. The velocity of the point B can be written as

$$v = [\omega R]$$

that is

$$v = [\omega (R_0 + r)] = v_0 + [\omega r] \tag{58.3}$$

where $v_0 = [\omega R_0]$ is the velocity of the axis of the cylinder (coinciding with the velocity of the translatory motion). It is obvious that the instantaneous axis moves with velocity v_0 in the plane and that at different time moments it passes through different points of the cylinder. The axis of rotation passing through the centre of the cylinder also moves in space but retains a fixed position relative to the particles of the cylinder.

When the cylinder rolls with sliding the instantaneous axis of rotation no longer passes through the point of contact. In the case $v_0 > \omega R_0$ (see Fig. 150*b*) the instantaneous axis passes through a point A lying below the plane and the

cylinder slides with velocity v_1.* In the case $v_0 < \omega R_0$ the instantaneous axis passes above the plane. In all the cases of the rolling motion of the cylinder the instantaneous axis moves in a line parallel to the plane with velocity v_0 (with which the geometrical axis of the cylinder moves).

Now let us discuss the dynamical aspect of a rolling motion of a cylinder on a plane. Let a cylinder of radius R_0 roll down an inclined plane forming an angle α with the horizontal plane (Fig. 151). The cylinder is acted upon by three forces: the gravity force P, the force of normal reaction of the plane N and the friction force F between the cylinder and the plane. We begin with determining the acceleration of the translatory part of the motion. Since the cylinder does not leave the plane in its motion, the component of the acceleration of the centre of mass of cylinder normal to that plane is equal to zero, that is

$$P \cos \alpha - N = 0 \tag{58.4}$$

The magnitude of the component of the resultant of all the external forces in the direction parallel to the inclined plane is equal to

$$f = P \sin \alpha - F \tag{58.5}$$

The latter force determines the acceleration of the centre of mass, that is the acceleration of the translatory motion along the inclined plane. The magnitude of the acceleration therefore is given by the expression

$$a = \frac{f}{m} = \frac{1}{m}(P \sin \alpha - F) \tag{58.6}$$

where m is the mass of the cylinder. The angular acceleration is determined by formula (57.11) and is completely specified by the moment of the friction force F (which is the only external force not passing through the centre of mass) and by the moment of inertia I of the cylinder with respect to its central axis. Thus, we have

$$\frac{d\omega}{dt} = \frac{M}{I} = \frac{FR_0}{I} \tag{58.7}$$

where R_0 is the radius of the cylinder.

Equations (58.6) and (58.7) are always valid irrespective of whether the cylinder rolls on the plane with sliding or without it. However, these equations do not enable us to find the three unknown quantities F, a, and $\frac{d\omega}{dt}$ and it is therefore required to establish an additional relation. When the cylinder rolls down the inclined plane without sliding the angular and the linear accelerations are connected by the relation

$$a = R_0 \frac{d\omega}{dt} \tag{58.8}$$

following from formula (58.1). On solving the three simultaneous equations (58.6), (58.7), and (58.8) we find the acceleration a. The last two equations give us

$$F = \frac{aI}{R_0^2} \tag{58.9}$$

* If we imagine a weightless disk of a sufficiently large radius attached to the rolling cylinder then the instantaneous axis passes through the point of the disk lying below the plane on a circle of radius OA with centre at O.

and the substitution of (58.9) in the first equation results in

$$ma = P \sin \alpha - \frac{Ia}{R_0^2}$$

whence

$$a = \frac{mg \sin \alpha}{m + \frac{I}{R_0^2}} = \frac{g \sin \alpha}{1 + \frac{I}{mR_0^2}} \tag{58.10}$$

where g is the acceleration of gravity.

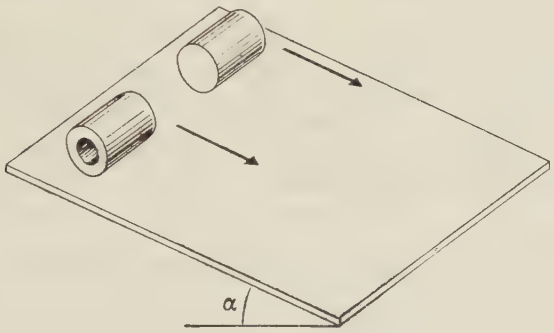

FIG. 152

Consequently, the acceleration a depends solely on the angle of inclination of the plane and on th ratio I/mR_0^2. The greater this ratio, the smaller the acceleration. The acceleration a cannot be smaller than $1/2\ g \sin \alpha$ because we have

$$I < mR_0^2$$

for any cylinder. Indeed, only in the case of an (imaginable) hollow cylinder of infinitesimal width and radius R_0 whose all mass is concentrated on its surface we have $I = mR_0^2$; for a real very thin hollow cylinder we have

$$I \approx mR_0^2 \quad (I < mR_0^2)$$

We see that in reality $I < mR_0^2$. The ratio I/mR_0^2 is independent of the mass of the cylinder and is specified only by the distribution of the mass over the volume of the cylinder.

For a homogeneous cylinder we have $I = 1/2mR_0^2$. Hence, the acceleration which such a cylinder has when it rolls without sliding down an inclined plane depends solely on the angle of inclination of the plane and is equal to $2/3g \sin \alpha$.

Suppose we have two homogeneous cylinders made of one material and having equal radii, one of them being solid and the other hollow (see Fig. 152). Which of them rolls down faster? It is obvious that the ratio I/mR_0^2 is greater for the hollow cylinder since the particles it consists of are at greater distances from its axis (see also formulas (59.4) and (59.5)). Therefore, the acceleration of the solid cylinder is greater and the cylinder rolls down faster.

From equations (58.9) and (58.10) we readily find the friction force:

$$F = \frac{mg \sin \alpha}{1 + \frac{mR_0^2}{I}} \tag{58.11}$$

The friction force depends on three parameters, namely the angle of inclination α, the gravity force $P = mg$ and the ratio mR_0^2/I.

As the angle of inclination α increases, the acceleration of the axis of the rolling cylinder grows proportionally to $\sin \alpha$. It is always less than the acceleration $g \sin \alpha$ with which a body moves sliding down the same inclined plane without friction (and without rotation) but it is greater than half that acceleration. The acceleration of the cylinder rolling down the plane is less than the acceleration of a body with the same mass sliding down the plane and this phenomenon can be interpreted (formally) as an increase of the "inertia" of the cylinder because of the rotation of the former. This increase of the "inertia" depends on the ratio I/mR_0^2.

All that has been said concerns a rolling motion of a cylinder without sliding. A rolling motion with sliding can be investigated in like manner. This can easily be done under the assumption that, as in the case of friction of rest, the force of rolling friction has a maximum value μN where μ is a constant proportionality coefficient and N is the force of normal pressure exerted by the cylinder on the plane. Then the cylinder rolls without sliding when

$$F < \mu N \tag{58.12}$$

where F is the friction force whence, by (58.4) and (58.11), we obtain

$$\tan \alpha < \mu \left(1 + \frac{mR_0^2}{I}\right) \tag{58.13}$$

Thus, for a cylinder to roll without sliding the angle of inclination must not exceed a definite value. A cylinder rolls with sliding only when the tangent of the angle of inclination of the plane is greater than

$$\mu \left(1 + \frac{mR_0^2}{I}\right) \tag{58.14}$$

In this case the magnitude of the acceleration of the axis of the cylinder is equal to

$$a = \frac{1}{m}(P \sin \alpha - \mu N) = g(\sin \alpha - \mu \cos \alpha) \tag{58.15}$$

Maxwell's Pendulum. Let us consider a small disk with a shaft to which two threads of equal lengths are attached (see Fig. 153). Such a device is known as Maxwell's pendulum. Imagine that the pendulum is suspended from the threads and that the threads are wound onto the shaft after which the pendulum moves downwards under the action of its weight. During the first part of the motion the threads wind off the shaft until the pendulum achieves its lowermost position (specified by the length of the threads). During the second part of the motion the disk rotates in the same direction and the threads again wind onto the shaft. This results in the upward motion of the pendulum. In the second part the rotation of the disk decelerates until the disk attains its uppermost position. Then the disk again moves downwards and so on. Because of the air friction the amplitude of the oscillation of the pendulum decreases and in some time the pendulum stops oscillating. However, since the friction forces are small the oscillation of the pendulum during several first upward and downward cycles of the motion can approximately be considered as a periodic process.

Let us set the dynamical equation of motion of the disk. Denoting by P the force of gravity, by f the magnitude of the tensile force in one thread, by R the radius of the shaft we can write the equation of the translatory motion of the disk in the form

$$P - 2f = ma \tag{58.16}$$

where a is the acceleration of the centre of masses. The equation of the rotation motion is

$$2fR = I\frac{d\omega}{dt} \tag{58.17}$$

where ω is the angular velocity of the disk and I is the moment of inertia of the pendulum. The velocity v of the centre of mass, that is the velocity of the axis of the shaft, and the angular velocity ω of the rotation of the disk are con-

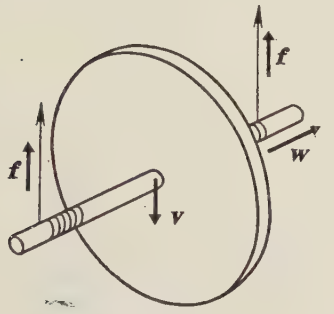

FIG. 153

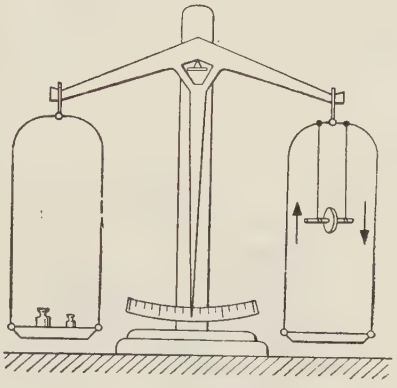

FIG. 154

nected by the relation $v = \omega R$. Consequently, the linear acceleration a and the angular acceleration $\beta = \frac{d\omega}{dt}$ are connected by the formula

$$a = R\frac{d\omega}{dt} \tag{58.18}$$

On solving the system of simultaneous equations (58.15), (58.16), and (58.17) in three unknowns we find the acceleration of the centre of mass

$$a = \frac{P}{m + \frac{I}{R^2}} \tag{58.19}$$

and the force of tension of the thread

$$f = \frac{P}{2}\frac{I}{I + mR^2} \tag{58.20}$$

It should be noted that the acceleration and the tensile force are independent of whether the disk moves upwards or downwards. In the oscillation process the velocity changes sign while the acceleration does not since the forces do not change their signs.

The fact that the force of tension of the threads remains constant in the upward and downward cycles of the motion can readily be confirmed experimentally. To this end we can attach the upper ends of the threads to a beam balance (see Fig. 154). On releasing the pans of the balance as the disk moves upwards or downwards we readily see that the "weight" of the pendulum is independent of the direction of motion. It is apparent that that "weight" is equal to $2f$ where f is the tension of each thread.

The pans must not be released when the pendulum is in its lowermost position because the change of the direction of motion is accompanied with impact which may destroy the balance.

59. Moments of Inertia of Some Bodies. Huygens-Steiner Parallel Axes Theorem

(1) *Moments of inertia of a homogeneous cylinder with respect to the geometrical axis.* A hollow or a solid cylinder can be divided (mentally) into (infinitely)

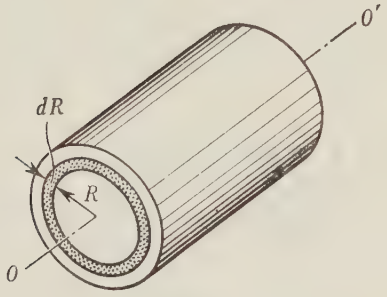

FIG. 155

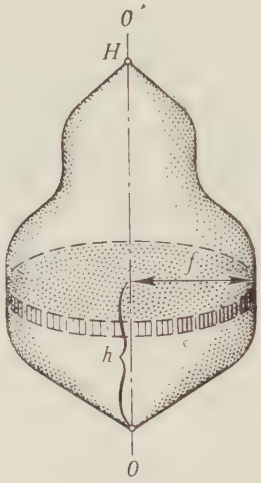

FIG. 156

thin cylindrical layers whose particles are at the same distance from the axis. Let a solid homogeneous cylinder of radius R_0 be divided into concentric layers of width dR as shown in Fig. 155. The mass of the particles forming a layer of radius R is equal to

$$dm = 2\pi Rh\rho\, dR \tag{59.1}$$

where h is the altitude of the cylinder and ρ is the mass density of the material of the cylinder. Since all the particles of the layer are at the same distance R from the axis the moment of inertia of the layer is equal to

$$dI = dm\, R^2 = 2\pi\rho h R^3\, dR \tag{59.2}$$

The moment of inertia of the whole cylinder can be found by adding together these infinitesimal moments of inertia, that is the total moment of inertia I is given by the formula

$$I = \int dI = \int R^2\, dm = 2\pi\rho h \int_0^{R_0} R^3\, dR = 2\pi\rho h \frac{R_0^4}{4} \tag{59.3}$$

Taking into account that the mass of the cylinder is $m = \pi R_0^2 h\, \rho$ we can rewrite (59.3) as

$$I = \frac{1}{2} m R_0^2 \tag{59.4}$$

Thus, the moment of inertia of a homogeneous solid cylinder is equal to its mass times half the square of its radius.

The moment of inertia of a hollow cylinder with inner radius R_1 and outer radius R_0 can easily be found with the aid of a formula similar to (59.3) in which the integration should be performed from the lower limit equal to R_1 to the upper limit equal to R_0:

$$I = 2\pi\rho h \int_{R_1}^{R_0} R^3\, dR = 2\pi\rho h \left(\frac{R_0^4}{4} - \frac{R_1^4}{4}\right) \tag{59.5}$$

Noting that the mass of such a cylinder is $m = \pi\rho\,(R_0^2 - R_1^2)\,h$ we can write (59.5) in the form

$$I = \frac{1}{2}\, m\,(R_0^2 + R_1^2) \tag{59.6}$$

In the same manner we can easily find the moment of inertia of any body representable as a collection of hollow cylinders, rings, disks, and the like.

(2) *Moment of inertia of a solid of revolution.* By a solid of revolution is meant a body whose surface is formed by the rotation of a plane curve about an axis lying in the plane of that curve. This curve is called the generatrix of the surface of revolution. Suppose that a curve $f(h)$, lying in one plane with an axis OO' (Fig. 156), whose end points are on the axis is rotated about that axis to form the surface of a homogeneous body. Given the density ρ of the material of the body and the function $f(h)$, it is required to compute the moment of inertia of the body with respect to the axis OO'. To this end we divide the body into infinitely thin disks of altitude dh. The moment of inertia of such a disk is equal to

$$dI = \frac{1}{2}\, dm\, f^2 = \frac{1}{2}\,\pi\rho f^4\, dh \tag{59.7}$$

Consequently, the moment of inertia of the whole solid of revolution is

$$I = \frac{1}{2}\int dm\, f^2 = \frac{\pi}{2}\,\rho \int_0^H f^4\, dh \tag{59.8}$$

The function $f(h)$ being known, we can readily compute the moment of inertia of the solid of revolution by formula (59.8).

Moments of inertia of a cone and of a ball. For a cone of altitude H the function $f(h)$ can be written as $f = (R_0/H)\,h$ (see Fig. 157*a*). On substituting this expression in formula (59.8) we find the moment of inertia of the cone:

$$I = \frac{\pi}{2}\,\rho \left(\frac{R_0}{H}\right)^4 \int_0^H h^4\, dh = \frac{\pi}{2}\,\rho \left(\frac{R_0}{H}\right)^4 \frac{H^5}{5} \tag{59.9}$$

As is known, the volume of such a cone is equal to $\frac{\pi R_0^2 H}{3}$, and formula (59.9) can therefore be rewritten as

$$I = \frac{3}{10}\, mR_0^2 \tag{59.10}$$

where m is the mass of the cone.

The moment of inertia of a ball can be found in like manner. To simplify the calculations, let us first compute the moment of inertia of the upper half of the ball cut from the whole ball by the plane passing through its centre perpendicularly to the axis of revolution (Fig. 157*b*). We have $f^2 + h^2 = R_0^2$ whence

$$f^2 = R_0^2 - h^2 \tag{59.11}$$

On substituting this expression in formula (59.8) and performing the integration from 0 to R_0 we obtain the moment of inertia of the half of the ball. Let

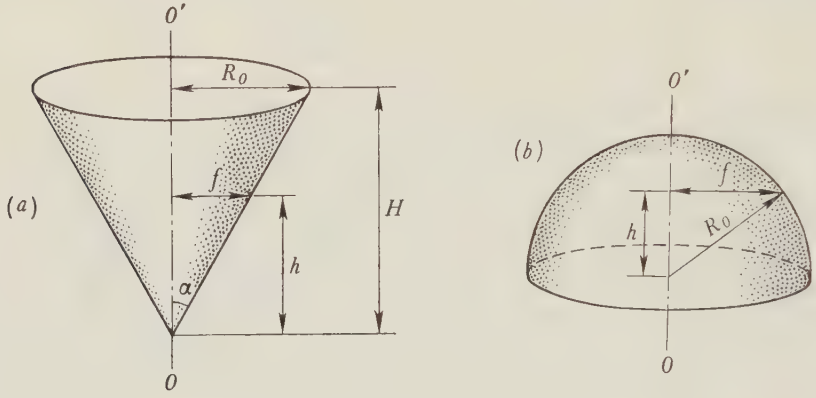

FIG. 157

the reader complete the calculations and compute the moment of inertia of the whole ball, which turns out to be

$$I = \frac{2}{5} mR_0^2 \tag{59.12}$$

(3) *Computing the moment of inertia of a body with respect to an axis passing through its centre of mass which is not an axis of symmetry of the body.* Up till now we have computed moments of inertia with respect to an axis of symmetry of a body. In the case of an arbitrary axis passing through the centre of mass the computation becomes more difficult. Therefore, we shall only consider a very simple example of the moment of inertia of a homogeneous rod of length l and mass m with respect to an axis forming an angle of α with the centre line of the rod and passing through the centre of mass (Fig. 158). Let x denote the distance of a particle of the rod of length dx from the midpoint of the rod. The mass of this particle is $dm = (m/l)\, dx$ and the distance f between the particle and the axis is $f = x \sin \alpha$. The moment of inertia of the particle is equal to

$$dI = dm\, f^2 = \frac{m}{l}\, dx\, x^2 \sin^2 \alpha$$

and the moment of inertia of the whole rod is

$$I = 2 \int_0^{l/2} dm\, f^2 = \frac{2m}{l} \sin^2 \alpha \int_0^{l/2} x^2\, dx = \frac{1}{12} ml^2 \sin^2 \alpha \tag{59.13}$$

It is obvious that if the rod is perpendicular to the axis ($\alpha = \pi/2$) then

$$I = \frac{1}{12} ml^2 \tag{59.14}$$

In these calculations the rod is supposed to be very thin; mathematically, this assumption means that the dimensions of the cross section of the rod are infinitesimal. Therefore, the result we have obtained has a practical meaning only when the dimensions of the cross section of a real rod are negligibly small in comparison with its length.

Moment of inertia of a thin disk with respect to an arbitrary axis lying in the plane of the disk. The above result obtained for a thin rod can be used for computing the moment of inertia of a disk. Let a disk of radius R_0 be split (mentally) into thin "rods" perpendicular to the given axis. The mass of such a "rod" of width dh lying at a distance h from the centre of the disk is equal to

$$dm = \frac{2m\sqrt{R_0^2 - h^2}}{\pi R_0^2}\, dh$$

and, by formula (59.14), its moment of inertia, is

$$dI = \frac{2}{12} \frac{m\sqrt{R_0^2 - h^2}}{\pi R_0^2} \cdot 4(R_0^2 - h^2)\, dh$$

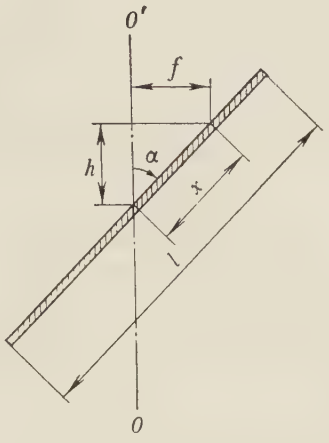

FIG. 158

Consequently, the moment of inertia of the whole disk can be computed by the formula

$$I = \frac{4}{3} \frac{m}{\pi R_0^2} \int_0^{R_0} (R_0^2 - h^2)^{3/2}\, dh = \frac{1}{4} mR_0^2 * \tag{59.15}$$

(4) *Computing the moment of inertia of a body with respect to an axis not passing through its centre of mass.* If we manage to find in a certain way the moment of inertia of a body with respect to an axis passing through its centre of mass then the moment of inertia with respect to any parallel axis can readily be found. Let the former moment of inertia be I_0 then the latter moment of inertia I is expressed by the formula

$$I = I_0 + ma^2 \tag{59.16}$$

where m is the mass of the body and a is the distance between the two axes. Formula (59.16) is known as the *Huygens-Steiner parallel axes theorem*. Let OO' be the axis passing through the centre of mass with respect to which the moment of

---

* When computing the integral we use the substitution $R_0 \sin \alpha = h$ and take into account that $\int_0^{\pi/2} \cos^4 \alpha\, d\alpha = \frac{3}{16}\pi$.

inertia I_0 is known. We take the line OO' as the z-axis of a rectangular coordinate system and draw the xy-plane through the centre of mass of the body, the origin being at the centre (Fig. 159*a*). Let the other axis $O''O'''$ parallel to the former axis pass through a point (x_0, y_0). Denoting the distance between the two axes by a we can write

$$a^2 = x_0^2 + y_0^2 \tag{59.17}$$

Let a particle of mass Δm_i have coordinates x_i and y_i; then the moment of iner-

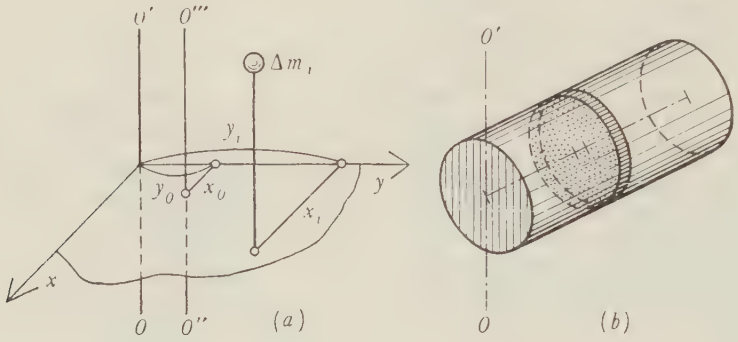

FIG. 159

tia of the body with respect to the axis OO' passing through the centre of mass is equal to

$$I_0 = \sum_i \Delta m_i (x_i^2 + y_i^2) \tag{59.18}$$

The square of the distance between the ith particle and the other axis $O''O'''$ is equal to

$$(x_i - x_0)^2 + (y_i - y_0)^2$$

and therefore the moment of inertia of the body with respect to the second axis is written in the form

$$I = \sum_i \Delta m_i [(x_i - x_0)^2 + (y_i - y_0)^2]$$

On removing the brackets and performing the summation we find

$$I = (x_0^2 + y_0^2) \sum_i \Delta m_i + \sum_i \Delta m_i (x_i^2 + y_i^2) - 2x_0 \sum_i \Delta m_i x_i - 2y_0 \sum_i \Delta m_i y_i \tag{59.19}$$

Taking into account formulas (59.17) and (59.18) and also the fact that, according to formulas (55.1) derived for the centre of mass, the last two sums on the right-hand side of (59.19) are equal to zero we arrive at the desired relation (59.16).

Thus, when the moment of inertia with respect to an axis passing through the centre of mass is known the moment of inertia with respect to any parallel axis can easily be found by formula (59.16).

Let us apply this result to the computation of the moment of inertia of a cylinder with respect to an axis passing through its generatrix. By formulas (59.4) and (59.16), this moment of inertia is readily found:

$$I = \frac{1}{2} mR_0^2 + mR_0^2 = \frac{3}{2} mR_0^2 \tag{59.20}$$

Formula (58.16) can also be used for computing the moment of inertia of a cylinder with respect to an axis OO' perpendicular to the geometrical axis of the cylinder (see Fig. 159*b*). To this end we divide the cylinder into infinitely thin disks, determine the moment of inertia of each disk with respect to the axis lying in the plane of the disk and parallel to OO' using formula (59.15) and then apply formula (59.16) to compute the moment of inertia of such a disk with respect to the axis OO'. Finally, adding together the moments of inertia of all the disks we determine the sought-for moment of inertia of the whole cylinder consisting of these disks. As an exercise, let the reader perform the calculations and find the moment of inertia of a cylinder of radius R and altitude H with respect to the axis OO' (Fig. 159*b*).

60. Kinetic Energy of a Body for Simultaneous Translatory and Rotational Motions

The kinetic energy of a rigid body is the sum of the kinetic energies of the particles the body is formed of. The kinetic energy of a particle of mass Δm_i is equal to

$$\frac{\Delta m_i v_i^2}{2}$$

where v_i is the linear velocity of the ith particle. It can be represented as a sum of two velocities: the velocity v_0 of the centre of mass and the velocity u_i of the ith particle relative to a coordinate system which is connected with the centre of mass and is in a translatory motion together with it:

$$v_i = v_0 + u_i \tag{60.1}$$

The kinetic energy of the whole body is therefore

$$E_k = \frac{1}{2} \sum_i \Delta m_i v_i^2 = \frac{1}{2} \sum_i \Delta m_i (v_0 + u_i)^2 =$$

$$= \frac{v_0^2}{2} \sum_i \Delta m_i + v_0 \sum_i \Delta m_i u_i + \frac{1}{2} \sum_i \Delta m_i u_i^2$$

that is

$$E_k = \frac{mv_0^2}{2} + \frac{1}{2} \sum_i \Delta m_i u_i^2 \tag{60.2}$$

because $\sum_i \Delta m_i u_i = 0$ (see Sec. 56).

This important relation applies to any motion of a rigid body and not only to one rigid body but to any system of rigid bodies as well.

Thus, the kinetic energy of a body consists of the two parts $\left(\frac{mv_0^2}{2}\right)$ and $\left(\sum_i \Delta m_i u_i^2\right)$, the former being the kinetic energy of the translatory motion and the latter the kinetic energy of the relative motion with respect to a reference

system which is connected with the centre of mass and is in the translatory motion together with the centre of mass of the body.

Any plane motion of a rigid body can be represented as a combination of two motions: the translatory motion together with the centre of mass and the rotational motion about the axis passing through the centre of mass and remaining parallel to a fixed direction in space during the motion. That is why in this case the kinetic energy of a motion relative to the centre of mass is the energy of the rotation of the body about the axis passing through the centre.

As was shown (see (54.2)), the kinetic energy of the rotation of a body with angular velocity ω is equal to

$$\frac{I\omega^2}{2} \tag{60.3}$$

where $I = \sum_i \Delta m_i r_i^2$ is the moment of inertia of the body with respect to the corresponding axis passing through the centre of mass.

Consequently, the total kinetic energy is

$$E_k = \frac{mv_0^2}{2} + \frac{I\omega^2}{2} \tag{60.4}$$

Thus, the kinetic energy of a body in an arbitrary plane motion is the sum of the kinetic energy $mv_0^2/2$ of the translatory motion and the energy $I\omega^2/2$ of the rotational motion.

We now consider an example in which the application of the law of conservation of energy provides a simple solution to the problem.

Let two cylinders of equal masses m and equal radii R roll down an inclined plane without sliding (see Fig. 152). In this case formula $v = \omega R$ is valid for both cylinders where ω denotes their angular velocities and v the velocities of their central axes. Suppose that the cylinders start rolling from the state of rest at the same height H over the horizontal plane and that the moments of inertia I_1 and I_2 of the cylinders are unequal. The question to be answered is which of the cylinders reaches the horizontal plane faster.

By the energy conservation law, the kinetic energies of the cylinders at the instants when they reach the horizontal plane are equal. Indeed, they must be equal to the potential energies of the cylinders in their initial positions, both potential energies being equal to PH where $P = mg$ is the weight of each cylinder. Consequently, we can write the equalities

$$PH = \frac{mv_1^2}{2} + \frac{I_1\omega_1^2}{2} = \frac{mv_2^2}{2} + \frac{I_2\omega_2^2}{2} \tag{60.5}$$

where v_1, v_2, and ω_1, ω_2 are, respectively, the terminal linear and the angular velocities of the first and the second cylinder when they roll down from the height H. Taking into account the relation $v = \omega R$ mentioned above, which indicates the absence of sliding, we can write $v_1 = \omega_1 R$ and $v_2 = \omega_2 R$ and therefore equalities (60.5) imply

$$v_1^2\left(m + \frac{I_1}{R^2}\right) = v_2^2\left(m + \frac{I_2}{R^2}\right) \tag{60.6}$$

Now it follows that the cylinder with a smaller moment of inertia reaches the horizontal plane faster than the other cylinder with a greater moment of inertia.

The energy conservation law expressed by formula (60.5) for the case of a cylinder rolling down an inclined plane without sliding establishes a one-to-one

relationship between the height from which the cylinder starts rolling and the velocity of its centre; this formula is analogous to the one derived earlier for the case of a free fall of a body.

61. Free Axes of Rotation

When a fixed axis about which a body rotates does not pass through the centre of mass of the body the centrifugal forces of inertia exert pressure on

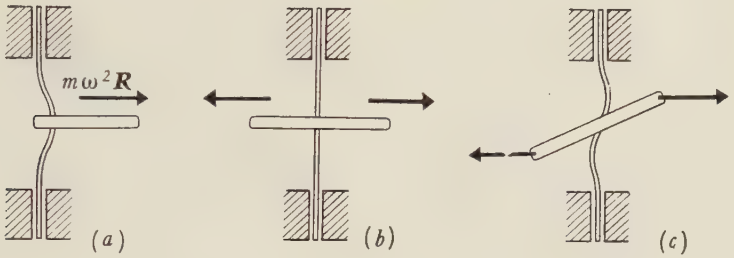

FIG. 160

the axis. For instance, if a thin rod (Fig. 160) is rotated about an axis passing near its end (see Fig. 160*a*) the centrifugal force of inertia which is equal to

$$m\omega^2 R$$

(where m is the mass of the rod and R is the distance between the centre of mass of the rod and the axis of rotation) bends the axis. It is evident that when the axis of rotation passes through the centre of mass of the rod (Fig. 160*b*) there are no bending forces. In this case the centrifugal force acting on one half of the rod is balanced by the centrifugal force acting on the other half. When the axis of rotation passes through the centre of mass and the rod is fixed on the axis as shown in Fig. 160*c* so that it forms an acute angle with the axis of rotation, the resultants of the centrifugal forces of inertia applied to both halves of the rod form a couple of forces whose moment also produces the bending of the axis.

Hence, a rotating body exerts no action on the axis of rotation when the axis passes through the centre of mass and the resultant moment of the inertia forces with respect to any axis perpendicular to the axis of rotation is equal to zero.

When a body possesses symmetry axes the directions of the axes of rotation on which the rotating body exerts no action can readily be determined. For instance, a rectangular parallelepiped made of a homogeneous material and shaped like a match-box (Fig. 161) possesses three mutually perpendicular symmetry axes which pass through the centres of the opposite faces of the parallelepiped. When this body is rotated about one of these axes the rotation produces no action on the bearings supporting that axis. That is why such axes of rotation are called *free axes*. To check the indicated property let us make this free parallelepiped rotate about one of these axes. If there are no external forces the rotation will last indefinitely. But when a body is made to rotate about an axis not coinciding with one of the free axes (for instance, when the parallelepi-

ped in Fig. 161 is made to rotate about an inclined axis *ab*) then the body, after it starts moving by itself, will no longer be in a pure rotational motion about that axis and the motion of the body will be compound.

The axis *I-I'* (see Fig. 161) passing through the largest faces of the parallelepiped corresponds to the greatest moment of inertia while the axis *II-II'* corresponds to the smallest moment of inertia. As will be shown in Sec. 64, any body possesses three mutually perpendicular free axes passing through its centre of mass. In the general theory it is proved that the rotation about the free axis

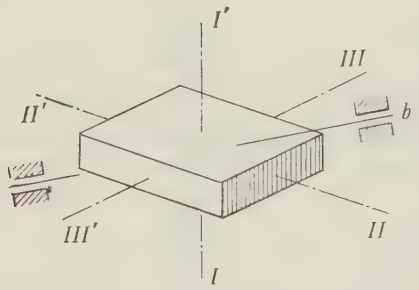

FIG. 161

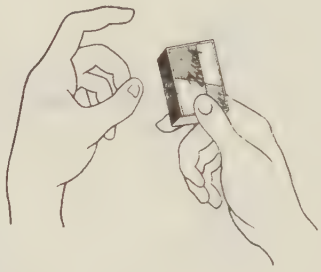

FIG. 162

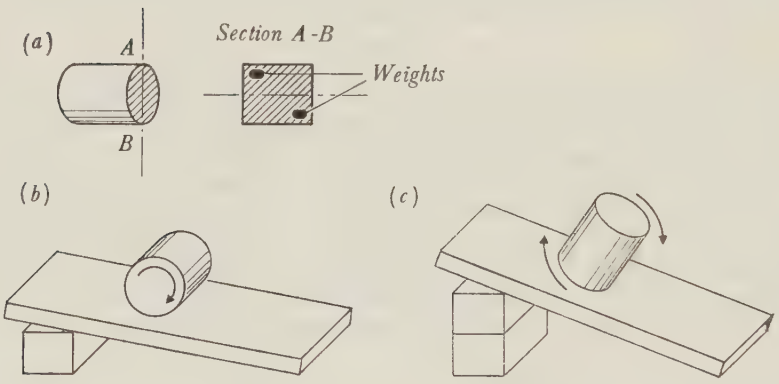

FIG. 163

corresponding to the moment of inertia having the intermediate value (in comparison with the greatest and the smallest moments of inertia) is unstable. For example, this is the case when the body shown in Fig. 161 rotates about the axis *III-III'*.

The stability of the rotation about the free axis with the greatest (or with the smallest) moment of inertia can be demonstrated by throwing an empty match-box with rotation (see Fig. 162). The box will be in a stable rotation (and, simultaneously, in a translatory motion which is of no importance for our discussion) when it is thrown so that it rotates about the axis *I-I'* or *II-II'* (compare with Fig. 161). On the contrary, when the box is made to rotate about the

free axis *III-III'* it does not retain a stable rotation and, due to the instability, the motion no longer remains plane.

A simple and visual demonstration of the role of the free axes gives a cylinder falling "head over heels" down inclined plane (Fig. 163*c*). Let the cylinder be made of a light material (for instance, of foam plastic) and let, as shown in Fig. 163*a*, two small lead loads be inserted into its face planes. The centre of mass of the cylinder lies on its axis; however, because of the loads, this axis is obviously no longer a free axis of the cylinder. The cylinder rolls down the

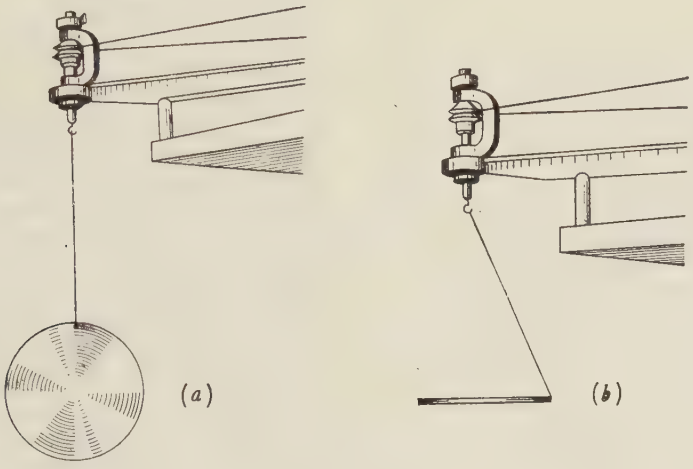

FIG. 164

inclined plane. When the inclination of the plane is small the cylinder rolls down like an ordinary homogeneous cylinder (see Fig. 163*b*). But when we try to make it roll down the plane with a sufficiently large angle of inclination the cylinder, when its speed becomes sufficiently high starts to move irregularly, to jump up and to fall "head over heels" (Fig. 163*c*).

A stable rotation of a body about its free axis can also be demonstrated as follows. The given body, for instance, a metal disk, is suspended from a thread attached to a rotating shaft as shown in Fig. 164. When the velocity of the rotation is low the disk rotates as is shown in Fig. 164*a*. As the angular velocity is increased, the disk, after several swings, starts rotating about the free axis of stable rotation. The motion becomes so stable that even the action of the force of weight of the body (which tends to change the position of the axis of rotation) does not affect the rotation significantly.

62. Kinematics of a Rigid Body

In the general case when the line of action of the resultant of the external forces applied to a body does not pass through the centre of mass the body is no longer in a pure translatory motion. The motion of the body becomes compound and is a combination of a translatory motion and a rotational motion. For instance, this can be observed in a rotation of a body about an axis not coinciding with one of its free axes.

When the resultant of the external forces passes through the centre of mass the law of motion has a simple form analogous to that describing the motion of a material point. In this case all particles of the body have the same acceleration and the body is in a translatory motion so that every line connected with the body remains parallel to itself in the course of the motion.

In the general case the motion of a body can be represented as a combination of a translatory motion (completely specified by the motion of the centre of mass) and a rotational motion about an axis passing through the centre of mass. Generally speaking, in the course of the motion this axis may change its position relative to the body and its position in space.

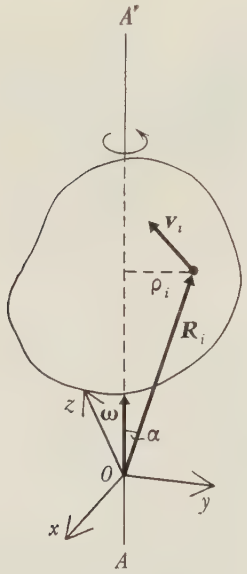

FIG. 165

Here we do not discuss the complex laws of motion for the general case of a compound motion of a body and limit ourselves to some examples in which one point of the body is fixed. These examples will play an important role in what follows.

First of all we note that when a rotation of a body about a fixed axis is investigated it is convenient to represent the velocities of different points of the body in the following way. Let us choose a point O on the axis of rotation AA' (Fig. 165) and take it as the origin of a fixed coordinate system $Oxyz$. Then the position of every particle of the body at any instant is specified by its radius vector $\boldsymbol{R}_i$. The velocity of the ith particle can be written as

$$\boldsymbol{v}_i = [\boldsymbol{\omega}\boldsymbol{R}_i] \qquad (62.1)$$

where the vector of angular velocity $\boldsymbol{\omega}$ is directed along the axis of rotation in accordance with the right-hand screw rule. Indeed, the distance from that particle to the axis is $\rho_i = R_i \sin \alpha$ and it is evident that formula (62.1) determines correctly both the magnitude and the direction of the velocity $\boldsymbol{v}_i$ of the particle. As is readily seen, the velocity of the ith particle is normal to the plane passing through the vectors $\boldsymbol{\omega}$ and $\boldsymbol{R}_i$, and its magnitude is equal to $\omega\rho_i$.

Let us draw some useful conclusions from the possibility of the representation of the velocities $\boldsymbol{v}_i$ of the particles by formula (62.1).

(a) The magnitude and the direction of the velocity $\boldsymbol{v}_i$ is independent of the position of the point O on the axis of rotation. For example, taking another point O' (see Fig. 166*a*) as the origin we obtain

$$\boldsymbol{v}_i = [\boldsymbol{\omega}\boldsymbol{R}_i'] = [\boldsymbol{\omega}\boldsymbol{R}_i] \quad \text{and} \quad v_i = \omega\rho_i$$

(b) Since the angular velocity $\boldsymbol{\omega}$ is a vector it can always be resolved into two vector components $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$. Accordingly, the rotation about the axis AA' with angular velocity $\boldsymbol{\omega}$ can be thought of as a combination of two simultaneous rotations about the corresponding axes OC and OE (Fig. 166*b*) with

angular velocities ω_1 and ω_2, respectively. Indeed, since $\omega = \omega_1 + \omega_2$ we have

$$v_i = [\omega R_i] = [\omega_1 R_i] + [\omega_2 R_i] = v_{i1} + v_{i2}$$

where $v_{i1} = [\omega_1 R_i]$ is the linear velocity in the rotation of the ith particle about the axis OC and $v_{i2} = [\omega_2 R_i]$ is the velocity of that particle in its rotation about the axis OE.

(c) There is an alternative representation of the rotation about the axis AA' which sometimes proves convenient. Let us choose an arbitrary point B connected with the body and take a reference system $Bx'y'z'$ (see Fig. 166c) which

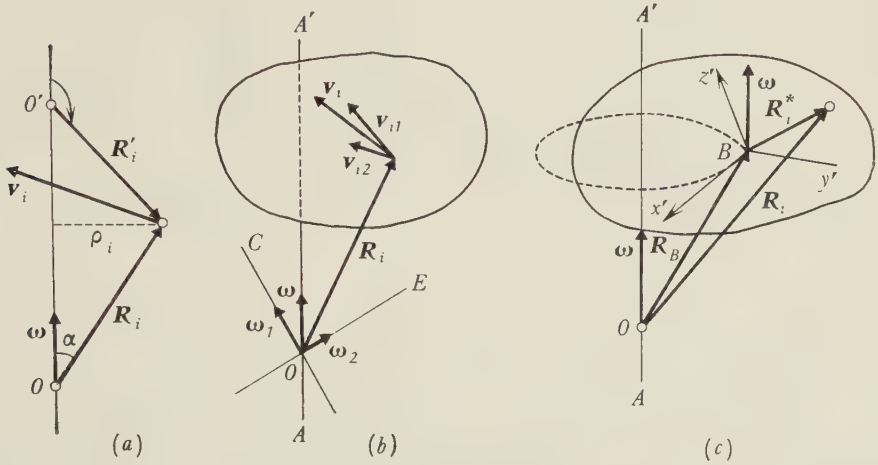

FIG. 166

is in a translatory motion (in a circle). Then the motion of the body can be thought of as a rotation with angular velocity ω about an axis passing through the point B. Indeed, since $R_i = R_B + R_i^*$ formula (62.1) implies

$$v_i = [\omega R_i] = [\omega R_B] + [\omega R_i^*] = v_B + [\omega R_i^*]$$

where $v_B = [\omega R_B]$. In other words, the velocity of every particle of the body can be represented as the sum of the velocity of the point B and the velocity resulting from the rotation of the body with angular velocity ω about the axis parallel to AA' and passing through the point B.

Now let us consider a more complex motion of a rigid body when an axis of rotation OA invariably connected with the body is itself in a rotational motion about a fixed axis OB. For instance, this is the case for a rotating globe attached to a rotating shaft as shown in Fig. 167.

Let a body rotate with an angular velocity ω_A about an axis OA connected with a moving body while the axis OA is itself in a rotational motion about a fixed axis OB with angular velocity ω_B (Fig. 168). The compound motion of the body can be resolved into two simple motions: a pure rotation about the axis OA with velocity ω_A (as if it were fixed, that is as if we had $\omega_B = 0$) and a pure rotation of the body about the axis OB with angular velocity ω_B (as if there were no rotation about the axis OA, that is as if we had $\omega_A = 0$). The

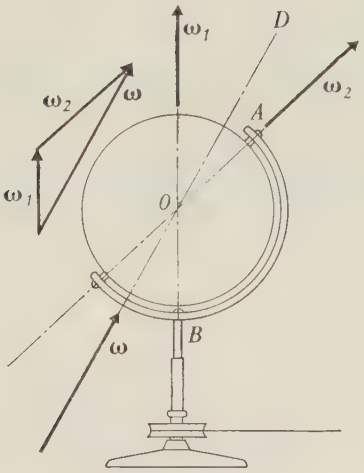

FIG. 167

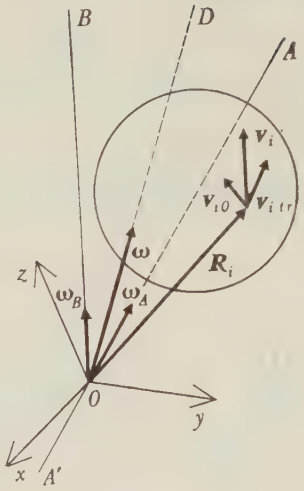

FIG. 168

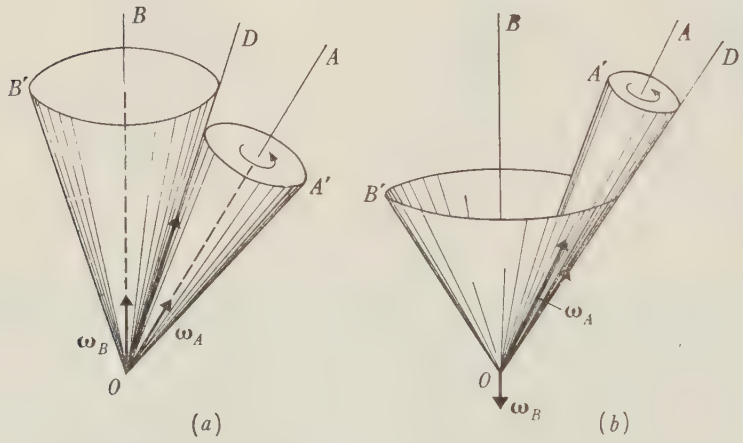

FIG. 169

linear velocity v_i of the ith particle in the resultant compound motion when both rotations about the axes OA and OB take place simultaneously can be found as the sum of the following two velocities: the velocity v_{i0} which the particle would have if ω_B were equal to zero and the velocity $v_{i\,tr}$ of the (transient) motion which would appear in its pure form if ω_A were equal to zero. Evidently,

$$v_{i0} = [\omega_A R_i] \quad \text{and} \quad v_{i\,tr} = [\omega_B R_i]$$

and therefore the linear velocity of the ith particle in the compound motion is written as

$$v_i = v_{i0} + v_{i\,tr} = [\omega_A R_i] + [\omega_B R_i] = [(\omega_A + \omega_B) R_i] = [\omega R_i] \qquad (62.2)$$

where $\omega = \omega_A + \omega_B$ is the geometric (vector) sum of the angular velocities ω_A and ω_B. The vector ω_B retains invariable position and is directed along the fixed axis OB while the vectors ω_A and ω change their positions in space. The vector ω lies in the plane passing through the fixed axis of rotation OB and the moving axis OA. Consequently, in this case, for any instant, the compound motion of the body can be represented as a rotation about an axis OD passing through the point O in the direction of the vector ω. This axis of rotation is spoken of as an *instantaneous axis*. Both its direction in space and its position

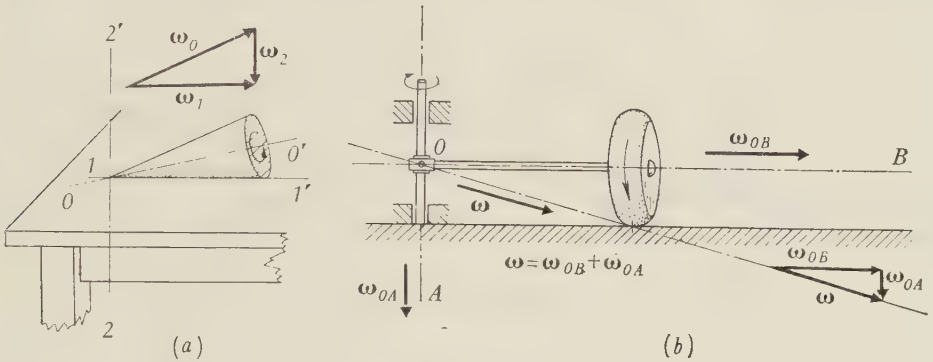

FIG. 170

relative to the body continuously vary. It is clear that those points of the body which are on the instantaneous axis at a given time t have a zero velocity $v_i = 0$, that is $v_{i\,tr} = -v_{i0}$, and hence for these points the linear velocities corresponding to simple constituent rotations are of equal magnitudes and opposite directions.

The compound motion of the globe resulting from its two rotations about the two axes OB and OD (Fig. 167) makes it possible to observe the position and the motion of the instantaneous axis: near the points at which the axis OD meets the surface of the globe the contours of the map are more clearly seen than in the other places on the globe.

The above described compound rotation about two axes with constant angular velocities ω_A and ω_B can be given a simple geometrical interpretation. The motion of the body can be represented with the aid of an imaginable cone OA', invariably connected with the body, which rolls on another imaginable cone OB' occupying a fixed position in space (Fig. 169*a*). The line of contact OD of the two cones is nothing but the instantaneous axis. At every instant the cone OA' turns about the instantaneous axis OD, that is it rolls without sliding on the fixed cone OB', the axis OD moving on the surface of the cone OB'. Depending on the magnitudes and direction of ω_A and ω_B, the axis OD may lie inside or outside the angle formed by the axes OA and OB. In the former case the cone OA' moves outside the cone OB' and rolls on its outer surface as shown in Fig. 169*a* while in the latter case (Fig. 169*b*) it is inside the cone OB' and rolls on its inner surface. Accordingly, in the former case the angle between the angular velocities ω_A and ω_B is acute and in the latter case it is obtuse.

Let us consider two more examples involving the addition of angular velocities.

A cone rolling on the surface of the table as shown in Fig. 170*a* is in a compound motion. At every instant it is in a rotational motion with an angular velocity ω_1 about the instantaneous axis *1-1'* lying in the line of contact of the cone and the plane surface of the table. The same motion can be thought of as a combination of simultaneous rotations about two axes: the vertical axis *2-2'* (occupying a fixed position in space) and the inclined axis *OO'* (coinciding with the geometrical axis of the cone), the angular velocities of the rotations being respectively ω_2 and ω_0. The vector ω_1 is equal to the sum of the vectors ω_2 and ω_0.

The motion of a millstone (see Fig. 170*b*) can be interpreted in like manner. When rolling on the plane the millstone is in a compound motion which can be regarded as a rotation about the fixed point *O*. This rotation can be represented as a motion of a cone with the point *O* as vertex and the millstone as base (see Fig. 170*b*). The circle bounding the base of the cone rolls on the horizontal plane. The directions of the axes of rotation *OA* and *OB* and the relationship between the corresponding angular velocities are indicated in the figure.

63. Moment of a Force About a Point and Angular Momentum of a Rigid Body

When the line of action of a force applied to a rigid body fixed at one point passes through that point the force obviously cannot affect the motion of the body since its action is balanced by the reaction force applied to the body at the point of support. Hence, only a force whose line of action does not pass through the point of support can change a state of rest or a motion of the body.

As before, in the latter case the force produces rotation of the body about the axis passing through the point of support and perpendicular to the plane containing the line of action of the force and the point of support, the measure of the action of the force being the moment of the force.

The moment $\boldsymbol{M}$ of a force about a point *O* (see Fig. 171*a*) is a vector quantity defined as the vector product of the radius vector $\boldsymbol{R}$ (drawn from the point *O* to the point of application of the force) by the force $\boldsymbol{F}$:

$$\boldsymbol{M} = [\boldsymbol{RF}], \quad M = RF \sin \alpha = Fh \tag{63.1}$$

where h is the *arm* of the force $\boldsymbol{F}$ ($h = R \sin \alpha$). This definition of the moment of a force as vector quantity is convenient because the direction of that vector indicates the axis about which the force tends to turn the body and the arm h of the force understood as the distance from the point of support to the line of action of the force is also taken into account.

It is obvious that a body fixed at a point can only be in equilibrium when the sum of the moments of the forces acting on it about the point of support is equal to zero or, which is the same, when the polygon formed of the vectors representing the moments of the forces (in the general case it is a spatial figure) is closed.

The above definition of the moment of a force about a point was stated in connection with the equilibrium problem for a body with one fixed point. In many other mechanical problems an important role is played by a more general notion which is not necessarily a point of support. Given a force $\boldsymbol{F}$ and the point of its application, the moment of the force about a pole *O* is defined as the vector

product $[\boldsymbol{RF}]$ where $\boldsymbol{R}$ is the radius vector joining the point O to the point of application of the force $\boldsymbol{F}$.

In the special case of a plane motion (see Sec. 57) we have already dealt with the definition of the moment of a force about a pole; in this case the moment of the force has a definite direction in space and coincides with the moment with respect to an axis which is always perpendicular to the plane of motion.

The moment of a force with respect to a given axis OO' whose definition was stated in Sec. 54 is equal to the projection M_0 onto the axis OO' of the moment

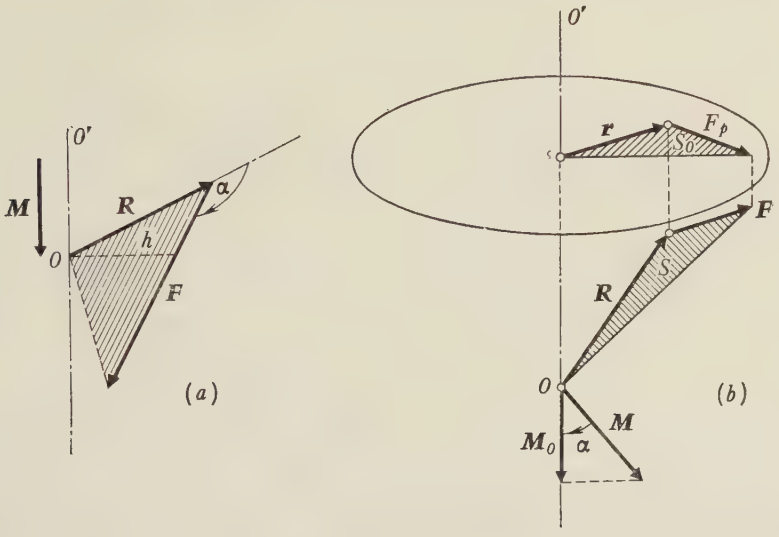

FIG. 171

$\boldsymbol{M}$ about an arbitrary point O lying on the axis OO' (see Fig. 171*b*). Indeed, the area S_0 equal to the projection of the area S onto the plane perpendicular to the axis OO' is the same for all the points O lying on the axis. The number $2S$ is equal to the modulus of the moment $\boldsymbol{M}$ of the force $\boldsymbol{F}$ about the point O and $2S_0$ is equal to the moment M_0 of the same force about the axis OO'. Equality $S_0 = S \cos \alpha$ therefore implies $M_0 = M \cos \alpha$ (see Fig. 171*b*).

In the analysis of the laws of motion of a rigid body an important role is played by the angular momentum (moment of momentum) of the body relative to a pole O. By the angular momentum of a separate particle of mass Δm_i about a given point is meant the vector product of the radius vector $\boldsymbol{R}_i$ of the particle drawn from that point by the (linear) momentum $\Delta m_i \boldsymbol{v}_i$ of the particle (Fig. 172). The angular momentum of the whole body is defined as the sum of the momenta of the particles it consists of:

$$\boldsymbol{N} = \sum_i \Delta m_i \, [\boldsymbol{R}_i \boldsymbol{v}_i] \tag{63.2}$$

The angular momentum of a rigid body is obviously a vector quantity. The angular momentum of a body with respect to a fixed axis was defined in Sec. 53 as the quantity $I\omega$ where I is the moment of inertia of the body with respect to that axis and ω is the angular velocity of rotation. As in the case of the moment

of a force, the projection of the angular momentum of a body about a point onto an axis passing through the point is equal to the angular momentum with respect to that axis in the sense of the definition stated in Sec. 53, that is this projection is equal to $I\omega$.

Generally speaking, the direction of the angular momentum does not necessarily coincide with the direction of the instantaneous axis or with that of a fixed axis. For instance, let a body consisting only of two particles of masses m_1 and m_2 (see Fig. 173) be in a rotation about a point O and let the instanta-

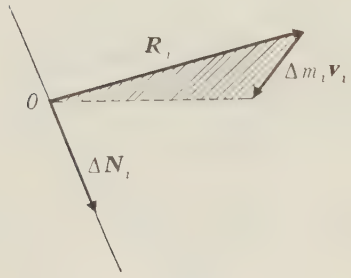

FIG. 172

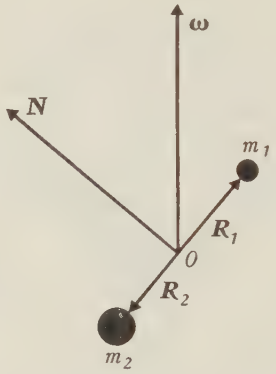

FIG. 173

neous axis go along the vector ω and form an angle with the line connecting the two particles. The axis of rotation and the two particles lie in the plane of the figure. It obviously follows that the velocities v_1 and v_2 of the particles are perpendicular to the plane of the figure while the vector N (the angular momentum with respect to O) is perpendicular to the line joining the particles and lies in the plane of the figure. For a homogeneous ball rotating about its centre the directions of its angular momentum and of instantaneous axis coincide.

As has already been indicated, every body has three *free axes* which are mutually perpendicular and pass through the centre of mass. They are also called the *principal axes of inertia.* When the axis of rotation coincides with a free axis the angular momentum and this axis have the same direction.

64. Angular Momentum of a Rigid Body and Moment of Inertia

When a rigid body rotates about an axis ω passing through a fixed point O (Fig. 174) the moment of momentum (the angular momentum) of a particle of mass Δm is

$$\Delta \boldsymbol{N} = \Delta m\,[\boldsymbol{R}\boldsymbol{v}] = \Delta m\,[\boldsymbol{R}\,[\boldsymbol{\omega}\boldsymbol{R}]] \tag{64.1}$$

The vector $\Delta \boldsymbol{N}$ lies in the plane of the vectors ω and $\boldsymbol{R}$ and is normal to $\boldsymbol{R}$. The projection of ΔN onto ω is equal to $\Delta N_\omega = \Delta m \rho^2 \omega$. Indeed, we have

$$\Delta N_\omega = \Delta N \cos\alpha = \Delta m R v \cos\alpha =$$
$$= \Delta m\, vR \sin(90° - \alpha) = \Delta m\, v\rho = \Delta m\, \rho^2 \omega$$

since $v = \omega\rho$. Consequently, in our previous study of a rotation of a body about a fixed axis we did in fact deal only with the projection ΔN_ω of the angular

momentum on the axis of rotation. This projection is equal to $\Delta m\,\rho^2\omega$, that is to the moment of inertia $\Delta I = \Delta m\,\rho^2$ of the particle multiplied by ω. The quantity ΔN_ω obviously does not depend on the position of the point O on the axis of rotation. Equality $\Delta N = \Delta I\,\omega$ expresses a simple relationship between the corresponding numbers (scalars).

We now proceed to the relationship between the vectors $\Delta \boldsymbol{N}$ and $\boldsymbol{\omega}$ for the general case. It turns out that this relationship involves a triple vector product of vectors which can be conveniently represented by a new physical quantity known as a *tensor*. The notion of a tensor generalizes the notion of a vector quantity.

To derive the desired relationship between the vectors $\Delta \boldsymbol{N}$ and $\boldsymbol{\omega}$ we first find the connection between the projections of $\Delta \boldsymbol{N}$ and of $\boldsymbol{\omega}$ on the coordinate axes. Using the well-known formula of vector algebra

$$[\boldsymbol{a}\,[\boldsymbol{bc}]] = \boldsymbol{b}\,(\boldsymbol{ac}) - \boldsymbol{c}\,(\boldsymbol{ab})$$

we rewrite relation (64.1) as

$$\Delta \boldsymbol{N} = \Delta m\,R^2\boldsymbol{\omega} - \Delta m\,\boldsymbol{R}\,(\boldsymbol{\omega R}) \quad (64.2)$$

Equation (64.2) expresses the resolution of $\Delta \boldsymbol{N}$ into components along the vectors $\boldsymbol{\omega}$ and $\boldsymbol{R}$ (see Fig. 175). Now we can write down the projections of $\Delta \boldsymbol{N}$ on the axis Ox of an orthogonal coordinate system connected with the body:

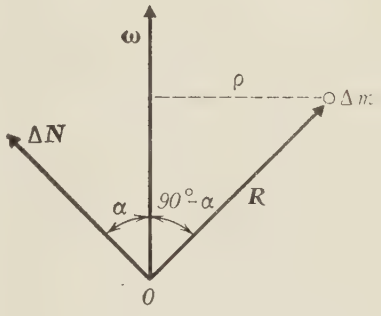

FIG. 174

$$\Delta N_x = \Delta m\,R^2\omega_x - \Delta m\,x\,(x\omega_x + y\omega_y + z\omega_z)$$

that is

$$\Delta N_x = \Delta m\,(R^2 - x^2)\,\omega_x - \Delta m\,xy\omega_y - \Delta m\,xz\omega_z$$

The projections on the other two axes Oy and Oz are found similarly, and thus we obtain

$$\begin{aligned}
\Delta N_x &= \Delta m\,(R^2 - x^2)\,\omega_x - \Delta m\,xy\omega_y - \Delta m\,xz\omega_z \\
\Delta N_y &= -\Delta m\,yx\omega_x + \Delta m\,(R^2 - y^2)\,\omega_y - \Delta m\,yz\omega_z \\
\Delta N_z &= -\Delta m\,zx\omega_x - \Delta m\,zy\omega_y + \Delta m\,(R^2 - z^2)\,\omega_z
\end{aligned} \quad (64.3)$$

where, as usual, x, y, and z are the projections of the vector $\boldsymbol{R}$. Each projection of $\Delta \boldsymbol{N}$ depends linearly on all the projections of $\boldsymbol{\omega}$. The expressions of the coefficients in ω_x, ω_y, and ω_z can easily be memorized since each of them has a dimension of the moment of inertia of a particle of mass Δm.

The collection of these nine factors standing in front of the projections of $\boldsymbol{\omega}$ in formulas (64.3) is called the *inertia tensor* of the particle of mass Δm. Denoting the tensor by one symbol $\Delta \mathcal{J}$ and arranging the nine numbers in a table (*matrix*) we can write

$$\Delta \mathcal{J} = \begin{pmatrix} \Delta m\,(R^2 - x^2) & -\Delta m\,xy & -\Delta m\,xz \\ -\Delta m\,yx & \Delta m\,(R^2 - y^2) & -\Delta m\,yz \\ -\Delta m\,zx & -\Delta m\,zy & \Delta m\,(R^2 - z^2) \end{pmatrix} \quad (64.4)$$

The three equalities (64.3) can be written (symbolically) as one formula

$$\Delta \boldsymbol{N} = \Delta \mathcal{J}\, \boldsymbol{\omega} \tag{64.5}$$

Formula (64.5) is analogous to relation $\Delta N_\omega = \Delta I\, \omega$, the distinction being that $\Delta \mathcal{J}$ is no longer a number but a tensor of rank two. The multiplication of $\Delta \mathcal{J}$ by the vector $\boldsymbol{\omega}$ is carried out in accordance with the rule of matrix multiplication. This rule is clearly seen from (64.3): the projection of the vector $\Delta \boldsymbol{N}$ on the x-axis is equal to the sum of the products of the elements in the first row of $\Delta \mathcal{J}$ by the corresponding projections of the vector $\boldsymbol{\omega}$, the projection of $\Delta \boldsymbol{N}$ on the y-axis is equal to the sum of the products of the elements of the second row of $\Delta \mathcal{J}$ by the projections of $\boldsymbol{\omega}$, etc.

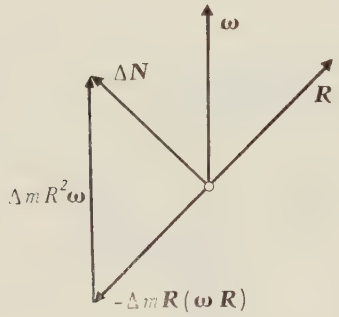

FIG. 175

Generally, a second-rank tensor is an ordered collection of nine numbers (representing some physical quantities) called the *components* of the tensor. In a transformation from one coordinate system to another these components are transformed as products of coordinates. We remind the reader that by a vector is meant an ordered triple of numbers which are transformed like coordinates. A scalar (a number) does not change when the coordinates are transformed. The multiplication of a tensor by a number reduces to the multiplication of each component of the tensor by that number.

The sum and the difference of two tensors are understood as a new tensor whose components are, respectively, the sums and the differences of the corresponding components of the two given tensors.

Among the second-rank tensors there is a unit tensor whose property is that the multiplication of this tensor by a vector does not alter the latter. The unit tensor $\mathcal{E}$ is obviously expressed by formula

$$\mathcal{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The unit tensor remains invariable under any transformation of coordinates. The multiplication of the tensor $a\mathcal{E}$ where a is a scalar by a vector results in the multiplication of the vector by that scalar a. The multiplication rule for a tensor by a vector was stated above.

The components of the tensor $\Delta \mathcal{J}$ expressed by formula (64.4) are pairwise products of the coordinates of the point multiplied by the constant mass Δm. That is why we call the quantity $\Delta \mathcal{J}$ a *tensor*; its every component is transformed according to the transformation rule for tensors stated above.

For a separate particle the relationship between $\Delta \boldsymbol{N}$ and $\boldsymbol{\omega}$ is seen directly from equality (64.2) (not involving the tensor notation $\Delta \mathcal{J}$) whose meaning is quite evident: the vector $\Delta \boldsymbol{N}$ lies in the plane containing $\boldsymbol{\omega}$ and $\boldsymbol{R}$; its construction is demonstrated in Fig. 175. The tensor $\mathcal{J}$ constructed for the whole rigid

body proves extremely convenient for describing the rotation of the body about a fixed axis. In this case the angular momentum is equal to

$$\boldsymbol{N} = \sum \Delta m_i [\boldsymbol{R}_i [\boldsymbol{\omega} \boldsymbol{R}_i]] = \sum \Delta m_i R_i^2 \boldsymbol{\omega} - \sum \Delta m_i \boldsymbol{R}_i (\boldsymbol{\omega} \boldsymbol{R}_i)$$

Using the tensor notation we can represent $\boldsymbol{N}$ in the form

$$\boldsymbol{N} = \mathcal{J} \boldsymbol{\omega} \tag{64.6}$$

where $\mathcal{J} = \sum \Delta \mathcal{J}_i$. We can write

$$\mathcal{J} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

where the components of the tensor $\mathcal{J}$ are

$$\left.\begin{aligned} I_{xx} &= \sum \Delta m_i (R_i^2 - x_i^2) \\ I_{yy} &= \sum \Delta m_i (R_i^2 - y_i^2) \\ I_{zz} &= \sum \Delta m_i (R_i^2 - z_i^2) \\ I_{xy} &= I_{yx} = -\sum \Delta m_i x_i y_i \\ I_{xz} &= I_{zx} = -\sum \Delta m_i x_i z_i \\ I_{yz} &= I_{zy} = -\sum \Delta m_i y_i z_i \end{aligned}\right\} \tag{64.7}$$

The components with two similar indices are obviously equal to the moments of inertia with respect to the corresponding axes. For instance,

$$I_{xx} = \sum \Delta m_i (R_i^2 - x_i^2) = \sum \Delta m_i (y_i^2 + z_i^2)$$

is the moment of inertia of the body with respect to the x-axis. The components with different indices are termed *products of inertia*. The tensor $\mathcal{J}$ has only six independent components since $I_{xy} = I_{yx}$, $I_{xz} = I_{zx}$, and $I_{yz} = I_{zy}$. Such a tensor is called *symmetric*.

The components of a tensor depend on the choice of the coordinate system, more precisely, they are specified by the directions of the axes. For a reference system $Oxyz$ connected with the body the components of the tensor $\mathcal{J}$ are constant quantities (they do not vary with time). The components of $\boldsymbol{N}$ and $\boldsymbol{\omega}$ are then determined with respect to the same axes. Thus, the six quantities I_{xx}, I_{xy}, I_{xz}, I_{yy}, I_{yz}, I_{zz} determine the set of the values of $\boldsymbol{N}$ corresponding to all the directions of $\boldsymbol{\omega}$, and, generally, the direction of $\boldsymbol{N}$ does not coincide with that of $\boldsymbol{\omega}$.

However, it turns out that for any body and for any point O there are at least three mutually perpendicular directions of $\boldsymbol{\omega}$ specifying three axes of rotation for which the vector $\boldsymbol{N}$ and the vector $\boldsymbol{\omega}$ have the same directions. These are the *principal directions* of the tensor $\mathcal{J}$. For the principal directions we have

$$\boldsymbol{N} = \lambda \boldsymbol{\omega}$$

where λ is a number. Let us rewrite the general equality (64.6) in terms of the projections on the coordinate axes:

$$\left.\begin{aligned} N_x &= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\ N_y &= I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\ N_z &= I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \end{aligned}\right\} \tag{64.8}$$

For the principal direction these equations take the form

$$\left.\begin{aligned}\lambda\omega_x &= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z\\ \lambda\omega_y &= I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z\\ \lambda\omega_z &= I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z\end{aligned}\right\} \tag{64.9}$$

We thus arrive at a system of three homogeneous linear algebraic equations in the three unknowns ω_x, ω_y, and ω_z. Such a system possesses non-zero solutions if and only if the determinant of the system is equal to zero, that is

$$\begin{vmatrix} I_{xx}-\lambda & I_{xy} & I_{xz}\\ I_{yx} & I_{yy}-\lambda & I_{yz}\\ I_{zx} & I_{zy} & I_{zz}-\lambda \end{vmatrix} = 0 \tag{64.10}$$

On computing the determinant in (64.10) we arrive at a third-degree algebraic equation with one unknown λ of the form

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 \tag{64.11}$$

where a_1, a_2, and a_3 depend on I_{xx}, I_{xy}, . . ., I_{zz} (on the component of the tensor).

Suppose that we managed to solve equation (64.11) and found its three roots λ_1, λ_2, and λ_3 which turned out to be pairwise different. On substituting the numbers λ_1, λ_2, and λ_3 in system (64.9) we find the corresponding principal directions. For instance, on substituting λ_1 in (64.9) we determine the principal directions corresponding to $\boldsymbol{\omega}_1$ for which $\boldsymbol{N} = \lambda_1\boldsymbol{\omega}_1$. Let us write down the first two equations of system (64.9) (the third one is a consequence of these equations):

$$\left.\begin{aligned}\lambda_1 - I_{xx} &= I_{xy}\frac{\omega_y}{\omega_x} + I_{xz}\frac{\omega_z}{\omega_x}\\ I_{yx} &= (I_{yy}-\lambda_1)\frac{\omega_y}{\omega_x} + I_{yz}\frac{\omega_z}{\omega_x}\end{aligned}\right\}$$

From these equations we find the ratios ω_y/ω_x and ω_z/ω_x specifying the principal direction corresponding to $\boldsymbol{\omega}_1$. The magnitude of the vector $\boldsymbol{\omega}_1$ can be quite arbitrary. The other two principal directions can similarly be found for the roots λ_2 and λ_3.

Let $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, and $\boldsymbol{n}_3$ be, respectively, the unit vectors in the principal directions corresponding to λ_1, λ_2, and λ_3. We will show that if λ_1, λ_2, and λ_3 are pairwise different, the vectors $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, and $\boldsymbol{n}_3$ are mutually perpendicular.

For the components (projections) of $\boldsymbol{n}_1$ we can write the vector equation $\lambda_1\boldsymbol{n}_1 = \mathcal{J}\boldsymbol{n}_1$ or, which is the same, the system of three scalar equations

$$\left.\begin{aligned}\lambda_1 n_{1x} &= I_{xx}n_{1x} + I_{xy}n_{1y} + I_{xz}n_{1z}\\ \lambda_1 n_{1y} &= I_{yx}n_{1x} + I_{yy}n_{1y} + I_{yz}n_{1z}\\ \lambda_1 n_{1z} &= I_{zx}n_{1x} + I_{zy}n_{1y} + I_{zz}n_{1z}\end{aligned}\right\} \tag{64.12}$$

involving the projections n_{1x}, n_{1y}, and n_{1z} of n_1. On multiplying the first equation (64.12) by n_{2x}, the second by n_{2y}, and the third by n_{2z} and adding together the results we find

$$\lambda_1\boldsymbol{n}_1\boldsymbol{n}_2 = A_1$$

where A_1 is the scalar product of the vectors $\boldsymbol{n}_2$ and $\mathcal{J}\boldsymbol{n}_1$: $A_1 = \boldsymbol{n}_2\mathcal{J}\boldsymbol{n}_1$. We similarly write the vector equation $\lambda_2\boldsymbol{n}_2 = \mathcal{J}\boldsymbol{n}_2$ which can also be represented

as a system of three scalar equations in terms of the projections of $\boldsymbol{n}_2$. On multiplying these equations by the corresponding components of $\boldsymbol{n}_1$ and adding them up we obtain $\lambda_2 \boldsymbol{n}_1 \boldsymbol{n}_2 = A_2 = \boldsymbol{n}_1 \mathcal{J} \boldsymbol{n}_2$. The tensor $\mathcal{J}$ being symmetric (i.e. $I_{xy} = I_{yx}$, $I_{xz} = I_{zx}$, $I_{yz} = I_{zy}$), it can easily be shown that $A_1 = A_2$. Therefore

$$\lambda_1 \boldsymbol{n}_2 \boldsymbol{n}_1 = \lambda_2 \boldsymbol{n}_1 \boldsymbol{n}_2 \tag{64.13}$$

whence $(\lambda_1 - \lambda_2)\, \boldsymbol{n}_1 \boldsymbol{n}_2 = 0$. If $\lambda_1 \neq \lambda_2$ the last equality can only be satisfied when $\boldsymbol{n}_1 \boldsymbol{n}_2 = 0$, which means that the vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are orthogonal. It can be shown analogously that if $\lambda_2 \neq \lambda_3$ then $\boldsymbol{n}_2 \boldsymbol{n}_3 = 0$ and that if $\lambda_1 \neq \lambda_3$ then $\boldsymbol{n}_1 \boldsymbol{n}_3 = 0$.

It should be noted that when the coordinate axes are directed along $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, and $\boldsymbol{n}_3$ the inertia tensor assumes a particularly simple form. Indeed, in this case we have $\boldsymbol{\omega} = \omega_1 \boldsymbol{n}_1 + \omega_2 \boldsymbol{n}_2 + \omega_3 \boldsymbol{n}_3$ whence

$$\begin{aligned} \boldsymbol{N} = N_1 \boldsymbol{n}_1 + N_2 \boldsymbol{n}_2 + N_3 \boldsymbol{n}_3 = \mathcal{J}\boldsymbol{\omega} = \\ = \mathcal{J}\,(\omega_1 \boldsymbol{n}_1 + \omega_2 \boldsymbol{n}_2 + \omega_3 \boldsymbol{n}_3) = \omega_1 \mathcal{J} \boldsymbol{n}_1 + \omega_2 \mathcal{J} \boldsymbol{n}_2 + \omega_3 \mathcal{J} \boldsymbol{n}_3 = \\ = \omega_1 \lambda_1 \boldsymbol{n}_1 + \omega_2 \lambda_2 \boldsymbol{n}_2 + \omega_3 \lambda_3 \boldsymbol{n}_3 \end{aligned}$$

Consequently, for such a coordinate system we have

$$\left.\begin{aligned} N_1 = \lambda_1 \omega_1, \quad N_2 = \lambda_2 \omega_2, \quad N_3 = \lambda_3 \omega_3 \\ N^2 = N_1^2 + N_2^2 + N_3^2 = \lambda_1^2 \omega_1^2 + \lambda_2^2 \omega_2^2 + \lambda_3^2 \omega_3^2 \end{aligned}\right\} \tag{64.14}$$

It follows that for the chosen coordinate system the inertia tensor takes the form

$$\mathcal{J} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

in which only the diagonal components are non-zero.

The numbers λ_1, λ_2, and λ_3 are obviously equal to the moments of inertia of the body with respect to the corresponding principal axes of inertia. They are therefore called the *principal moments of inertia*. The principal moments of inertia are specified by the distribution of the mass of the body over the spatial region occupied by it and are independent of the choice of the coordinate system. The principal axes determined for the centre of mass of the body coincide with its free axes.

Furthermore, it can be shown that if two of the roots λ_1, λ_2, and λ_3, for instance, λ_1 and λ_2, coincide ($\lambda_1 = \lambda_2$) there are an infinite number of principal axes lying in the plane normal to the vector $\boldsymbol{n}_3$. This is the case when the body is homogeneous and symmetric with respect to an axis and $\boldsymbol{n}_3$ goes along that axis. Such homogeneous bodies as a cylinder (with circular or square cross section), a solid of revolution, and the like possess axial symmetry. A homogeneous ball is centrally symmetric, and any axis passing through its centre is its principal axis.

We now derive an expression for the kinetic energy of a rigid body rotating with angular velocity $\boldsymbol{\omega}$ about an axis passing through a fixed point O at which the body is supported (Fig. 176). It turns out that the kinetic energy T is connected with $\boldsymbol{\omega}$, $\boldsymbol{N}$, and $\mathcal{J}$ by the relation

$$T = \frac{1}{2}\,\boldsymbol{\omega}\boldsymbol{N} = \frac{1}{2}\,\boldsymbol{\omega}\mathcal{J}\boldsymbol{\omega} \tag{64.15}$$

Indeed, taking into account (64.6) we write

$$\boldsymbol{\omega}\boldsymbol{N} = \sum \Delta m_i \boldsymbol{\omega}\,[\boldsymbol{R}_i\,[\boldsymbol{\omega}\boldsymbol{R}_i]]$$

and the product $\boldsymbol{\omega}\,[\boldsymbol{R}_i\,[\boldsymbol{\omega}\boldsymbol{R}_i]]$ can be expressed as

$$\boldsymbol{\omega}\,[\boldsymbol{R}_i\,[\boldsymbol{\omega}\boldsymbol{R}_i]] = \boldsymbol{\omega}\,\{\boldsymbol{R}_i^2\boldsymbol{\omega} - \boldsymbol{R}_i\,(\boldsymbol{\omega}\boldsymbol{R}_i)\} =$$
$$= \omega^2 R_i^2 - (\boldsymbol{\omega}\boldsymbol{R}_i)^2 = \omega^2 R_i^2\,(1 - \cos\alpha_i) = \omega^2\rho_i^2 = v_i^2$$

Therefore $\boldsymbol{\omega}\boldsymbol{N} = \sum \Delta m_i v_i^2 = 2T$.

The kinetic energy can also be represented in the form

$$T = \frac{1}{2}\sum \Delta m_i v_i^2 = \frac{1}{2}\,\omega^2 \sum \Delta m_i \rho_i^2 = \frac{1}{2}\,I_\omega \omega^2 \tag{64.16}$$

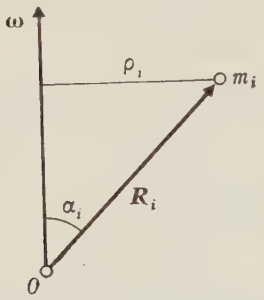

FIG. 176

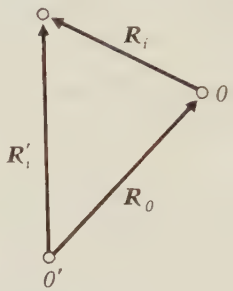

FIG. 177

as in the case of a rotation about a fixed axis where $I_\omega = \sum \Delta m_i \rho_i^2$ is the moment of inertia with respect to the axis coinciding with $\boldsymbol{\omega}$ (I_ω is a scalar).

It should be noted that

$$2T = \boldsymbol{\omega}\boldsymbol{N} = \omega N_\omega = I_\omega \omega^2 \tag{64.17}$$

whence follows the familiar equality

$$N_\omega = I_\omega \boldsymbol{\omega} \tag{64.18}$$

Hence, the projection of $\boldsymbol{N}$ on $\boldsymbol{\omega}$ is equal to the moment of inertia with respect to the axis coinciding with $\boldsymbol{\omega}$ multiplied by ω. We see that the relations derived earlier for the case of a rotation about an axis occupying a fixed position relative to the body remain valid for any instantaneous axis.

It can be shown that for the angular momentum $\boldsymbol{N}$ with respect to any point O' we have

$$\boldsymbol{N} = m\,[\boldsymbol{R}_0\,[\boldsymbol{\omega}\boldsymbol{R}_0]] + \boldsymbol{N}_0 \tag{64.19}$$

where $\boldsymbol{N}_0$ is the angular momentum for the rotation with angular velocity $\boldsymbol{\omega}$ about the axis passing through the centre of mass (see Fig. 177; here $m = \sum \Delta m_i$ is the total mass of the body and $\boldsymbol{R}_0$ is the radius vector of the centre of mass drawn from the point O'). Indeed, putting $\boldsymbol{R}_i' = \boldsymbol{R}_0 + \boldsymbol{R}_i$ we can write

$$\boldsymbol{N} = \sum \Delta m_i\,[\boldsymbol{R}_i'\,[\boldsymbol{\omega}\boldsymbol{R}_i']] = \sum \Delta m_i\,[(\boldsymbol{R}_0 + \boldsymbol{R}_i)\,[\boldsymbol{\omega}\,(\boldsymbol{R}_0 + \boldsymbol{R}_i)]]$$

On applying the formula for the triple vector product and taking into account that $\sum \Delta m_i \boldsymbol{R}_i = 0$ we arrive at

$$\boldsymbol{N} = m\,[\boldsymbol{R}_0\,[\boldsymbol{\omega}\boldsymbol{R}_0]] + \sum \Delta m_i\,[\boldsymbol{R}_i\,[\boldsymbol{\omega}\boldsymbol{R}_i]]$$

The last sum on the right-hand side (denoted as N_0 in (64.19)) is the angular momentum for the rotation about the axis passing through O with angular velocity $\boldsymbol{\omega}$ while the first term on the right is the angular momentum of the rotation of the body as a whole (as a particle with mass m concentrated at the centre of mass) with angular velocity $\boldsymbol{\omega}$ about the axis passing through O'.

The angular momentum for a rotation about an axis passing through the centre of mass can be written as

$$\boldsymbol{N}_0 = \mathcal{J}_0 \boldsymbol{\omega} \tag{64.20}$$

where $\mathcal{J}_0$ is the inertia tensor in a coordinate system with origin at the centre of mass. The components of $\mathcal{J}_0$ are invariable when the coordinate system is rigidly connected with the body.

Knowing $\mathcal{J}_0$ we can find the moment of inertia with respect to any axis passing through the centre of mass O of the body. Let the direction of the axis of rotation be specified by a unit vector $\boldsymbol{n}$. Then, by (64.18) and (64.20), the projection of $\boldsymbol{N}$ onto $\boldsymbol{n}$ can be expressed as

$$N_n = \boldsymbol{N}_0 \boldsymbol{n} = \boldsymbol{n} \mathcal{J}_0 \boldsymbol{n} \omega = I_n \omega$$

where I_n is the moment of inertia with respect to the axis $\boldsymbol{n}$. Thus,

$$I_n = \boldsymbol{n} \mathcal{J}_0 \boldsymbol{n} \tag{64.21}$$

The general expression of the number I_n in terms of the components of the tensor $\mathcal{J}_0$ and of the vector $\boldsymbol{n}$ is rather lengthy. We give a shorter formula for the special case of the coordinate axes directed along the principal directions $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, and $\boldsymbol{n}_3$. In this case we have $\boldsymbol{n} = \alpha_1 \boldsymbol{n}_1 + \alpha_2 \boldsymbol{n}_2 + \alpha_3 \boldsymbol{n}_3$ where α_1, α_2, and α_3 are the direction cosines of $\boldsymbol{n}$ relative to the axes with unit vectors $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, and $\boldsymbol{n}_3$. Consequently,

$$\mathcal{J}_0 \boldsymbol{n} = \alpha_1 \mathcal{J}_0 \boldsymbol{n}_1 + \alpha_2 \mathcal{J}_0 \boldsymbol{n}_2 + \alpha_3 \mathcal{J}_0 \boldsymbol{n}_3 = \alpha_1 \lambda_1 \boldsymbol{n}_1 + \alpha_2 \lambda_2 \boldsymbol{n}_2 + \alpha_3 \lambda_3 \boldsymbol{n}_3$$

Therefore

$$I_n = \boldsymbol{n} \mathcal{J}_0 \boldsymbol{n} = \alpha_1^2 \lambda_1 + \alpha_2^2 \lambda_2 + \alpha_3^2 \lambda_3 \tag{64.22}$$

where λ_1, λ_2, and λ_3 are the moments of inertia with respect to the principal axes passing through the centre of mass. Given λ_1, λ_2, λ_3, α_1, α_2, and α_3, we can readily determine the moment of inertia I_n with respect to any axis $\boldsymbol{n}$ using formula (64.22).

The distribution of the values of I_n for various directions can be visualized with the aid of the *ellipsoid of inertia*. Let us divide the basic relation (64.22) by I_n and denote $\alpha_1/\sqrt{I_n} = x_1$, $\alpha_2/\sqrt{I_n} = x_2$, and $\alpha_3/\sqrt{I_n} = x_3$. Then equality (64.22) can be rewritten as

$$\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 = 1$$

This is the equation of an ellipsoid in coordinates x_1, x_2, and x_3 reckoned along the principal axes $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, $\boldsymbol{n}_3$ of the inertia tensor. The semiaxes of the ellipsoid are obviously equal to $1/\sqrt{\lambda_1}$, $1/\sqrt{\lambda_2}$, and $1/\sqrt{\lambda_3}$ and the distance ρ from the centre of the ellipsoid to its surface in the direction of $\boldsymbol{n}$ is given by the formula

$$\rho^2 = x_1^2 + x_2^2 + x_3^2 = 1/I_n$$

since $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$. Therefore, ρ is the reciprocal of the square root of the moment of inertia I_n with respect to the axis going in the direction of the vector $\boldsymbol{n}$. The surface of the ellipsoid thus shows the distribution of the values of

the possible moments of inertia with respect to the axes passing through the centre of mass.* The shortest axis of the ellipsoid is in the direction to which corresponds the greatest possible moment of inertia while the longest axis indicates the direction to which corresponds the smallest moment of inertia.

It can be shown that the shape of the ellipsoid of inertia resembles to a certain extent the shape of the corresponding homogeneous body. For example, for a homogeneous rectangular parallelepiped with edges of different lengths (see Fig. 178 where such a "match-box-shaped" body is shown) the ellipsoid of iner-

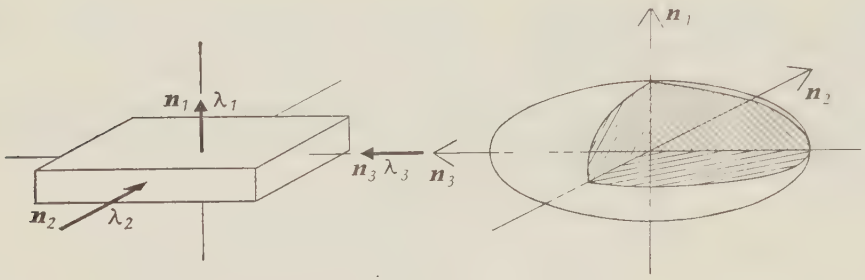

FIG. 178

tia is of the form shown on the right of the figure. The longest semiaxis goes along n_3 while the shortest semiaxis goes along n_1. The principal directions of the inertia tensor for the centre of mass coincide with the free axes. For an arbitrary point of the body the principal axes of the inertia tensor are no longer free axes because when the body rotates about one of them it is acted upon by the centrifugal inertia forces and the direction of N does not coincide with that of ω.

For a solid of revolution the ellipsoid of inertia is an ellipsoid of revolution and one of the principal axes coincides with the axis of symmetry of the solid. Any axis perpendicular to this axis and passing through the centre of mass is a principal axis. Any two mutually perpendicular axes in the plane perpendicular to the axis of revolution can be taken as the other two principal axes of the ellipsoid. For a homogeneous ball any direction passing through its centre is principal, that is its ellipsoid of inertia degenerates into a sphere.

If an axially symmetric body has two mutually perpendicular principal axes with equal moments of inertia the corresponding ellipsoid of inertia is an ellipsoid of revolution. This is the case for a rod with square cross section; the symmetry conditions indicate that such a rod possesses two principal directions with equal moments of inertia. The same considerations lead to the conclusion that the ellipsoid of inertia of a cube degenerates into a sphere.

In conclusion we note that the Huygens-Steiner theorem is a consequence of equality (64.19). For, on multiplying this equality by unit vector $\boldsymbol{n}$ and taking into account (64.20) and the relations $\omega = n\omega$ we obtain

$$N_n = nN = m\omega n\,[R_0\,[nR_0]] + n\mathcal{J}_0 n\omega = n\mathcal{J}n\omega$$

* The ellipsoid of inertia can also be constructed for the rotation about any axis not necessarily passing through the centre of mass.

where $\boldsymbol{n}\mathcal{J}_0\boldsymbol{n} = I_{0n}$ is the moment of inertia with respect to the axis going in the direction of $\boldsymbol{n}$ and passing through the centre of mass O and $\boldsymbol{n}\mathcal{J}\boldsymbol{n} = I_n$ is the moment of inertia with respect to the axis passing through an arbitrary point O' in the direction of $\boldsymbol{n}$ (Fig. 179). The scalar product in the last formula can be written as

$$\boldsymbol{n}\,[\boldsymbol{R}_0\,[\boldsymbol{n}\boldsymbol{R}_0]] = R_0^2 - (\boldsymbol{n}\boldsymbol{R}_0)^2 = R_0^2\,(1 - \cos^2\alpha) = h^2$$

where h is the distance between the axes going in the direction of $\boldsymbol{n}$ and passing

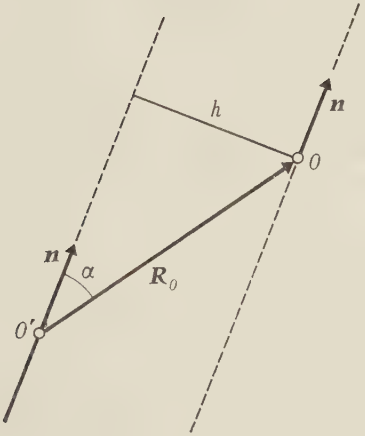

FIG. 179

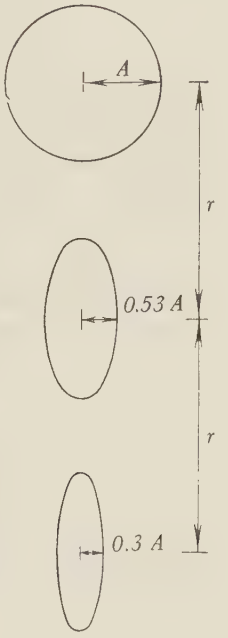

FIG. 180

through the points O and O', respectively. The substitution of this expression in the basic equality leads (after we cancel by ω) to

$$mh^2 + I_{0n} = I_n$$

which is nothing but the Huygens-Steiner parallel axes theorem.

We apply this theorem to find the ellipsoid of inertia in the following example. Let a ball rotate about a point O' lying at a distance of h from its centre. The symmetry conditions indicate that one of the principal axes passes through the point O' and through the centre of the ball and that the moments of inertia with respect to any axes passing through O' and normal to OO' are equal to each other. Any two axes lying in the normal plane to OO' correspond to equal moments of inertia and are principal axes. The ellipsoid of inertia is therefore an ellipsoid of revolution about the first principal axis OO'. It is evident that the moment of inertia with respect to that axis is equal to the moment of inertia of the ball I_0. The moments of inertia with respect to the second and the third

principal axes are equal to $I_0 + mh^2$. Thus, for the point O' the ratio of the semiaxes of the ellipsoid of inertia is equal to

$$\frac{\sqrt{I_0}}{\sqrt{I_0+mh^2}} = \frac{\sqrt{2/5}}{\sqrt{2/5+h^2/r^2}} = \left(1+\frac{5}{2}\frac{h^2}{r^2}\right)^{-1/2}$$

since, as we know, $I_0 = 2/5mr^2$ where r is the radius of the ball (see (59.12)). In Fig. 180 are shown the shapes of the ellipsoids of inertia for the three cases $h = 0$, $h = r$, and $h = 2r$. Here $A = 1/\sqrt{I_0}$.

65. The Fundamental Law of Dynamics of a Rigid Body

Let a rigid body move relative to a coordinate system with origin at a point O and let the forces acting on the body be known. Suppose that a particle of mass Δm_i of the body is at a point with radius vector $\boldsymbol{R}_i$ (drawn from the point O) and is acted upon by a force $\boldsymbol{F}_i$. By the force $\boldsymbol{F}_i$ we mean the sum of all forces acting on that particle, both internal and external.

By the second law of dynamics,

$$\Delta m_i \frac{d\boldsymbol{v}_i}{dt} = \boldsymbol{F}_i \tag{65.1}$$

On performing the vector multiplication of both sides of (65.1) by the radius vector $\boldsymbol{R}_i$ of the ith particle we obtain

$$\Delta m_i \left[\boldsymbol{R}_i \frac{d\boldsymbol{v}_i}{dt}\right] = [\boldsymbol{R}_i \boldsymbol{F}_i] = \Delta \boldsymbol{M}_i \tag{65.2}$$

Now let us compute the angular momentum of that particle about the point O:

$$\Delta \boldsymbol{N}_i = \Delta m_i [\boldsymbol{R}_i \boldsymbol{v}_i] \tag{65.3}$$

The time derivative of this angular momentum is

$$\frac{d}{dt}\Delta \boldsymbol{N}_i = \frac{d}{dt}(\Delta m_i [\boldsymbol{R}_i \boldsymbol{v}_i]) = \Delta m_i \left[\frac{d\boldsymbol{R}}{dt}\boldsymbol{v}_i\right] + \Delta m_i \left[\boldsymbol{R}_i \frac{d\boldsymbol{v}_i}{dt}\right] \tag{65.4}$$

The derivative of the radius vector of the particle is equal to its velocity, that is

$$\frac{d\boldsymbol{R}_i}{dt} = \boldsymbol{v}_i$$

thus we find

$$\left[\frac{d\boldsymbol{R}_i}{dt}\boldsymbol{v}_i\right] = [\boldsymbol{v}_i \boldsymbol{v}_i] = 0$$

The comparison of (65.2) and (65.4) shows that

$$\frac{d}{dt}\Delta \boldsymbol{N}_i = \Delta m_i \left[\boldsymbol{R}_i \frac{d\boldsymbol{v}_i}{dt}\right] = \Delta \boldsymbol{M}_i \tag{65.5}$$

Hence, the moment of a force acting on a particle is equal to the angular momentum of that particle.

On adding together the moments of all forces acting on all particles of the body and also summing the derivatives of the angular momenta of the particles (which are equal to the corresponding moments of forces) we obtain

$$\frac{d}{dt}\sum_i \Delta \boldsymbol{N}_i = \sum_i \Delta \boldsymbol{M}_i$$

that is

$$\frac{d\boldsymbol{N}}{dt} = \boldsymbol{M} \tag{65.6}$$

where

$$\sum_i \Delta \boldsymbol{N}_i = \boldsymbol{N} \quad \text{and} \quad \sum_i \Delta \boldsymbol{M}_i = \boldsymbol{M}$$

Since the internal forces form pairs of forces equal in the magnitudes and opposite in the directions (with common lines of action), the sum of the moments of the internal forces about any point is equal to zero and therefore the sum of the moments of all the forces reduces to the moment of the resultant of the external forces alone.

Consequently, the fundamental law of dynamics of a rigid body can be stated as follows: *the moment of the resultant of all the external forces about any fixed point O is equal to the time derivative of the corresponding angular momentum of the body.*

This law can also be restated in the following way: the increment $d\boldsymbol{N}$ of the angular momentum during time dt coincides (in its direction and magnitude) with the product of the moment of the resultant of all external forces $\boldsymbol{M}$ by time dt (here dt and $d\boldsymbol{N}$ are understood as infinitesimals).

To complete the description of the motion of a rigid body we should add to law (65.6) the law of motion for the centre of mass which holds in all cases (see its derivation in Sec. 56):

$$m \frac{d\boldsymbol{v}_0}{dt} = \boldsymbol{F} \tag{65.7}$$

where $\boldsymbol{v}_0$ is the velocity of the centre of mass and $\boldsymbol{F}$ is the sum (the resultant) of all external forces.

The law of variation of the angular momentum expressed by (65.6) can also be transformed to a non-rotating coordinate system which is in a translatory motion and is connected with the centre of mass (its origin is at the centre). This results in the equation

$$\frac{^{\mathrm{r}}d\boldsymbol{N}_0}{dt} = \boldsymbol{M}_0 \tag{65.8}$$

where $\boldsymbol{N}_0$ is the angular momentum about the centre of mass and $\boldsymbol{M}_0$ is the sum of the moments of the external forces about the centre of mass. Formula (65.8) is derived in the same way as it was done in Sec. 57 where an analogous formula was deduced for the case of a plane motion (see (57.8)). The only distinction is that in the general case treated here all vectors ($\boldsymbol{R}_i$, $\boldsymbol{v}_i$, etc.) have arbitrary orientation in space.

Thus, formulas (65.7) and (65.8) provide a complete description of the laws of dynamics governing an arbitraty motion of a rigid body. In practical problems the application of formula (65.8) to the analysis of a concrete motion may be rather intricate because in many cases it is difficult to determine the direction of the angular momentum using only elementary techniques.

The derivation of the basic laws of dynamics for a rigid body clearly indicates that the laws

$$\frac{d\boldsymbol{K}}{dt} = \boldsymbol{F} \quad \text{and} \quad \frac{d\boldsymbol{N}}{dt} = \boldsymbol{M} \tag{65.9}$$

are also valid for any system of material points or bodies (not only for a rigid body). The notation used in formulas (65.9) is familiar to the reader: $K = \sum K_i$ is the sum of the momenta of all particles (bodies), $F = \sum F_i$ is the sum (the resultant) of all internal forces, etc.

The character of the variation of the momentum (see (65.6)) for an arbitrary system of particles or for a rigid body as the position of the pole is changed should be discussed in more detail. Suppose that N, N_i, R_i, etc. have been computed for a point A (see Fig. 181) relative to a fixed reference system. The question is how the law of variation of the momentum changes when we pass to another pole O whose position is determined by an arbitrary radius vector R_0.

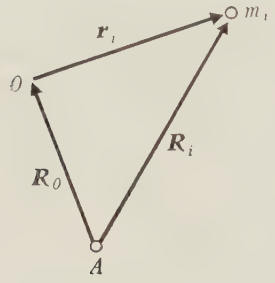

FIG. 181

For the sake of generality let us suppose that R_0 is a function of time t, that is the point O can be in a motion. For each particle we have (see Fig. 181)

$$R_i = R_0 + r_i$$

Let us show that the sum of the momenta of the particles of the system (relative to the point O) which is equal to

$$N_0 = \sum [r_i K_i]$$

satisfies the condition

$$\frac{dN_0}{dt} = M_0 - \left[\frac{dR_0}{dt} K\right] \quad (65.10)$$

where $M_0 = \sum [r_i F_i]$. Indeed, substituting the expression of R_i in the formula $N = \sum [R_i K_i]$ we can write

$$N = \sum [(R_0 + r_i) K_i] = [R_0 \sum K_i] + \sum [r_i K_i]$$

that is

$$N = [R_0 K] + N_0$$

It follows that

$$\frac{dN}{dt} = \left[\frac{dR_0}{dt} K\right] + \left[R_0 \frac{dK}{dt}\right] + \frac{dN_0}{dt}$$

and, by (65.6), this expression must be equal to

$$M = \sum [R_i F_i] = \sum [(R_0 + r_i) F_i] = [R_0 \sum F_i] + \sum [r_i F_i] = [R_0 F] + M_0$$

Now taking into account the equality $F = \frac{dK}{dt}$ we arrive at relation (65.10). It means the following: in the case of a fixed pole O chosen quite arbitrarily we have $\frac{dR_0}{dt} = 0$, and the law describing the variation of the angular momentum has the ordinary form (65.6); in the case of a moving point O relation (65.6) remains valid when $\frac{dR_0}{dt}$ is parallel to K, that is when pole O moves with the velocity of the centre of mass of the system of particles (of the body); finally, the relation $m \frac{dR_0}{dt} = K$ means that the pole O is placed at the centre of mass.

The definition of N_0 was stated on condition that the velocity (and the momentum) are taken relative to a fixed reference frame. However, it can be shown that the sum of the momenta will be equal to the same quantity N_0 if the velocities of the particles are taken relative to a reference system which is in a translatory motion together with the centre of mass.

Let the momentum of the ith particle (in the above mentioned reference system) be k_i; then

$$K_i = m_i \frac{dR_0}{dt} + k_i$$

where $\frac{dR_0}{dt}$ is the velocity of the centre of mass. Consequently,

$$N_0 = \sum [r_i K_i] = \sum \left[r_i \left(m_i \frac{dR_0}{dt} + k_i \right) \right] =$$

$$= \left[\sum m_i r_i \frac{dR_0}{dt} \right] + \sum [r_i k_i] = \sum [r_i k_i]$$

The point O being at the centre of mass, there must be $\sum m_i r_i = 0$. Therefore, the formula (65.7) describing the motion of the centre of mass always holds both for any system of particles and for any rigid body.

66. Gyroscopes

When a body is in a very fast rotation about a given axis and simultaneously rotates about some other axes with comparatively low angular velocities it is possible to determine approximately the direction of the angular momentum, which simplifies considerably the investigation of the motions in such cases.

The physical phenomena taking place in the rotation of tops and gyroscopes usually satisfy the above condition. This makes it possible to elaborate a simple theory for the motion of the gyroscopes.

Let us consider a top. Making the top rotate rapidly about its axis one can observe the amazing stability of the rotation of the top as it spins on a point (on the sharp lower end of its axis). We can take a top spinning on a board and throw it up to watch its stable rotation after it has fallen back on the board (see Fig. 182). These phenomena can be explained on the basis of the law of variation of the angular momentum (formula (65.8)), which will be done later after the analysis of the laws governing the motion of the gyroscopes has been completed. A *gyroscope* is a body symmetric with respect to its axis of rotation (it is usually shaped as a disk) about which it spins with a high angular velocity. The analysis of the motion of a gyroscope is simplified when the gyroscope has one fixed point coinciding with the centre of mass of the disk. This kind of support can be realized practically by using gimbals (a cardan suspension) supporting the gyroscope with the aid of two rings as shown in Fig. 183.

Figure 183 shows a disk-shaped body (a gyroscope) fixed on the axis AA' which can rotate without friction (practically, with as little friction as possible). The axis AA' is supported by the bearings (at the points A' and A' in the figure) attached to the inner supporting ring. The inner ring is in its turn supported by two bearings attached to the outer ring and can rotate about the axis BB'. We shall conditionally refer to the axis BB' as the "horizontal" one (in the course of motion its position may of course change); this axis forms

an angle of 90° with the axis AA' of the gyroscope. Finally, the outer ring can freely rotate about the axis DD' supported by two bearings attached to the post. The axis DD' is perpendicular to the horizontal axis; that is why we shall conditionally call DD' the "vertical" axis. Thus, the intersection point of the three axes AA', BB', and DD' always coincides with the centre of mass of the gyroscope.

The rings are symmetric with respect to their axes and therefore when the disk does not rotate the system is in a state of neutral equilibrium for any

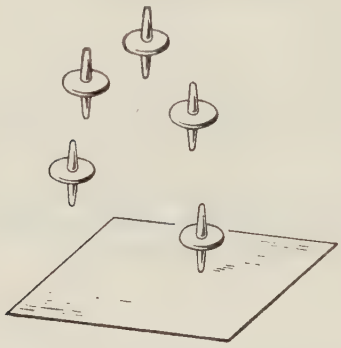

FIG. 182

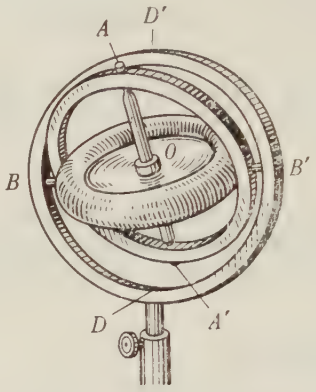

FIG. 183

configuration of the bodies involved since the line of action of the resultant force of gravity always passes through the point of intersection of the three axes. Hence, the gyroscope supported in this way can be regarded as a symmetric body with one fixed point (with one point of support). The axis AA' of the gyroscope can freely turn about the vertical and the horizontal axes and can thus occupy any direction in space. Besides, the disk itself can freely rotate about its axis. In other words, the gyroscope can occupy any position relative to the fixed point O. Such a body is spoken of as a *free* (or two-degree-of-freedom) *gyroscope* (on condition that the forces of friction in the bearings of all the three axes are negligibly small and that the moments of inertia of the rings together with the bearings are also very small).

Applying very little effort we can easily make the disk together with rings rotate about any axis when the disk does not rotate about its own axis. Now let us make the disk rotate about its axis (relative to the supporting system) with a high angular speed (this can be done by pulling a thread wound onto the axis of the gyroscope as shown in Fig. 184). Then the gyroscope receives a certain angular momentum N. The axis of symmetry of the gyroscope retaining its position in space, the direction of the angular momentum N obviously coincides with that of the axis of the disk and its magnitude N is equal to $I\omega$, that is to the moment of inertia I times the angular speed ω.

Now let us take the whole device, turn it in various ways, lift it, and lower it. It is then seen that under all these manipulations the direction of the axis of the gyroscope remains invariable in space. Furthermore, when we strike the inner ring we find that even a considerably strong push does not change the direction of the axis of the gyroscope. The axis behaves as if its direction were fixed. We can also observe that when the gyroscope does not rotate about

its axis the same pushes produce a fast rotation of the axis of the gyroscope together with the ring.

All these facts can readily be explained on the basis of the fundamental law of motion of a body with one point of support. The moments of the friction forces in the bearings being negligibly small and the moment of the gravity force about the point of support being zero, there are no moments of external forces acting on the rotating disk when the whole device is being moved. Therefore, *the vector representing the angular momentum must retain both a constant*

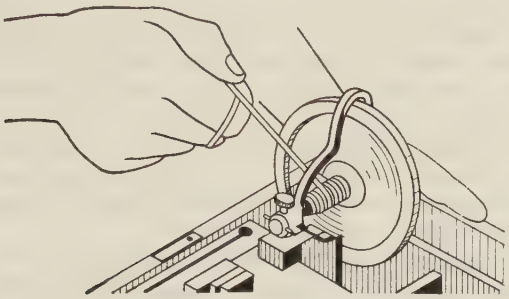

FIG. 184

magnitude and an invariable direction in space. The axis of the gyroscope coincides with the angular momentum in its direction at the initial instant and it continues to coincide with it and to retain its direction in space in the course of motion. The same explanation applies to the case when the top is thrown up and is flying (see Fig. 182). During the flight the top is free, the moment of the gravity force about its centre of mass is equal to zero and the gravity force itself cannot change the character of the rotation of the body. That is why the direction of the axis remains invariable during the flight: the angular momentum of the top retains both its magnitude and its direction in space.

When the inner ring of the gyroscope is struck a moment of forces $\boldsymbol{M}$ is produced during time dt and the increment of the angular momentum during that time is $d\boldsymbol{N}$. But since the gyroscope is in a very fast rotation and possesses an angular momentum with a considerably large magnitude this small increment of the momentum can only result in a turn of the vector $\boldsymbol{N}$ through a very small angle $d\alpha$. In other words, the increment $d\boldsymbol{N}$ of the angular momentum is negligibly small relative to the original value $\boldsymbol{N}$ of the momentum and therefore even when $d\boldsymbol{N}$ is perpendicular to $\boldsymbol{N}$ the angle of the deviation of the axis of the gyroscope from the direction of the angular momentum is very small. Hence, practically the axis remains invariable.

It is quite clear, theoretically, that a free gyroscope can be used for the construction of a compass. If there were no friction in the bearings supporting the axis of the gyroscope and the axes of the rings, the axis of such a gyrocompass would always show a definite direction in space irrespective of the rotation of the Earth about its axis and of the annual motion of the Earth round the Sun. In such a case the axis of the gyroscope would not take part

in the motion of the Earth and would show an invariable direction; for instance, it could be permanently oriented to the Polaris. But since it is practically impossible to eliminate friction completely, the moments of friction forces, however small, eventually change the direction of the axis of the gyroscope and of its angular momentum. That is why a free gyroscope can only be used for the construction of a gyroscopic compass working during some finite time period.

67. Motion of the Axis of a Gyroscope

In this section we investigate the motion of the axis of a rotating gyroscope. The disk of the gyroscope can be in a compound motion representable as a combination of a rotation about the axis of the gyroscope and a motion of the axis itself. In the general case the direction of the angular momentum of the gyroscope must not necessarily coincide with the axis of the gyroscope. But in case the gyroscope is in a very fast rotation about its axis while its rotation connected with the motion of the axis of the disk is comparatively slow it is allowable to assume that the directions of the angular momentum and of the axis of the disk are approximately coincident.

For example, in ordinary demonstrative experiments with a gyroscope its disk rotates about the axis with an angular speed of the order of ten revolutions per second while the angular speed of the rotation of the axis itself is less than one revolution per ten seconds. For such motions the angle of the deviation of the vector representing the angular momentum from the direction of the axis of rotation is less than 1/100 radian, that is less than 1°.

Indeed, as was pointed out in Sec. 64, the angular momentum can always be resolved into components along the principal axes and therefore, as was indicated, the magnitudes of these components can simply be computed as the products of the corresponding moments of inertia by the corresponding components of the angular velocity vector. The axis of rotation of the disk and any two mutually perpendicular axes lying in the plane of the disk are principal axes. Let us resolve the instantaneous angular velocity of rotation $\boldsymbol{\omega}$ into the components $\boldsymbol{\omega}_0$ and $\boldsymbol{\omega}_p$ going, respectively, along the axis of the disk and perpendicularly to it (see Fig. 185). Then, knowing the moment of intertia I_0 of the disk with respect to its axis and the moment I_p with respect to the axis perpendicular to the former we can determine the direction of $\boldsymbol{N}$ and the angle β between the vector $\boldsymbol{N}$ and the axis of rotation (the direction of $\boldsymbol{N}$ coincides with that of $\boldsymbol{\omega}_0$). Obviously, $I_0 \gg I_p$ and therefore $\beta < \arctan \omega_p/\omega_0 \sim 0$. Therefore, the angular deviation of the direction of the angular momentum $\boldsymbol{N}$ from the direction of the axis indicated by the vector $\boldsymbol{\omega}_0$ is practically very small. When neglecting this deviation we introduce a negligibly small error not affecting the essence of the ultimate conclusions. In what follows we shall therefore assume that, practically, $\boldsymbol{N}$ coincides with $\boldsymbol{\omega}_0$ in its direction.

Let us set the axis of the disk horizontally and make the disk rotate very speedily. Then let us suspend a small load from the inner ring of the gyroscope near the axis of the disk (see Fig. 186*a*). It turns out that under the action of the gravity force $\boldsymbol{P}$ of the load the point of suspension of the load does not move downwards but rotates slowly together with the axis of the disk and the outer ring about the vertical axis in the counterclockwise direction. The moment we remove the load the rotation immediately stops. Increasing the weight of the load we observe the corresponding increase of the velocity of the rota-

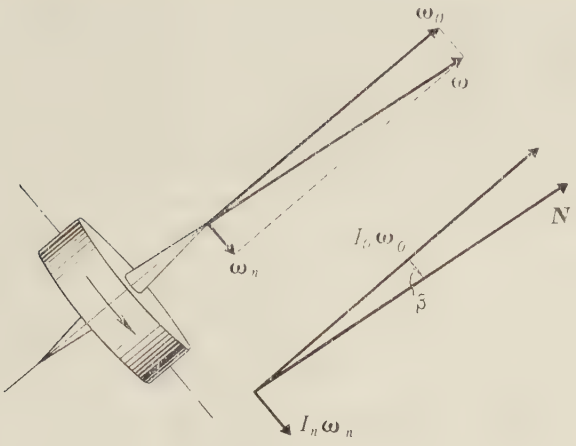

FIG. 185

tion of the ring about the vertical axis. Next, let us remove the load and press down the inner ring with a stick at the point from which the load was suspended. The ring will move counterclockwise as if it were acted upon by the force of weight of the load. If we push the ring upwards (see Fig. 186*b*) it also rotates in the horizontal plane but in the opposite direction.

Thus, although we exert a downward or upward action on the gyroscope it moves in the perpendicular direction in the horizontal plane. Another strange

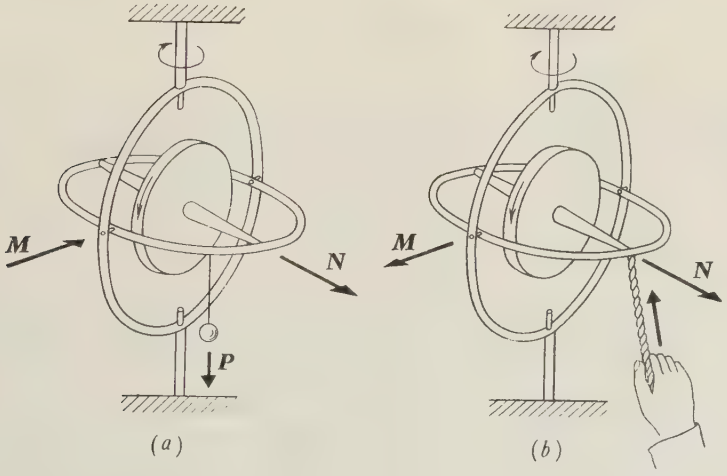

FIG. 186

phenomenon is the absence of "inertia": the moment the external action stops the motion also stops. These facts will be elucidated below.

The rotation of the axis of a gyroscope under the action of an external force, for instance, the rotation of the ring of the cardan suspension of the disk

under the action of the weight of the load is called *precession*. It can easily be explained on the basis of the law of motion of a rigid body expressed by formula (65.6) established earlier. The rotating disk whose axis is in a horizontal position is acted upon by the force of gravity having a non-zero moment about the point of support. The angular momentum of the disk must therefore vary. By formula (65.6), the increment dN of the angular momentum during time dt is equal to $\boldsymbol{M}\,dt$. The moment $\boldsymbol{M}$ of the force of gravity of the load about the point of intersection of the three free axes (the point of support) is represented by a vector lying in the horizontal plane. The angular momentum is also a vector; its direction coincides with that of the axis of the disk lying in the horizontal plane. The increment $d\boldsymbol{N} = \boldsymbol{M}\,dt$ is a vector perpendicular to the angular momentum $\boldsymbol{N}$ (Fig. 187). Consequently, during time dt the angular momentum vector turns through an angle

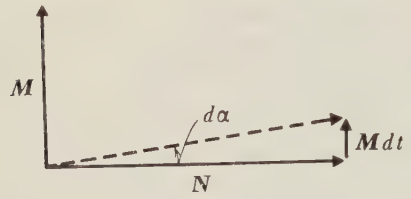

FIG. 187

$$d\alpha = \frac{M\,dt}{N} \tag{67.1}$$

in the horizontal plane and the axis of the disk of the gyroscope turns together with that vector through the same angle $d\alpha$. Formula (67.1) expresses precisely the angle of rotation of $\boldsymbol{N}$ and gives an approximate value of the angle of rotation of the axis of the gyroscope. The moment of the external force acts permanently and is always perpendicular to the angular momentum; therefore, the axis of the disk is in a rotation about the vertical axis with angular speed

$$\Omega = \frac{d\alpha}{dt} = \frac{M}{N} \tag{67.2}$$

Note that here, in contrast to what we are used to, the moment of the external force specifies the angular velocity of the gyroscope and not the acceleration; that is why the removal of the force results in stopping the motion of the axis. The angular velocity Ω of the rotation of the axis is called the *precession velocity*; it is equal to the ratio of the magnitude of the moment of the external force to the magnitude of the angular momentum of the gyroscope.

When the external force is applied so that its moment is not perpendicular to the angular momentum, for instance, when the load is suspended from a point on the inner ring not lying on the axis of rotation of the gyroscope, the precession velocity is determined solely by the component of the moment of the force of gravity $\boldsymbol{P}$ normal to the angular momentum, that is in this case $d\boldsymbol{N} = \boldsymbol{M}_n\,dt$ where $\boldsymbol{M}_n$ is the normal component. The component $\boldsymbol{M}_0$ of the moment going along the axis is balanced by the reaction force exerted by the outer ring.

Thus, the end point of the axis of the gyroscope moves in the direction of the moment of the force and not in the direction of the force itself, which accounts for the "strange" behaviour of the gyroscope.

Similar phenomena can be observed when another type of suspension of a gyroscope shown in Fig. 188 is used. In this case the ring carrying the bearings supporting the axis of the disk of the gyroscope is attached to the rod AB which is (geometrically) the extension of the axis of the disk of the gyroscope. The rod is supported at a point O so that it can rotate about a horizontal axis per-

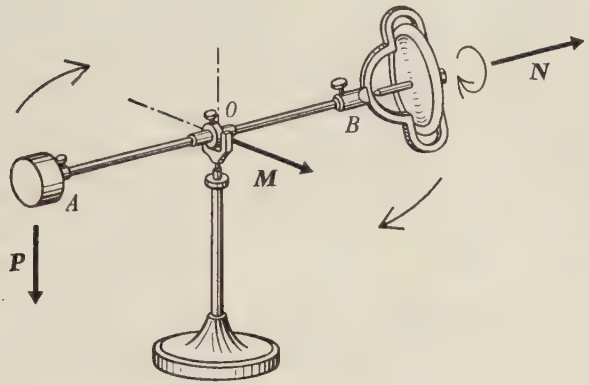

FIG. 188

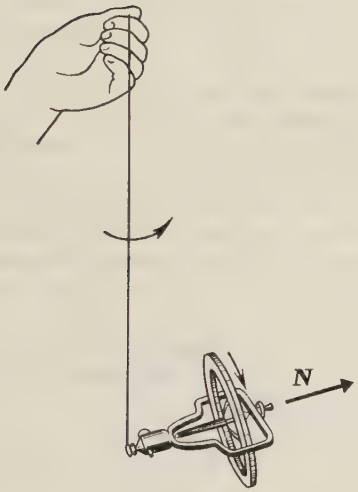

FIG. 189

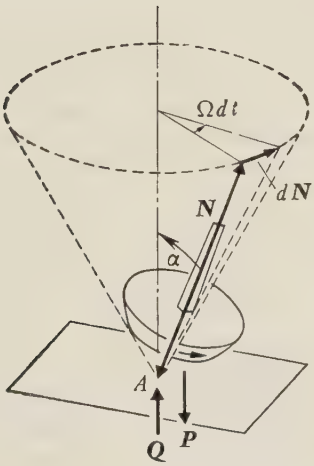

FIG. 190

pendicular to the rod (the vector $\boldsymbol{M}$ in the figure goes in the direction of that axis). This horizontal axis is supported by a semiring which can rotate about the vertical axis relative to the post. All the three axes of rotation meet at the point O and hence the disk of the gyroscope can be in any position in space (at a certain distance from the point O which is supposed to be in a fixed position on the rod). The moment of the force of weight of the gyroscope is balanced by the moment of the gravity force $\boldsymbol{P}$ of the counter weight at the end A of the rod AB.

When this system is unbalanced (i.e. when the counter weight is decreased or increased) there appears a rotating moment about the horizontal axis acting on the disk and producing the precession of the gyroscope. When the balancing weight is decreased the precession is in one direction and when it is increased the precession is in the opposite direction.

The phenomenon of precession can also be demonstrated as follows. Let the ring carrying the gyroscope or the rod to which the ring with the gyroscope is attached be suspended from a thread as shown in Fig. 189 so that the axis of the rotating disk is at an angle to the vertical. If the disk did not rotate the gyroscope would simply "fall" down. In case it is in a rapid rotation there appears precession due to the moment of the force of gravity.

Precession can also be easily shown in an experiment with the top. Strictly speaking, a rotating top always shows a slight precession which, however, is very small when the angular velocity is large and when the top is almost vertical. Indeed, the top (see Fig. 190) is supported at the point A lying below its centre of mass and therefore when the axis of the top departs from the vertical (which practically is always the case) there arises the moment of a couple of forces, namely, of the reaction force $\boldsymbol{Q}$ at the point of support and of the force of weight $\boldsymbol{P}$, which causes precession.

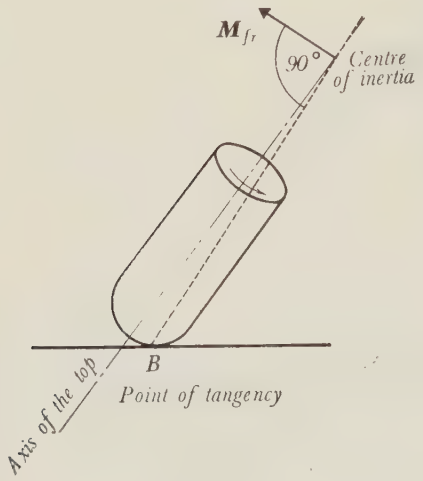

FIG. 191

When the top is in a rapid rotation its precession is such that the axis of the top remains on the lateral surface of a cone with vertex at the point A and the angular momentum vector rotates together with the axis.* Denoting by h the distance from the point A to the centre of mass of the top, by I the moment of inertia of the top and by ω the angular speed of rotation we can write the relation $M = Qh \sin \alpha = Ph \sin \alpha$ for the moment M of the couple. The increment of the magnitude of the angular momentum is $dN = M\,dt = \Omega\,dt\,N \sin \alpha$ where Ω is the precession velocity (see Fig. 190). On replacing N by $I\omega$ we receive the formula for the precession velocity:

$$\Omega = \frac{Ph \sin \alpha}{I\omega \sin \alpha} = \frac{Ph}{I\omega} \tag{67.3}$$

When the angular speed ω of the top is large the precession velocity Ω is negligibly small. As the rotation of the top gets slower (this is always the case due to the friction forces and air resistance) the velocity of the precession becomes larger and can be more easily observed.

It should also be noted that besides the couple of forces indicated above, the top is also acted upon by the moment of the force of friction whose action makes the top tend to the vertical position. This can easily be observed when the top is made to rotate speedily with its axis inclined to the vertical. It can be seen that in some time the axis becomes (almost) vertical. This effect is particulary strong when the lower end of the axis of the top is blunt.

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* This imaginable cone described by the vector N with origin at the point A is shown in dotted line in Fig. 190.

To explain this effect let us consider Fig. 191 where the lower end of the axis of such a top is shown. The vector N is not shown in the figure but it is meant to be directed upwards along the axis. The point B of contact between the end of the axis and the horizontal surface does not lie on the geometrical axis of the top and therefore the friction force applied to that end (it is directed to the reader perpendicularly to the plane of the figure) has a moment M_{fr} about the centre of mass of the top. The vector M_{fr} is in the plane of the figure and is directed towards the vertical. Consequently, the increment $dN = M_{fr}\,dt$ of the angular momentum of the top is also directed towards the vertical, which makes the axis of the top tend to the vertical position perpendicular to the supporting plane.

The inclined top is under the action of two moments: the moment of the couple of forces (the couple is formed by the reaction force of the supporting plane and the force of weight) and the moment of the friction force. If the air resistance is negligibly small the motion of the top is determined by these two moments.

68. Gyroscopic Effect

Experiments with the precession of a gyroscope lead to the following conclusion: since the axis of the gyroscope rotates in a certain direction there must be some forces acting on the gyroscope whose resultant moment is in the direction of motion of the positive end of the gyroscope.*

For instance, the rotating millstone shown in Fig. 170*b* is acted upon by a force exerted by the plane on which it rolls. This force is directed upwards and creates a moment which determines the rotation of the axis of the millstone in the horizontal plane. The millstone itself acts on the plane with the force of the same magnitude and opposite direction. The magnitude of this gyroscopic force can be found from the precession law expressed by formula (67.2). If the angular velocity of the millstone is ω the angular velocity of the rotation about the vertical axis is $\omega\frac{R}{H}$ (Fig. 192). Let us regard the millstone as a homogeneous cylinder of radius R placed at a distance of H from the vertical axis. As will be seen, the force of pressure the millstone exerts on the plane can be much greater than the weight of the millstone in the absence of rotation. This accounts for the usage of such a construction.

The angular momentum of the millstone is always in the plane passing through the axes of rotation and it goes at an angle to the vertical (see Fig. 192). We are only interested in the horizontal component of the angular momentum whose magnitude is

$$N_h = I\omega = 1/2\, mR^2\omega \tag{68.1}$$

where m is the mass of the millstone. It is evident that only the direction of the horizontal component of the angular momentum changes; the vertical component retains both its magnitude N_v and its direction. Therefore, dN is equal to the increment of the horizontal component while the latter is mea-

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* By the positive end of the axis of a gyroscope we mean the one from which the arrow representing the angular momentum vector goes out. The positive directions of that vector and of the vector of angular velocity of rotation coincide (according to the right-hand screw rule mentioned in their definitions).

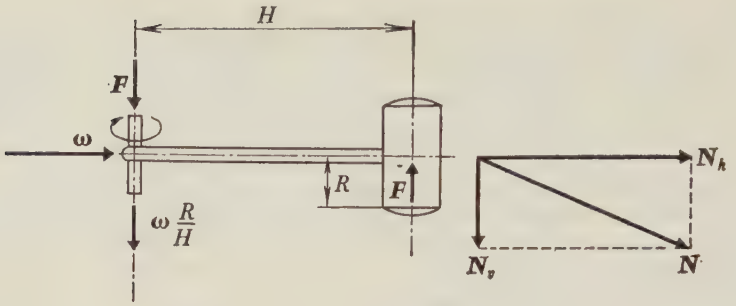

FIG. 192

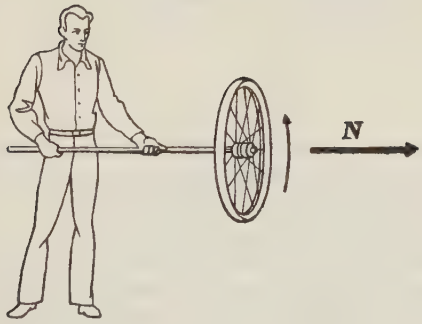

FIG. 193

sured by the change of the direction of the axis of the millstone. Thus

$$dN = N_h \omega \frac{R}{H}\, dt = FH\, dt \qquad (68.2)$$

Substituting the expression of N_h given by (68.1) in (68.2) we obtain

$$\frac{1}{2} mR^2\omega^2 \frac{R}{H} = FH \qquad (68.3)$$

whence the magnitude F of the force $\boldsymbol{F}$ is found:

$$F = \frac{1}{2} m \frac{R^3}{H^2} \omega^2 \qquad (68.4)$$

Consequently, the force of pressure of the millstone on the horizontal plane exceeds its weight by F.

One can readily feel the gyroscopic effect by taking the axis of a rapidly rotating body and trying to turn it in a certain direction (see Fig. 193). Applying a moment of a force to turn the axis one feels as if the axis were tearing away and tending to turn in the perpendicular direction. To compensate for this tendency one should apply considerable effort which increases together with the speed of the turn of the axis.

The same gyroscopic forces act on the bearings of the axes of rapidly rotating parts of a machine when the machine itself is turned. This accounts for the forces acting upon the bearings of a turbine on a ship or in an airplane engine when the ship or the airplane changes the direction of motion. The magnitudes of these forces increase together with the rate of change of the direction of motion.

69. Rotation of the Axis of a Constrained Gyroscope

Let a gyroscope be supported so that it has only one degree of freedom, for instance, let it be able to rotate only about one axis (different from the axis of its rotating disk). Then the reaction of the system to a force trying to turn it about that allowable axis is quite different from its reaction in the previous cases studied in Sec. 68. It turns out that in the case under consideration the gyroscope no longer resists the action of the external force and behaves like an ordinary body; it simply turns in the direction of the force applied to it.

Suppose that the horizontal axis of the gyroscope shown in Fig. 188 is fixed. On being slightly pushed the gyroscope thus suspended rotates about the vertical axis. The explanation of this fact is that in the case under consideration the moment of forces which is needed for the turn of the angular momentum vector is created by the reaction forces in the bearing supporting the vertical axis and not by external forces.

When the gyroscope in Fig. 188 is free and is acted upon by a horizontal force rotating the rod AB in the clockwise direction (we mean the top view) the ring with the gyroscope moves downwards since the external moment is directed downwards. When the horizontal axis is fixed (see the scheme in Fig. 194) such a motion is impossible. For, the bearing of the vertical axis exerts the forces F and F' on the axis and the moment M of these forces is normal to the horizontal axis; this moment results in the displacement of the axis of the gyroscope in the direction of the push.

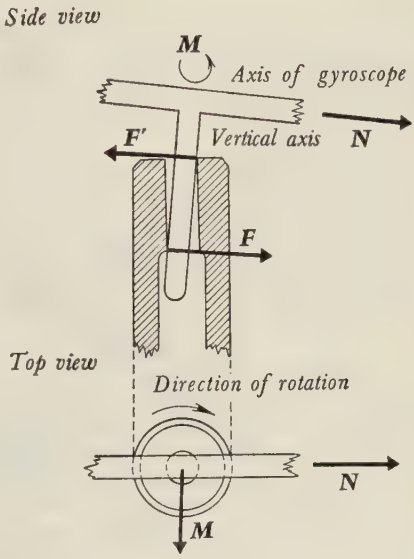

FIG. 194

Let us consider the experiment whose scheme is shown in Fig. 195. Here the vertical axis of the gyroscope is fixed and the post is rigidly connected with the horizontal disk. When the horizontal disk is rotated we observe that the axis of the gyroscope moves so that the direction of the rotation of the disk of the gyroscope tends to coincide with the direction of the rotation of the horizontal disk. The phenomenon is explained quite simply: as the horizontal disk is rotated the post and the vertical axis of the gyroscope transmit to the gyroscope a moment of forces M represented by a vector parallel to the axis of rotation of the horizontal disk (see Fig. 195). This moment can only produce the rotation about the horizontal axis until the direction of the axis of rotation of the gyroscope coincides with the direction of the moment M.

Now let us fix the post with the gyroscope so that the "vertical" axis of the gyroscope is inclined to the axis of rotation of the horizontal disk and let us fix the "horizontal" axis of the gyroscope so that it can only be in rotation about its own axis and about the "vertical" axis. The scheme of the experiment is shown in Fig. 196. In these conditions the gyroscope is acted upon by a moment of forces which is always in the vertical plane passing through the axis of rotation of the horizontal disk. Indeed, the angular velocity ω of the horizontal disk can be resolved into the two components ω_v and ω_h. The rotation of the post with angular velocity ω_v does not affect the motion of the gyroscope since the "vertical" axis is free. The rotation with the angular velocity ω_h creates a moment M whose direction coincides with that of the vector ω_h lying in the above indicated plane.

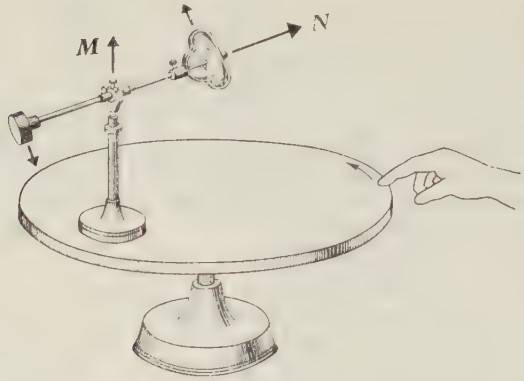

FIG. 195

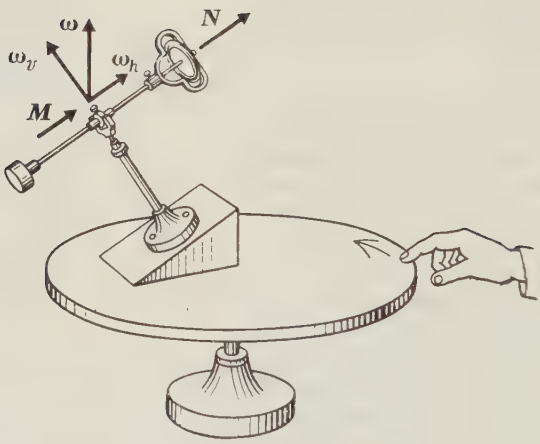

FIG. 196

These simple experiments are visual because when the direction of the rotation of the horizontal disk is changed the gyroscope turns in the opposite direction. They show how the construction of a gyro-compass can be designed.

70. Motion of a Free Gyroscope

In this section we give an example of the solution of a problem on a motion of a free gyroscope by which is meant a gyroscope with the centre of mass as its single point of support.

Let the moment $\boldsymbol{M}$ of the external forces be equal to zero. Then we have

$$\boldsymbol{N} = \text{const} \quad \text{and} \quad \frac{d\boldsymbol{N}}{dt} = 0 \tag{70.1}$$

where $\boldsymbol{N}$ is the angular momentum of the gyroscope. Let us write down the last condition in a reference frame connected with the rotating disk of the gyroscope. To this end we apply the differentiation rule for the vectors in a rota-

ting coordinate system derived in Sec. 48. The derivative of the vector $\boldsymbol{N}$ in (70.1) is taken in an inertial (fixed) reference system. In the rotating system of coordinates connected with the rotating disk of the gyroscope whose angular velocity is $\boldsymbol{\omega}$ we have, according to the general formula (48.10),

$$\frac{(d\boldsymbol{N})_0}{dt} = \frac{d\boldsymbol{N}}{dt} + [\boldsymbol{N}\boldsymbol{\omega}]$$

In the case under consideration this reduces to

$$\frac{(d\boldsymbol{N})_0}{dt} + [\boldsymbol{\omega}\boldsymbol{N}] = 0 \tag{70.2}$$

This equation completely specifies the motion of a freely rotating body with a single point of support at its centre of mass. The vector $\boldsymbol{N}$ is constant in its magnitude and retains its direction in space. However, its position relative to the rotating body may vary and equation (70.2) describes this variation. In the special case when the vectors $\boldsymbol{\omega}$ and $\boldsymbol{N}$ coincide we have $\frac{(d\boldsymbol{N})_0}{dt} = 0$ and the body rotates about the axis of rotation going along the vector $\boldsymbol{\omega}$ and occupying invariable positions in space and relative to the body.

Let us take the free axes of the body as the coordinate axes and rewrite equation (70.2) in terms of the projections on these axes. The reader should remember that for the projections of $\boldsymbol{N}$ on the axes we have

$$N_1 = \lambda_1\omega_1, \qquad N_2 = \lambda_2\omega_2, \text{ and } N_3 = \lambda_3\omega_3$$

where λ_1, λ_2, and λ_3 are the principal moments of inertia, N_1 and ω_1 are the projections of $\boldsymbol{N}$ and $\boldsymbol{\omega}$ on the first principal axis to which corresponds the moment of inertia λ_1, etc. The quantities λ_1, λ_2, and λ_3 are constant in the chosen coordinate system. Therefore

$$\frac{dN_1}{dt} = \lambda_1\frac{d\omega_1}{dt}, \quad \frac{dN_2}{dt} = \lambda_2\frac{d\omega_2}{dt} \quad \text{and} \quad \frac{dN_3}{dt} = \lambda_3\frac{d\omega_3}{dt} \tag{70.3}$$

because

$$\frac{(d\boldsymbol{\omega})_0}{dt} = \frac{d\boldsymbol{\omega}}{dt} \quad \text{and} \quad (d\boldsymbol{N})_0 = dN_1\boldsymbol{n}_1 + dN_2\boldsymbol{n}_2 + dN_3\boldsymbol{n}_3$$

where $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, and $\boldsymbol{n}_3$ are unit vectors in the directions of the principal axes.

Taking into account (70.3) we can rewrite equation (70.2) in the form

$$\left.\begin{aligned} \lambda_1\frac{d\omega_1}{dt} + \omega_2N_3 - \omega_3N_2 = 0 \\ \lambda_2\frac{d\omega_2}{dt} + \omega_3N_1 - \omega_1N_3 = 0 \\ \lambda_3\frac{d\omega_3}{dt} + \omega_1N_2 - \omega_2N_1 = 0 \end{aligned}\right\}$$

that is

$$\left.\begin{aligned} \lambda_1\frac{d\omega_1}{dt} + \omega_2\omega_3(\lambda_3 - \lambda_2) = 0 \\ \lambda_2\frac{d\omega_2}{dt} + \omega_3\omega_1(\lambda_1 - \lambda_3) = 0 \\ \lambda_3\frac{d\omega_3}{dt} + \omega_1\omega_2(\lambda_2 - \lambda_1) = 0 \end{aligned}\right\} \tag{70.4}$$

The equations we have obtained are a special case (for $\boldsymbol{M} = 0$) of the *Euler equation.*

The general analysis of these equations is rather intricate. We will limit ourselves to the case of a free gyroscope shaped as a solid of revolution. In this

case two principal moments of inertia of the gyroscope are equal. For definiteness, let $\lambda_2 = \lambda_3$. Then equations (70.4) assume the form

$$\frac{d\omega_1}{dt} = 0, \quad \frac{d\omega_2}{dt} + \omega_3\omega_0 = 0 \quad \text{and} \quad \frac{d\omega_3}{dt} - \omega_2\omega_0 = 0 \tag{70.5}$$

where $\omega_0 = \omega_1 \frac{\lambda_1 - \lambda_2}{\lambda_2}$.

The first of these equations implies $\omega_1 = \text{const}$, that is the projection of the vector $\boldsymbol{\omega}$ on the first coordinate axis remains invariable. Hence the quantity ω_0 is constant and the remaining two simultaneous equations can easily be solved. By the direct substitution we can readily verify that

$$\omega_2 = A \cos(\omega_0 t + \varphi) \text{ and}$$
$$\omega_3 = A \sin(\omega_0 t + \varphi) \tag{70.6}$$

are the solutions (A and φ are constants dependent on the initial conditions).

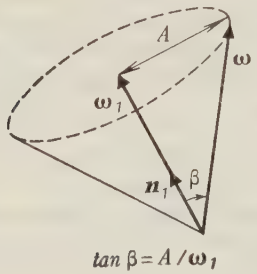

FIG. 197

The analysis of the solution we have obtained indicates that the vector $\boldsymbol{\omega}$ remains invariable in its magnitude and is in a motion on the surface of a cone (Fig. 197). It rotates about the axis of symmetry of the body with angular speed ω_0. The expression $A = \sqrt{\omega_2^2 + \omega_3^2}$ depends on the initial conditions while the angle β specifying the shape of the cone is determined by the ratio A/ω_1. The ratio ω_0/ω_1 depends solely on the ratio of the principal moments of inertia, that is on the distribution of the mass of the gyroscope.

To visualize the motion of the axis of the gyroscope it is expedient to pass from the rotating coordinate system to a fixed system of reference. This can be performed in a simple way as follows.

Let us use formula (64.14). We always have

$$\boldsymbol{N} = \lambda_1\omega_1\boldsymbol{n}_1 + \lambda_2\omega_2\boldsymbol{n}_2 + \lambda_3\omega_3\boldsymbol{n}_3$$

and in the case under consideration $\lambda_3 = \lambda_2$. Therefore

$$\boldsymbol{N} = \lambda_1\omega_1\boldsymbol{n}_1 + \lambda_2(\omega_2\boldsymbol{n}_2 + \omega_3\boldsymbol{n}_3) + \lambda_2\omega_1\boldsymbol{n}_1 - \lambda_2\omega_1\boldsymbol{n}_1 =$$
$$= (\lambda_1 - \lambda_2)\,\omega_1\boldsymbol{n}_1 + \lambda_2\boldsymbol{\omega} \tag{70.7}$$

It follows that the vectors $\boldsymbol{n}_1$ (going along the axis of the body), $\boldsymbol{N}$, and $\boldsymbol{\omega}$ lie in one plane. Taking this into account and using equality (70.7) we can determine the motion of the axis of the gyroscope. To this end we rewrite (70.7) as

$$\boldsymbol{\omega} = \frac{\boldsymbol{N}}{\lambda_2} - \omega_0\boldsymbol{n}_1 \tag{70.8}$$

where $\frac{\boldsymbol{N}}{\lambda_2} = \boldsymbol{\Omega}$ is the component of the angular velocity along the vector $\boldsymbol{N}$. The vector $\boldsymbol{\Omega}$ retains its magnitude and direction. Since the moduli of the vectors $\boldsymbol{\omega}$ and $-\omega_0\boldsymbol{n}_1$ are constant and the vectors themselves lie in a plane passing through the vector $\boldsymbol{\Omega}$, they rotate with angular velocity $\boldsymbol{\Omega}$ and the angle between the vectors remains constant. Therefore, the vector $\boldsymbol{\Omega}$ repre-

sents the *precession velocity* of the gyroscope. The precession phenomenon is shown in Figs. 198 and 199 for the cases $\lambda_1 > \lambda_2$ and $\lambda_1 < \lambda_2$, respectively.

Suppose that the angle β between ω and $\boldsymbol{n}_1$ (the latter vector goes along the axis of the gyroscope) is known. Then we find $\omega_0 = \frac{\lambda_1 - \lambda_2}{\lambda_2}\omega_1$. It should be noted that in the case $\lambda_1 > \lambda_2$ (Fig. 198) we have $\omega_0 > 0$ and therefore the angular velocity of rotation $-\omega_0 \boldsymbol{n}_1$ of the gyroscope about its axis is opposite to the precession velocity Ω. The axis of the gyroscope is in this case in preces-

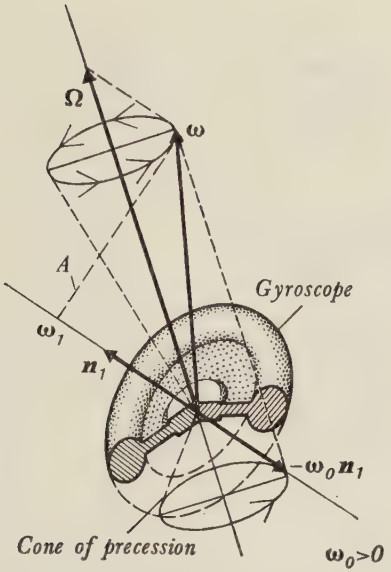

FIG. 198

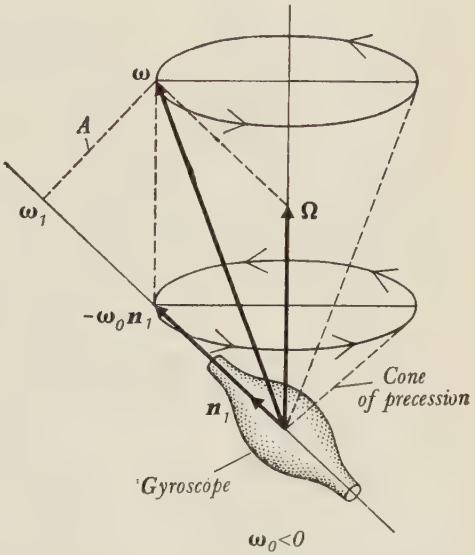

FIG. 199

sion whose direction is opposite to the rotation of the gyroscope itself. The condition $\lambda_1 > \lambda_2$ means that the body in question has a shape flattened in the direction of its symmetry axis, which is clearly seen in the figure.

Figure 199 demonstrates another case when the body is elongated in the direction of its symmetry axis: $\lambda_1 < \lambda_2$. In this case the precession is in the direction of the rotation of the body about its axis since $\omega_0 < 0$. The angle β (or the quantity A) depends on the initial conditions.

Such a rotation of the axis of a gyroscope about the direction of the angular momentum vector $\boldsymbol{N}$ in the surface of a cone is called *regular* precession in contrast to *pseudoregular* precession occurring under the action of a constant moment of the gravity force. When the rotation of a gyroscope about its own axis is very fast (this was assumed in the foregoing section) the character of the phenomenon of presudoregular precession is almost undistinguishable from that of the phenomenon of regular precession when these phenomena are observed in an ordinary manner. In the general case the pseudoregular precession involves small oscillation of the axis of the gyroscope on the surface of the precession cone (this is called the *nutation phenomenon*). However, oscillation can hardly be observed when the angular speed of the rotation of the gyroscope is high.

It should be noted that a gyroscope can also move without any precession when $A = 0$ and the directions of N and ω coincide. There can be no precession for a ball-shaped gyroscope because in this case we always have $N = \lambda\omega$.

71. On Gyroscopic Forces

The analysis of the gyroscopic phenomena presented in the foregoing sections was based on the general law of motion of a rigid body (deduced in Sec. 65)

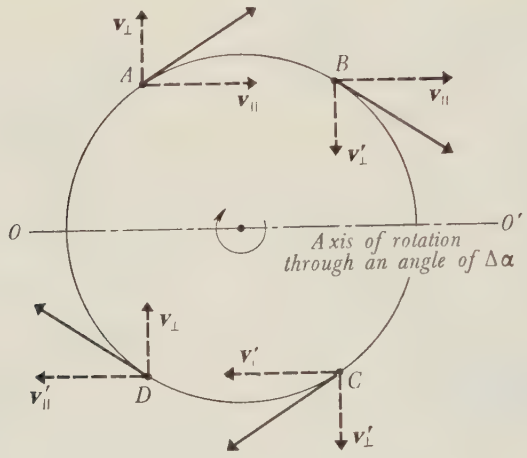

FIG. 200

which is expressed by the formula $M = \frac{dN}{dt}$. Using this law we managed to explain all the peculiarities of the behaviour of a gyroscope. But it is also possible to consider directly the accelerations of the particles of a rotating body in order to find what forces act on them; such forces are usually referred to as gyroscopic.

Let the axis of a rotating disk turn through an angle $d\alpha$ during time dt. Let us find the accelerations of various points of the disk in this motion. We shall consider the four typical points A, B, C, and D indicated in Fig. 200 which are symmetric, at time t, with respect to the axis OO' about which the axis of the disk turns. Let us resolve the velocities of each of the four points into two components: parallel to the axis OO' and perpendicular to it. The parallel components of the velocities of the points A and B have equal values $v_{\parallel}$ and the parallel components of the velocities of the points C and D are equal to the same vector $v'_{\parallel}$. The perpendicular components of the velocities of the points A and D have equal values $v_\perp$ and the perpendicular components of the velocities of the points C and B are equal to $v'_\perp$. Besides the parallel and perpendicular components lying in the plane of the disk, each of the points has a component normal to the plane of the disk which appears because of the rotation about the axis OO'. Let us suppose that the rotation about the axis OO' is much slower than the rotation about the axis of the disk. Then, for the sake of simplicity, we can neglect the components normal to the plane of the disk.

We now consider the increments of the velocities during time dt. All the points change their positions during that time interval dt: they gain some displacements in the direction of the rotation of the disk along the corresponding arcs of the circle; besides, because of the rotation about the axis OO' they leave the plane of the figure: the points A and B move towards the reader while the points C and D move in the opposite direction (see Fig. 200).

Let us consider the corresponding increments of the components of the velocity of each point. For the points A and B the components $v_{\parallel}$ receive the incre-

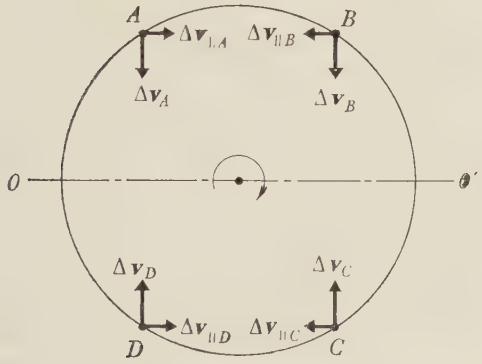

FIG. 201

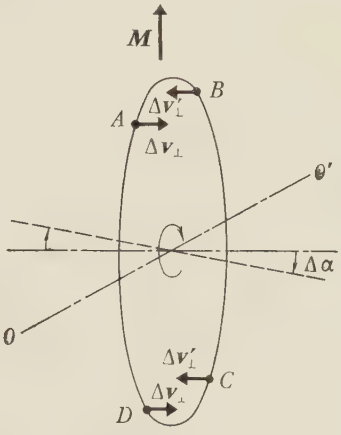

FIG. 202

ments $\Delta v_{\parallel A}$ and $\Delta v_{\parallel B}$, respectively, lying in the plane of the disk and having equal magnitudes and opposite directions (Fig. 201), which follows from the symmetry of the disk. The same applies to the points C and D. The increments of the components $v_{\perp}$ and $v'_{\perp}$ are not in the plane of the disk. Let us denote the components of these increments in the direction normal to the disk as $\Delta v_{\perp}$ and $\Delta v'_{\perp}$ (Fig. 202), the components of the increments parallel to the plane of the disk as Δv_A, Δv_B, Δv_C, and Δv_D (see Fig. 201). All the components of the increments can be divided into pairs of equal vectors, which follows from the symmetry of the configuration of the points A, B, C and D. Therefore, the sum of the components of the increments lying in the plane of the disk is equal to zero.

The magnitudes of the increments can easily be determined. The increments of the velocity lying in the plane of the disk (Fig. 201) correspond to the uniform motion of the points in the circle. Therefore, the sum of these increments specifies the centripetal acceleration of the given point. Only the perpendicular components $v_{\perp}$ and $v'_{\perp}$ change their directions because of the rotation of the disk about the axis OO'; therefore, the magnitudes of the increments of the velocities of the points A, B, C, and D in the direction normal to the disk are equal to $\Delta v_{\perp} = \Delta\alpha\, v_{\perp}$.

The components of the increments of the velocities normal to the disk (see Fig. 202) are of equal magnitudes; their directions are opposite for the points A, D and B, C. The same applies to any other four points. Consequently, all the

points of the disk must have accelerations whose directions coincide with the directions of the increments of the velocities shown in Fig. 202. The disk must therefore be subject to the action of a couple of forces whose moment is perpendicular to the axis of the disk and to the axis OO'. This allows us to draw the reverse conclusion: if a rotating disk is acted upon by a couple of forces its axis rotates so that the general relation

$$\boldsymbol{M} = \frac{d\boldsymbol{N}}{dt}$$

is fulfilled for which we assume that the axis of the disk and the vector $\boldsymbol{N}$ have the same direction.

CHAPTER 8

Rolling Friction

72. Forces of Rolling Friction. Forces of Sliding Friction in Rolling Motion of a Cylinder

In physics and engineering an important role is played by forces appearing when a wheel (a cylinder) is rolling on a plane. Generally, these forces are called friction forces. To describe the phenomenon in more detail we distinguish between the three types of friction forces which are due to *rolling*, *sliding*, and *adhesion* of wheels. Forces of rolling friction, like friction forces in a translatory motion, always exist and always decelerate the motion. Forces of sliding friction and adhesive forces appearing in a rolling motion can both accelerate and decelerate the rolling body. Like forces of friction of rest, adhesive forces must not necessarily be connected with transformation of mechanical energy into heat.

Let a homogeneous cylinder roll with sliding down inclined plane. Then, generally speaking, all the three types of friction forces are present. In the absence of sliding only forces of rolling friction and adhesive forces act; in many cases the force of rolling friction is very small and then only the adhesion should be taken into account (which was done in Sec. 58). When a rolling motion is accompanied by the action of adhesive forces alone, that is when the point of contact of the cylinder and the plane does not slide relative to the plane, there are no losses of mechanical energy.

Let a cylinder roll on a horizontal plane without rolling friction so that the axis of the cylinder is perpendicular to the velocity of motion. If the cylinder rolls without sliding and its motion is uniform then obviously there is no tangential force of interaction between the cylinder and the plane (it is assumed of course that there is no air resistance).

Now imagine that we take a cylinder rotating about its axis with angular velocity ω_0, put it on a horizontal plane and let it move freely the moment it touches the plane (Fig. 203). Then the cylinder rolls on the plane and the observation shows that at the beginning of its motion there is an acceleration along the direction of motion. This means that the cylinder is acted upon by an external force. This force is exerted by the horizontal plane and the magnitude of the force is equal to the mass of the cylinder times the acceleration of the points lying on the axis of the cylinder. The initial value of the linear velocity of the cylinder is equal to zero and the linear velocity increases as the motion of the cylinder starts whence it follows that during that period of motion sliding takes place. The force of sliding friction thus accelerates the linear velocity of the axis of the cylinder and decelerates its rotation about the axis. Hence, the velocity v of the translatory motion increases while the angular velocity ω decreases. The velocity v_{sl} of the sliding of the points of the cylinder adjoin-

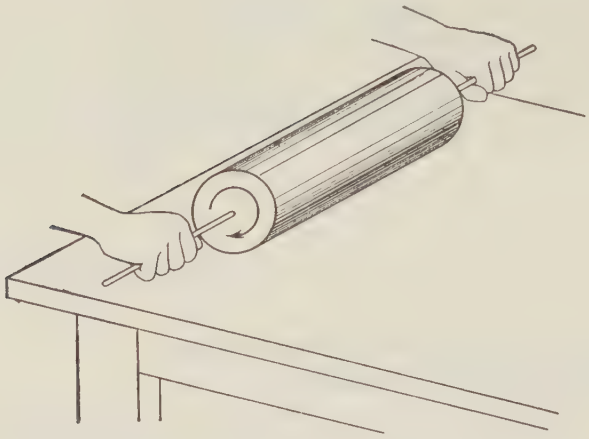

FIG. 203

ing the plane decreases in its modulus because $\omega R > v$ and

$$v_{sl} = \omega R - v$$

where R is the radius of the cylinder. The moment the velocity v_{sl} turns into zero the force of sliding friction stops acting and, beginning with that moment, the motion becomes uniform.

The motion of the cylinder can be investigated quantitatively if we assume that the force of sliding friction in the phenomenon approximately obeys the

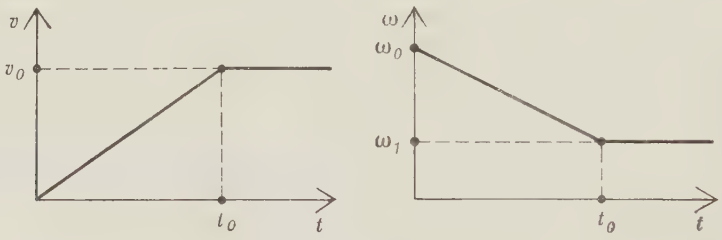

FIG. 204

Coulomb law (see Sec. 42) and is equal to $f_0 = \mu N$ where N is the weight of the cylinder. The moment the cylinder is put on the plane and allowed to move the accelerating force is equal to f_0 and it retains the same value during the motion. Therefore, the linear acceleration of the cylinder is determined by the formula

$$a = \frac{f_0}{m} \tag{72.1}$$

where m is the mass of the cylinder. The cylinder simultaneously possesses a negative angular acceleration $\beta = -|\beta|$ with

$$|\beta| = \frac{f_0 R}{I} \tag{72.2}$$

where I is the moment of inertia of the cylinder with respect to its symmetry axis. Consequently, the velocity of the axis of the cylinder increases according to the linear law

$$v = at = \frac{f_0}{m} t \tag{72.3}$$

while the angular velocity decreases linearly:

$$\omega = \omega_0 - |\beta| t = \omega_0 - \frac{f_0 R}{I} t \tag{72.4}$$

where ω_0 is the initial angular velocity of the cylinder at initial instant $t = 0$ when the rotating cylinder is put on the plane.

It is evident that there is an instant $t = t_0$ ($t_0 > 0$) at which the linear velocity v becomes equal to ωR. At that instant the sliding disappears and, consequently, the force of sliding friction stops acting on the cylinder. That is why after the time instant $t = t_0$ the cylinder rolls without sliding, its axis moving uniformly with velocity

$$v_0 = \frac{\omega_0 R}{1 + \frac{mR^2}{I}} \tag{72.5}$$

Formula (72.5) can readily be obtained by eliminating t from expressions (72.3) and (72.4) and putting $\omega R = v_0$. The magnitude t_0 is found from formulas (72.3) and (72.5):

$$t_0 = \frac{mv_0}{f_0} = \frac{m\omega_0 R}{f_0 \left(1 + \frac{mR^2}{I}\right)} \tag{72.6}$$

The time diagrams for the linear and angular velocities are shown in Fig. 204. The friction force f_0 acts only from $t = 0$ to $t = t_0$ when sliding takes place. For $t > t_0$ the cylinder rolls uniformly without sliding.

It should be noted that in the phenomenon we have discussed a part of the initial kinetic energy of the rotating cylinder is transformed into heat. Let the reader show that this part of the initial energy is equal to

$$\frac{mR^2 I}{(I + mR^2)^2}$$

We stress once again that at the initial time $t = 0$ there must appear sliding. The uniform motion of the cylinder can only start without sliding at the initial time when there is an impact of the cylinder against the plane at the initial instant, the force of the impact being infinite (more precisely, this force must be sufficiently large). Such an impact can take place when the surface of the plane has some irregularities. After the impact the points of the cylinder contacting the plane stop moving (relative to the plane) under the action of the impulse $P = \int f\, dt$ of the force f exerted by the plane on the cylinder at the points of contact. The impulse P of the force f gives the cylinder a momentum of magnitude $mv = P$ and, besides, decelerates the rotation of the cylinder. The corresponding decrement $\Delta\omega$ of the angular velocity ω can be found from the condition $I\Delta\omega = RP$. After the impact the cylinder is in a pure rolling motion on the plane and $v = (\omega - \Delta\omega) R$. On eliminating P and $\Delta\omega$ we find

$$v \quad \frac{\omega_0 R}{1 + \frac{mR^2}{I}}$$

When the velocity of sliding v_{sl} is directed backwards (i.e. $\omega R > v$) the force of sliding friction is directed forwards; when the velocity v_{sl} is directed forwards the force of sliding friction is directed backwards ($\omega R < v$). In the latter case the friction force decelerates the motion of the cylinder.

Let us make the cylinder roll along the plane with an initial velocity v_0 so that it does not rotate at the initial instant. Then the force of sliding friction decelerates the translatory motion. The velocity v of the translatory motion decreases and the angular velocity ω of the rotational motion increases until ωR becomes equal to v after which a uniform motion without sliding starts.

In a rolling motion without sliding there may appear adhesive forces. In the examples we have discussed there are no such forces only because the cylinder is not acted upon by horizontal external forces. When a cylinder rolling along a plane without sliding is under the action of an external force the corresponding force of adhesion appears. Consequently, adhesive forces, like forces of friction of rest, are specified by the magnitude of the external forces acting on the moving body.

73. Adhesion of Wheels

When an automobile or a locomotive moves and its wheels roll without sliding an important role is played by the forces produced by *adhesion of wheels*. In a rolling without sliding the contacting surfaces of the wheels and of the road or rails do not move relative to each other and there may appear an adhesive force. As in the case of friction of rest, this force does not exceed a definite maximum value f_0. The magnitude f_0 is proportional to the force N with which the wheel presses the supporting surface. The dependence of the adhesive force on the force N can be of a more complex character for a moving automobile because in this case the pressure results in a deformation of the automobile tyres.

Let us consider the scheme of the motion of a self-propelling vehicle (or an automobile) shown in Fig. 205*a*. The driving wheel is under the action of a rotational moment of magnitude $M = f \cdot b$ (Fig. 205*b*) produced by the engine and transmitted to the wheel through the body of the vehicle. The body of the vehicle is in its turn acted upon by a moment of magnitude $M' = f' \cdot b$ directed opposite to M and, by the third law of dynamics, $f = f'$. Since the wheel rolls without sliding the forces applied to it determine its acceleration in accordance with the equalities

$$M - f_{ad}R = I\beta, \quad f_{ad} - f_1 = ma \quad \text{and} \quad a = \beta R \tag{73.1}$$

where f_1 is the force exerted by the body of the vehicle on the axis of the wheel (see Fig. 205*b*), m is the mass of the wheel, f_{ad} is the adhesive force applied to the wheel, I is the moment of inertia of the wheel, and R is its radius. These equations make it possible to establish the relationship between the force f_1 and the moment M:

$$f_1 = \frac{M}{R} - \left(m + \frac{I}{R^2}\right) a \tag{73.2}$$

The force applied to the body of the vehicle is equal to f_1 in the magnitude and opposite in the direction. Hence, f_1 is the magnitude of the driving force producing the motion of the vehicle. It is less than the adhesive force f_{ad} by the pro-

duct ma. When the motion is uniform, that is $a = 0$, formula (73.1) results in

$$f_1 = f_{ad} = \frac{M}{R} \tag{73.3}$$

Consequently, for a uniform motion the driving force f_1 is equal to the adhesive force f_{ad} and its magnitude depends solely on the moment M produced by the engine and on the radius R of the wheel. Formulas (73.1), (73.2) and (73.3) have been derived under the assumption that $f_{ad} < f_0$, which means that there is

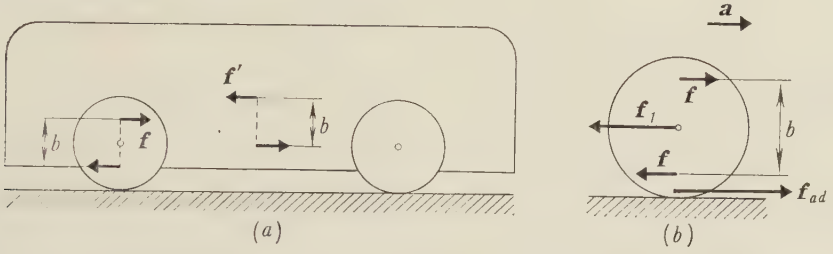

FIG. 205

no sliding of the wheel on the supporting surface. If otherwise, the relation $\omega R = v$ does not apply and, consequently, the equality $\beta R = a$ does not hold either. The external force applied to the driving wheel cannot exceed f_0. When the moment M produced by the engine increases so that the force exceeds that value f_0 sliding sets in. In this case the first two equalities (73.1) remain valid while the last one (i.e. $a = \beta R$) no longer holds; for the case of sliding we have $a < \beta R$. For instance, when $a = 0$ and $\beta > 0$ a part of the rotational moment is expended on the increase of the angular velocity of the wheel but not on the increase of the driving force.

When an automobile starts moving on a slippery road with a sharp increase of the rotational moment M produced by the engine the *slipping* of the wheels may occur. In this case the wheels rotate rapidly and slip while the driving force is small and the automobile does not move or starts moving very slowly. The sharp decrease of the driving force is accounted for by the fact that the magnitude of the force of sliding friction decreases as the velocity of sliding increases. That is why in such cases the moment M should be increased gradually as the automobile starts moving.

In all these considerations we have supposed that the rotational moment M is directed so that it gives the automobile a forward acceleration. When the moment produced by the engine changes sign during the motion the above equations remain valid with the only distinction that the signs of the acceleration a and of the driving force f_1 change. It is evident that in this case the force f_1 is directed backwards and decelerates motion of the automobile.

74. Braking and Skidding

When the brakes of an automobile are applied, the brake shoes (Fig. 206) exert friction forces f' acting on the wheels. These forces produce braking couples with moments M_{br} acting on the wheels and decelerating the automobile.

The moment M_{br} satisfies equations analogous to (73.2) and involving the decelerative force f_d (provided that there is no sliding of the wheels). As has been said, when the automobile starts the rotational moment produced by the engine should be increased gradually in order to avoid the slipping of the wheels and the decrease of the driving force. Similarly, when the brakes are applied, the moment M_{br} of the braking couple must not be increased sharply.

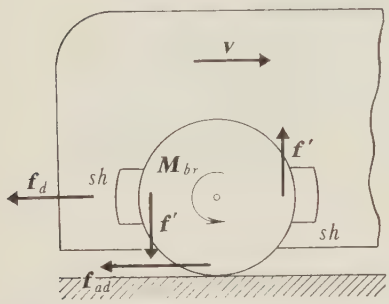

FIG. 206

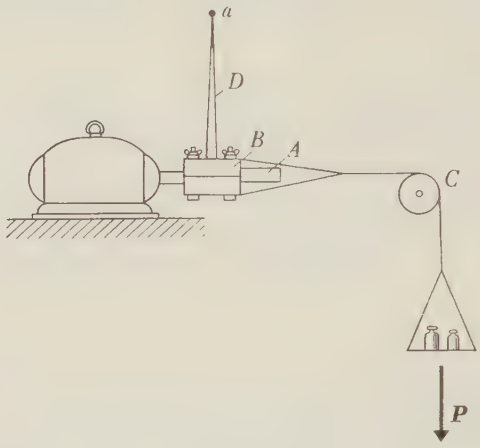

FIG. 207

The sliding of the wheels of an automobile which arises when the brakes are applied may be dangerous because of the phenomenon of *skidding* which leads to the loss of control. Skidding often occurs when the brakes of an automobile are applied on a slippery road. This can also be explained by a specific dependence of the forces of sliding friction on the velocity and the direction of the motion.

When a body slides on a horizontal plane in the direction of its velocity v it turns out that even a very small force f acting perpendicularly to v may produce a considerable displacement of the body in the direction of f. The matter is that a body sliding in a certain direction can produce only a very small friction force as it is displaced in the perpendicular direction. This phenomenon is due to the absence of adhesion between the surfaces of the body and of the plane because the body moves relative to the plane. This effect can be visualized in an experiment suggested by S. E. Khaikin. The scheme of this experiment is shown in Fig. 207.

In the figure we see an electric motor with a clutch B on its axle A. A load of weight P is suspended from a thread which is passed around the pulley C and is attached to the clutch. The force of weight of the load stretches the thread and tends to move the clutch B away from the axle. The clutch consists of two halves pressed together with the aid of screws. On regulating properly the tightening of the screws we can determine the weight P of the load which can remove the clutch from the axle of the electric motor when the axle does not rotate. The clutch is prevented from rotating together with the axle by the lever D attached to the clutch; the other end of the lever is fixed at the point a with the aid of another thread stretched perpendicularly to the plane of the figure. Then the motor is started and the axle rotates; it turns out that the clutch moves along

the axle as the latter rotates under the action of a considerably smaller force (approximately of magnitude of 0.1 P and even less).

This phenomenon of the decrease of the force of sliding friction can be explained rather simply if we take into account that the force of sliding friction is always directed opposite to the velocity of sliding. When the clutch moves along the axle with a velocity v_1 the resultant velocity of the sliding of the particles on the surface of the axle with respect to the clutch is directed at an angle of a small magnitude α to the plane perpendicular to the axle (see Fig. 208). Consequently, the friction force f_0 is directed at the angle α to that plane. Only a small component f_1 of the force f_0 acts in the direction of the axle:

$$f_1 = f_0 \tan \alpha \approx f_0 \alpha \approx f_0 \frac{v_1}{v} \ll f_0 \qquad (74.1)$$

because usually we have $v_1 \ll v$.

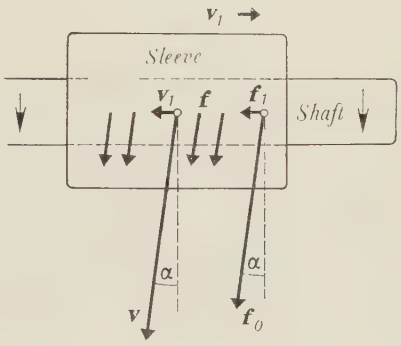

FIG. 208

We now can explain the skidding of an automobile on a slippery road when the brakes are applied sharply. The brake shoes stop the rotation of the wheels and the automobile starts sliding forwards by inertia. A small irregularity on the road or a slight difference in the friction forces acting on different wheels may produce a side force or a moment of forces with respect to the vertical or a lateral displacement of the automobile. For, there are no friction forces (or they are too small) that can prevent the wheels of the automobile from a motion perpendicular to the direction in which the automobile moved before. This is of course a simplified explanation, which however describes correctly the main features of the phenomenon.

75. Rolling Friction

In the foregoing sections we did not discuss the forces of rolling friction. We only dealt with adhesive forces in a rolling motion (analogous to the force of friction of rest) and with forces appearing when a cylinder rolls on a plane with sliding (these forces are analogous to the force of sliding friction between plane surfaces). An important distinction between them is that an adhesive force performs no work (its action is not accompanied by a transformation of mechanical energy into heat) while the force of sliding friction always performs some work connected with energy transformation from the mechanical form into heat.

When a cylinder rolls on a plane without sliding there are always energy losses, that is some amount of mechanical energy goes into heat. The velocity of such a cylinder rolling on a smooth horizontal plane without sliding gradually decreases and finally the cylinder stops. Since the forces of air resistance, which are always present, are very small we conclude that there must be some other force decelerating the cylinder. This is the force of rolling friction whose magnitude depends on the properties of the materials of the cylinder and of the plane.

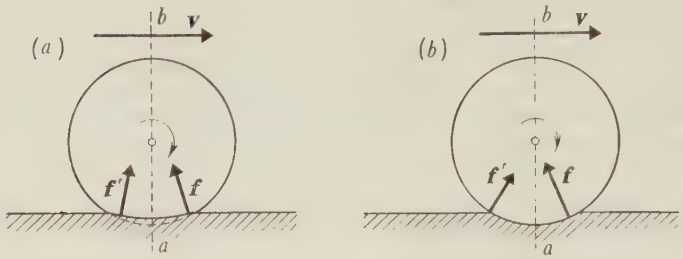

FIG. 209

When the cylinder rolls the plane undergoes a deformation (see Fig. 209*a*) produced by the action of the force pressing the cylinder to the plane. If these deformations are perfectly elastic the interaction forces between the cylinder and the plane must be symmetric with respect to the vertical plane *ab* passing through the axis of the cylinder. To each such force $\boldsymbol{f}$ there corresponds a force $\boldsymbol{f}'$ of the same magnitude which is symmetric to $\boldsymbol{f}$ and whose point of application is symmetric to that of the force $\boldsymbol{f}$. (Here we neglect the forces of sliding friction appearing because of the relative motion of the cylinder and the plane.*)

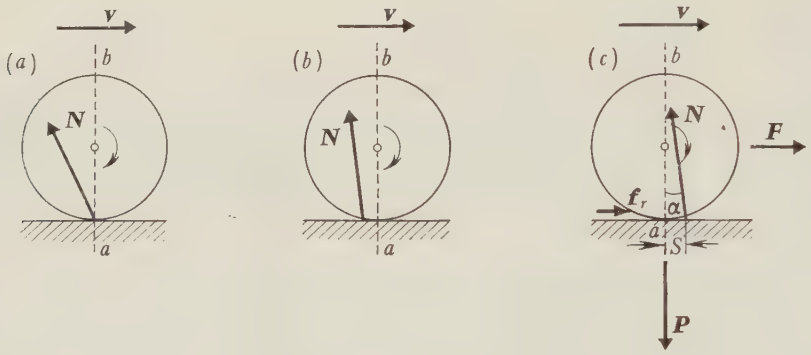

FIG. 210

Because of the symmetry, the resultant of all the forces of the elastic deformation of the supporting surface must be vertical and the sum of the moments of the deformation forces with respect to the axis of the cylinder is equal to zero. Therefore, the forces of elastic deformations of the cylinder and the plane cannot affect the velocity of the rolling motion, that is the motion is such as if there were no deformations at all. This means that no forces of rolling friction arise in such a case.

Thus to explain the appearance of the forces of rolling friction we should assume that the deformation of the cylinder and of the plane is non-elastic, which in fact is always the case.

* Generally speaking, the contacting areas of the cylinder and of the plane always move slightly relative to each other so that the force of sliding friction, however small, inevitably appears.

For the sake of simplicity we assume that the cylinder is non-deformable while the surface of the plane undergoes non-elastic deformation and has some residual deformation. (For the ultimate result of our investigation it is unimportant which of the bodies is in fact deformed, that is the cylinder or the plane or both.) For the model we have chosen it is quite clear that the forces of the action of the plane upon the cylinder are no longer symmetric with respect to the plane *ab* (see Fig. 209*b*): the force f must be of greater magnitude than the force f' applied to the symmetric area lying behind the plane *ab* (relative to the direction of the motion). The resultant force of these deformation forces must therefore have a non-zero horizontal component directed backwards and the sum of the moments of these forces with respect to the axis of the cylinder is also different from zero. The direction of the resultant moment is opposite to that of the direction of the rotation of the cylinder.

An exhaustive theory of rolling friction has not yet been elaborated because the laws describing the forces resulting from variable complex non-elastic deformations have not been investigated thoroughly. That is why we cannot present the exact calculation of forces of rolling friction.

However, taking into account that a cylinder rolling on a horizontal plane without sliding is gradually decelerated, we can draw some qualitative and quantitative conclusions concerning the direction and the magnitude of the friction force exerted on the rolling cylinder by the horizontal plane under the assumption that the area of contact is very small in comparison with the radius of the cylinder.

Let us suppose that the air resistance is negligibly small and can be neglected so that the deceleration of the cylinder can be thought of as being due only to the forces of rolling friction. The cylinder has a negative linear acceleration a and a negative angular acceleration β. These quantities are connected by the condition $a = \beta R$ expressing the absence of sliding. First of all we note that the resultant of all forces acting on the cylinder departs from the vertical in the backward direction (relative to the direction of motion) since the linear acceleration of the cylinder is negative. Now let us determine the position of that resultant force relative to the centre of the cylinder.

The point of application of the resultant force can neither be in the vertical plane *ab* passing through the centre nor behind it, that is the configurations shown in Fig. 210*a* and *b* are impossible. For, if otherwise, the resultant force would give the cylinder a positive acceleration. Consequently, the only possibility is the one shown in Fig. 210*c*: the point of application of the resultant force $\boldsymbol{N}$ must be before the cylinder and the line of action of the force $\boldsymbol{N}$ must intersect the vertical passing through the centre at a point lying above the centre of the cylinder. Indeed, if otherwise, this force would produce a positive acceleration of the cylinder. Thus, the force with which the plane acts upon the cylinder rolling on it without sliding with a deceleration produced by the forces of rolling friction must be directed and applied as shown in Fig. 210*c*.

The horizontal component of the force $\boldsymbol{N}$ is nothing but the force of rolling friction f_r. Since the distance s between the point of application of the force $\boldsymbol{N}$ and the point of contact b is practically very small in comparison with the radius R of the cylinder (this means that the angle of inclination α is very small) the magnitude N of the force $\boldsymbol{N}$ is approximately equal to that of the force pressing the cylinder to the plane, that is to the force of weight of the cylinder $\boldsymbol{P}$ in the case under consideration.

The relationship between the force of rolling friction and other parameters is found experimentally. In principle, this can be performed as follows.

To the axis of the cylinder rolling along the horizontal plane (Fig. 210*c*) a constant force F is applied in the direction of motion and the magnitude F of this horizontal force is chosen so that the motion becomes uniform. Then the magnitude of the force F is obviously equal to that of the force of rolling friction f_r (the air resistance is neglected). The rotation of the cylinder being uniform and its angular acceleration being equal to zero, the force N must pass through the axis of the cylinder. The other two forces, that is the force of weight P and the external force F pass through the axis of the cylinder according to the conditions of the experiment. Consequently,

$$P = N \cos \alpha \text{ and}$$
$$F = N \sin \alpha = f_r \qquad (75.1)$$

The angle α is practically very small and relations (75.1) can therefore be rewritten approximately as

$$P \approx N \text{ and } f_r \approx N\alpha \approx P \frac{s}{R} \qquad (75.2)$$

There are tables in which the experimental values of the quantity s are compiled. Instead of the expression

$$f_r \approx P \frac{s}{R} \qquad (75.3)$$

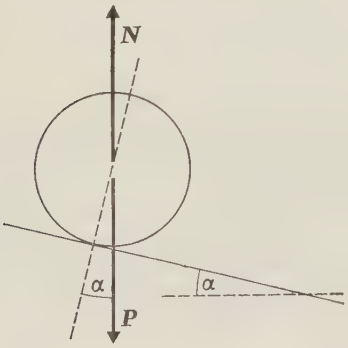

FIG. 211

of the force of rolling friction it is more convenient to write the formula for the moment of the friction force with respect to the axis of the rolling body:

$$f_r R \approx Ps \qquad (75.4)$$

Thus, the moment of the force of rolling friction is equal to the product of the force of normal pressure P by the quantity s, which is referred to as the *coefficient of rolling friction.*

Experiments show that, within some limits, for steel, other metals and hard wood the values of s are practically independent of the velocity of the rolling motion and of the radius of the cylinder although, in principle, it is clear that such a dependence, however slight, must exist. Thus, practically the coefficient s depends solely on the materials of the cylinder and of the plane.

The coefficient s can also be determined in experiments with a cylinder rolling down inclined plane. Suppose that the angle of inclination α of the plane (see Fig. 211) is chosen so that the cylinder rolls down the plane uniformly without sliding. In this case the force with which the plane acts upon the cylinder must have the vertical direction and pass through the axis of the cylinder because the cylinder rolls uniformly. Consequently, the force of rolling friction is given by the relation

$$f_r = P \sin \alpha \approx P\alpha \approx \frac{s}{R} P \qquad (75.5)$$

In these experiments it is necessary to check whether the resistance force produced by the viscosity of the air is in fact much smaller than the force of rolling friction.

When a cart is rolled on the ground the forces of rolling friction acting on its wheels obviously depend on the velocity; the experimental data concerning such a dependence are still rather poor. The problem of the computation of the

force of rolling friction for the wheel of an automobile is much more complex. In the first approximation it can be assumed that the moment of this friction force is also constant and that the value of the coefficient s needed for the computation can be taken from the corresponding tables and can be considered dependent solely on the parameters characterizing the tyres of the automobile and the road surface.

The force of rolling friction of a cylinder is much smaller than the force of sliding friction and therefore in modern machines ball and roller bearings are

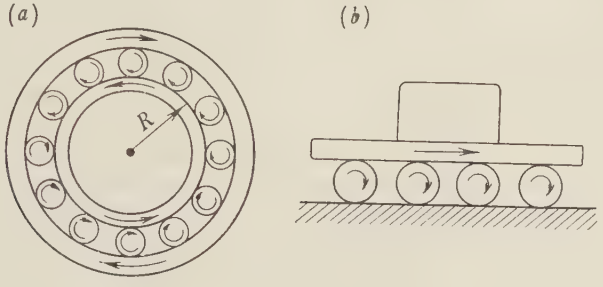

FIG. 212

widely used. As can be seen from Fig. 212, in the case of a plane surface of contact (Fig. 212*b*) the rollers are in a pure rolling motion while in an ordinary ball or roller bearing (Fig. 212*a*) the rollers cannot be in a pure rolling motion and there must appear sliding. However, the smaller the ratio of the radius of the balls (rollers) to the radius R of the inner race of the bearing, the weaker is the sliding. On the other hand, the balls (rollers) cannot have a very small radius because they press in the surface of the races of the bearing.

If the material of a cylinder and of an inclined plane down which the cylinder rolls is sufficiently hard the force of rolling friction is very small. This makes it possible to study the laws of a uniformly accelerated motion in experiments with a ball rolling down an inclined trough or with a cylinder rolling down an inclined plane by analogy with experiments with bodies falling in vacuum. This was very important for Galileo's experiments in which he studied the laws of free fall by observing a ball rolling down an inclined trough.

Let γ be the angle of inclination of the plane to the horizontal direction and let γ be greater than α (see (75.5)). Then the accelerating force is equal to $mg \sin \gamma$. The force of rolling friction is equal to $mg \sin \alpha$. If the air resistance is negligibly small then, by formulas (58.10) and (59.12), the acceleration of the ball is constant and is equal to

$$a = \frac{5}{7} g (\sin \gamma - \sin \alpha) \tag{75.6}$$

Choosing γ in an appropriate manner we can make the acceleration sufficiently small, which makes it possible to observe an accelerated motion with a small velocity; as was shown in Sec. 40, in such a case it is in fact legitimate to neglect air friction. *

* However, in Sec. 40 we dealt with a translatory motion while here we speak of a compound motion of a ball which is a combination of a translatory and a rotational motion. But when the angular velocity and the linear velocity of the ball are small the air resistance is of the same order as in the case of a pure translatory motion.

CHAPTER 9

Gravitational Attraction of Bodies

76. Law of Universal Gravitation

All physical bodies are subject to the action of the forces of mutual gravitational attraction. The basic law describing the gravitational forces was stated by I. Newton. It is called *Newton's law of universal gravitation.*

This law reads: *between any two particles of masses* m_1 *and* m_2 *at separation* R *from each other there exist attractive forces* $\boldsymbol{F}_{12}$ *and* $\boldsymbol{F}_{21}$ (see Fig. 213) *directed from one body to the other and equal in their magnitude which is directly proportional to the product of the masses of the bodies and inversely proportional to the square of the distance between them.* Thus,

$$F_{12} = F_{21} = \gamma \frac{m_1 m_2}{R^2} \tag{76.1}$$

where γ is a coefficient called the *gravitational constant.*

The law of gravitation thus stated applies to the case when the dimensions of the bodies are small relative to the distance between them, that is when the bodies can be treated as material points (particles).

The forces of mutual gravitational attraction of two bodies which cannot be regarded as material points can be found as follows. The bodies are (mentally) split into small parts which can be considered material points, the forces of attraction exerted on a particle of one body by all particles of the other body are found and added together; this results in the force of gravitational attraction with which the latter body acts on the given particle of the former body. Finally, these resultant forces are summed over all particles of the first body. This sum is the force of attraction exerted by the second body on the first one. By the third law of dynamics, the force with which the first body attracts the second body has the same magnitude and opposite direction.

Given two homogeneous balls, the computations of the above type show that the resultant gravitational forces are applied at the centres of the balls and have a magnitude equal to

$$\gamma \frac{m_1 m_2}{R^2} \tag{76.2}$$

where m_1 and m_2 are the masses of the balls, R is the distance between their centres, and γ is the same gravitational constant as in formula (76.1). Hence, formula (76.1) applies both to two particles and to two homogeneous balls (that is to two balls with uniform mass distribution).

In ordinary laboratory experiments the attractive force between the bodies is very small in comparison with their weights and therefore it cannot be obser-

ved. However, special more precise experiments allow scientists to demonstrate the presence of the force of gravitation and to measure it.

The first direct measurement of the force of gravitation was carried out by H. Cavendish in 1798 with the aid of a torsion balance. The same method was used in further more precise check experiments for the measurement of the gravitational attractive force.

The scheme of Cavendish's experiment is shown in Fig. 214. Two equal masses of magnitude m are placed at the ends of a comparatively light rocker arm A, the middle of the rocker being suspended from a sufficiently long and thin string. At the midpoint O of the rocker arm a mirror is fixed. The change in the direction of the light ray reflected from the mirror as the rocker arm turns can be directly observed and measured. The deflection of the ray makes it possible to determine the angle of twist of the string from which the rocker arm is suspended and to compute the corresponding forces producing the twist. Two big lead balls of a greater mass M (approximately, $M = 158$ kg) are brought up to the rocker arm from different sides. The forces of attraction exerted by the big balls on the small ones form a couple which rotates the rocker arm A until the moment of the couple of the gravitational forces is balanced by the torsion moment of the thread. The latter moment is computed from the known parameters of the system and from the angle of turn of the reflected light ray. Varying the distances between the masses m and M Cavendish determined the dependence of the force of gravitational attraction on that distance. The results of the experiments confirmed the validity of Newton's law of gravitation.

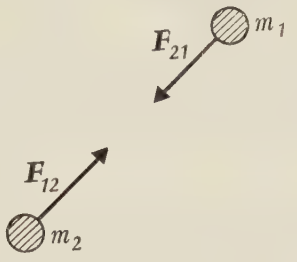

FIG. 213

As was mentioned Cavendish's results were checked many times by other scientists using various modifications of the experiment. At present the results of precise measurements give us the following value of the gravitational constant:

$$\gamma = 6.65 \times 10^{-11} \mathrm{m^3/(kg \cdot s^2)} \tag{76.3}$$

in the SI system and

$$\gamma = 6.65 \times 10^{-8} \mathrm{cm^3/(g \cdot s^2)}$$

in the cgs system.

The constant γ is measured in derived units.

It is possible to construct a system of units in which γ is a dimensionless quantity but this leads to the change of the dimensions of the other physical quantities; such a system is not used in practice.

Knowing the value of the gravitational constant we can determine the mass m_e of the Earth proceeding from the known value of the acceleration of gravity g. Indeed, by Newton's law of gravitation, a body of mass m is attracted by the Earth (near the Earth's surface) with the force (of weight) of magnitude

$$P = \gamma \frac{m \cdot m_e}{r_0^2}$$

where r_0 is the radius of the Earth. On the other hand,

$$P = mg$$

These equalities result in

$$m_e = \frac{gr_0^2}{\gamma} \tag{76.4}$$

On substituting the value $r_0 \approx 6.4 \times 10^8$ cm in this expression we find

$$m_e = \frac{981 \times 6.4^2 \times 10^{16}}{6.65 \times 10^{-8}} \approx 6 \times 10^{27} \text{g} = 6 \times 10^{21} \text{ ton} \tag{76.5}$$

Newton himself verified his law by analysing the motion of the Moon round the Earth. Assuming that the Moon is in a circular orbit under the action of

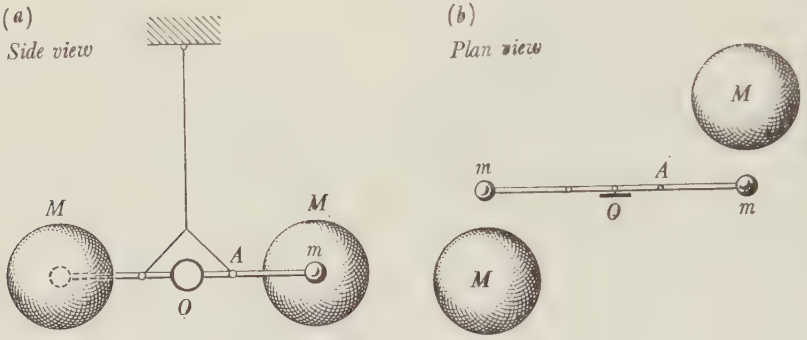

FIG. 214

the Earth's attraction solely and knowing the period of the Moon's rotation round the Earth (the lunar month) $T = 27.3$ days and the (average) distance $r = 3.844 \times 10^{10}$ cm between the Earth and the Moon he found the centripetal acceleration w of the Moon:

$$w = \frac{v^2}{r} = \frac{(2\pi r)^2}{T^2 r} = \frac{4\pi^2 r}{T^2} = \frac{4\pi^2 3.844 \times 10^{10}}{2.358^2 \times 10^{12}} \text{ cm/s}^2$$

because $T = 27.3$ days $= 2.358 \times 10^6$ s. On performing the calculation we obtain

$$w \approx 0.27 \text{ cm/s}^2 \tag{76.6}$$

This centripetal acceleration is given to the Moon by the Earth's gravitational attraction and therefore, by the second law of dynamics,

$$m_m w = \gamma \frac{m_m \cdot m_e}{r^2} \tag{76.7}$$

where m_m is the mass of the Moon. It follows that

$$w = \gamma \frac{m_e}{r^2} \tag{76.8}$$

On the other hand, the acceleration g near the Earth's surface can be written as

$$g = \gamma \frac{m_e}{r_0^2}$$

where r_0 is the radius of the Earth. On finding m_e from this relation and substituting this value into formula (76.8) we obtain

$$w = g\frac{r_0^2}{r^2} \tag{76.9}$$

and thus find the centripetal acceleration of the Moon:

$$w = 981\left(\frac{6.4}{3.844}\right)^2 \cdot \frac{10^{16}}{10^{20}} \approx 0.273 \text{ cm/s}^2 \tag{76.10}$$

This value of the acceleration coincides with the one found earlier purely kinematically (see (76.6)).

This comparison showed Newton that the law of gravitation he stated was correct. Astronomical observations and calculations also confirm the validity of the law of gravitation. Note that in the verification of the law of gravitation performed by Newton there is no need to know the value of the gravitational constant γ.

The discovery of the universal law of gravitation extended considerably the field of application of the laws of mechanics established originally in experiments with bodies on the Earth. It also made it possible to generalize these laws to all the physical bodies in the Universe.

77. Inertial Mass and Gravitational Mass

The statement of the law of gravitation expressed by formula (76.1) tacitly implies that the masses of the bodies entering into this formula coincide with the masses as measure of the inertia of the bodies. However, to justify this assumption some additional investigation is needed.

The results of experiments show that every body possesses a *gravitational* mass measuring its gravitational property and, as is known, every body has an *inertial* mass measuring its inertia property. These two quantities are of quite a different nature. It turns out however that these masses are in fact *directly proportional* to each other. It is therefore possible to construct the systems of units so that they become equal numerically. Such systems are used in physics.

The conclusion concerning the proportionality of the inertial mass and the gravitational mass can be drawn on the basis of experiments showing that the acceleration of gravity in the free fall of all bodies with different masses has the same value for a given place on the Earth's surface. A falling body has the inertia property measured by its inertial mass m_{in} and the gravitational property measured by its gravitational mass m_{gr}. The force of gravity can therefore be expressed as

$$P = km_{gr} \tag{77.1}$$

where k is a dimensional constant factor. Thus, the attractive force created by the Earth is proportional to the gravitational mass of the body. On the other hand, a free fall of a body is nothing but its motion under the action of the attractive force of gravitation. Therefore, by the second law of dynamics, we can write for the body in question the formula

$$P = m_{in}g \tag{77.2}$$

where g is the acceleration of gravity. On equating (77.1) to (77.2) we arrive at the relation

$$g = k\frac{m_{gr}}{m_{in}} \tag{77.3}$$

The acceleration g being the same for all the bodies irrespective of their materials and their dimensions, the inertial mass m_{in} is proportional to the gravitational mass m_{gr}. If we choose as the unit of the inertial mass one kilogram (1 kg), the unit of the gravitational mass can be taken so that the factor k is equal to 9.81 m/s^2. For such a choice of the units the magnitude of the gravitational mass exactly coincides with the magnitude of the inertial mass for the same body.

To verify the proportionality between the inertial and gravitational masses I. Newton carried out experiments with pendulums of different masses having the same lengths. The theory of oscillations (see Sec. 124) shows that the period of a simple pendulum depends solely on its length l, the constant k entering into formula (77.1), and the ratio $\frac{m_{\text{in}}}{m_{\text{gr}}}$, namely

$$T = 2\pi \sqrt{\frac{l}{k} \frac{m_{\text{in}}}{m_{\text{gr}}}} \qquad (77.4)$$

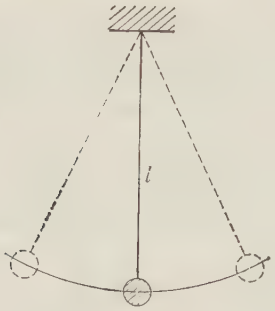

FIG. 215

By a simple pendulum is meant a body with dimensions considerably smaller than the length of the thread it is suspended from, the mass of the thread being negligibly small in comparison with the mass of the suspended body (Fig. 215). Experiments show that for any simple pendulum the period of oscillation is proportional to $\sqrt{l}$. Consequently, the ratio $m_{\text{in}}/m_{\text{gr}}$ remains invariable that is $m_{\text{in}}/m_{\text{gr}} = \text{const}$. For the indicated choice of the units we have $k = g$ and the ratio $m_{\text{in}}/m_{\text{gr}}$ becomes equal to 1. Therefore, formula (77.4) for the period of a simple pendulum can be written in the form

$$T = 2\pi \sqrt{\frac{l}{g}} \qquad (77.5)$$

Newton's experiments were checked with high accuracy by F. W. Bessel. At the beginning of the twentieth century they were again verified with a still greater accuracy by Academician Krylov who used more precise methods and more sophisticated apparatuses.

It is evident that on the basis of the results of the measurement of the oscillation period of a pendulum of a definite length it is also possible to determine the magnitude of the acceleration of gravity in the free fall, that is the magnitude of the force of gravity at the given place on the Earth's surface. This period admits of a very precise measurement, which makes it possible to determine the acceleration of gravity at the given place with high accuracy.

Measuring the period of oscillation of the same pendulum for various places on the Earth's surface we can investigate the variation of the gravity force from place to place. It turns out that because of the inhomogeneity of the Earth's crust the gravitational force varies from place to place even at one latitude. This variation enables geologists to draw certain conclusions concerning natural deposits (gravitational prospecting).

The physical significance of the proportionality law for the gravitational and inertial masses turns out to be of primary importance in the relativity theory. In this theory the proportionality law is referred to as *the equivalence law for*

the gravitational and inertial masses of a body. This equivalence principle makes it possible to draw the conclusion that for any sifficiently small spatial region it is possible to choose such an accelerated reference system in which there is no gravitation field.

78. Potential Energy of Gravitation

Since the forces of gravitational attraction between physical bodies or between the particles constituting a body depend on their mutual positions in space every collection of bodies forming a definite configuration possesses an amount of potential energy of gravitation. When the mutual positions of the bodies change the gravitational forces perform some work and therefore the potential energy of the system of the bodies also changes.

It should be noted that a variation of the potential energy of gravitation taken with opposite sign is equal to the corresponding work performed by the gravitational forces as the configuration of the system of bodies changes. According to the general definition, we say that a system of bodies possesses an amount of potential energy if the work performed by the interaction forces as the system passes from one configuration to another does not depend on the way in which this passage is carried out and depends solely on the initial and the terminal configuration of the system.

The potential energy of gravitation can easily be computed for a system of two particles of masses m_1 and m_2 (or two balls with the same masses). To this end we use formula (36.8) and substitute in it the expression

$$f(r) = -\gamma \frac{m_1 m_2}{r^2} \tag{78.1}$$

where r is the distance between the particles and γ is the gravitational constant. It follows that the difference in the potential energy for two arbitrary states of the system of the two material points is characterized completely by the corresponding values $\boldsymbol{r} = \boldsymbol{r}_1$ and $\boldsymbol{r} = \boldsymbol{r}_2$ of the vector $\boldsymbol{r}$ joining the points and is equal to

$$U_2 - U_1 = \gamma \int_{r_1}^{r_2} \frac{m_1 m_2}{r^2}\, dr = -\gamma m_1 m_2 \frac{1}{r}\Big|_{r_1}^{r_2} = \gamma m_1 m_2 \left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \gamma \frac{m_1 m_2}{r_1} - \gamma \frac{m_1 m_2}{r_2} \tag{78.2}$$

Since the potential energy is determined to within an arbitrary constant it is only possible to find the difference between the values of the potential energy corresponding to two given states of the system in question. For the sake of convenience, the potential energy corresponding to an infinite distance ($r \to \infty$) is usually assumed to be zero: $U(\infty) = 0$. Under this assumption formula (78.2) can be written in the form

$$U_1 = U(r_1) = -\gamma \frac{m_1 m_2}{r_1} \tag{78.3}$$

For such a choice of the value of $U(\infty)$ the potential energy of gravitation of a system of two particles or balls is always negative (Fig. 216). The energy increases with the distance. For, the gravitational interaction between the bodies produces attractive forces and, consequently, to remove the bodies from one another some work must be expended. In other words, the potential energy must increase. The potential energy attains its maximum value for an infinite

separation of the particles and has a minimum value for the smallest possible distance between them. If the system in question consists of two balls of radii r_{10} and r_{20} the minimum value of the potential energy of their gravitational interaction is

$$U_{\min} = -\gamma \frac{m_1 m_2}{r_{10} + r_{20}} \tag{78.4}$$

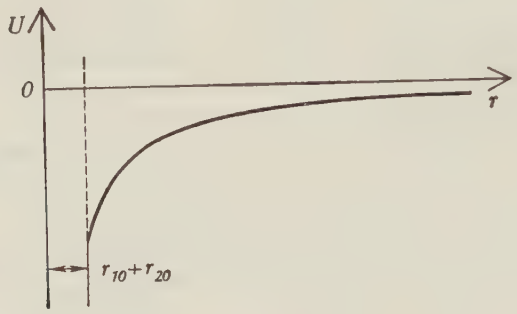

FIG. 216

The potential energy of gravitation of non-homogeneous bodies or of systems consisting of many material bodies is computed in a more complex manner but the principal idea of the computation remains the same. When the distance between the bodies increases the potential energy of gravitation also increases.

79. Basic Laws of Celestial Mechanics

The laws of motion of celestial bodies and, in particular, the laws governing the motion of the planets in the solar system are a simple consequence of the general laws of mechanics (often called Newton's laws by which are meant the three laws of dynamics and the law of universal gravitation).

Newton's studies of the laws of mechanics were preceded by the discoveries of the German astronomer J. Kepler who deduced, on the basis of the observation data found by the Danish astronomer Tycho Brahe, the laws of planetary motion round the Sun. These are known as Kepler's laws. They read:

1. *The orbits of all the planets are ellipses, the Sun being at one of their focuses.*
2. *The motion of each planet is such that the radius vector of the planet drawn from the Sun sweeps over equal areas during equal time intervals* (Fig. 217).
3. *The squares of the periods of revolution of the planets are in the ratio of the cubes of the semimajor axes of their elliptical orbits.*

Kepler's first law follows from the solution of the problem of the determination of the orbit of a planet and from the law of motion of a planet in the orbit. To solve this problem one must find the trajectory of motion of a material point under the action of a central force* whose magnitude is inversely proportional to the square of the distance from the centre. The solution of the problem indi-

* By a central force is meant a force which is always directed to one point called the centre.

cates that the trajectories of all the celestial bodies are plane curves which are ellipses, or parabolas or hyperbolas.

In the special case when the orbit of a planet is a circle it can easily be proved using elementary methods that such a motion under the action of the central force of this kind is in fact possible. For, a planet can be in a circular motion only when it is attracted by the Sun with a force equal to the centripetal force.

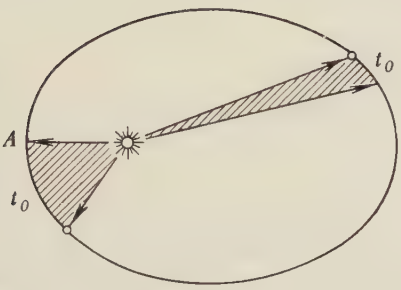

FIG. 217

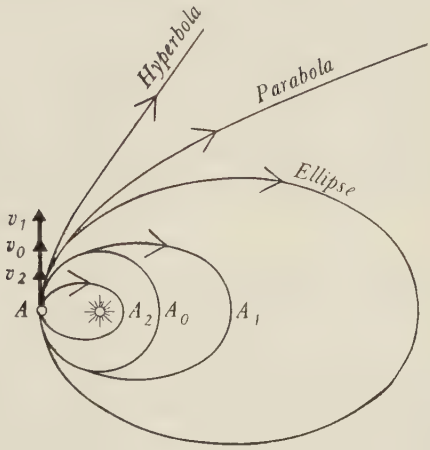

FIG. 218

If a planet starts moving in a circle from a given place it is necessary that it should have a definite velocity directed perpendicularly to the radius vector and equal to

$$v_0 = \sqrt{\frac{\gamma M}{R}} \tag{79.1}$$

where M is the mass of the Sun and γ is the gravitational constant*. Thus, the velocity of motion in a circular orbit and the radius of the orbit are connected with each other, the velocity being independent of the mass of the planet.

More complicated calculations show (and observations confirm the result) that the shape and the type of the orbit are connected with the initial velocity. For example, when the velocity of motion v_2 at the ("initial") point A (see Fig. 218) is less than the value v_0 given by formula (79.1) the planet is in an elliptic orbit so that the Sun is at the farther focus of the ellipse (see the orbit AA_2 in Fig. 218). If the velocity v_1 exceeds the circular velocity v_0 the planet also moves in an elliptic orbit but in this case the Sun is at the nearest focus (see the orbit AA_1 in Fig. 218). The motion of a planet in an elliptic orbit is only possible when the velocity at the point A is less than the value

$$v_{\text{par}} = \sqrt{\frac{2\gamma M}{R}} \tag{79.2}$$

When the velocity at the point A is equal to v_{par} the planet moves along a parabola. When the velocity of motion at the point A exceeds the parabolic velo-

* Formula (79.1) follows from the equality $mv_0^2/R = \gamma Mm/R^2$ where m is the mass of the planet.

city given by formula (79.2) the celestial body (in this case it can no longer be called a planet) moves in a hyperbola and never returns to the initial point.

In a motion of a planet along an orbit the sum of the kinetic and the potential energies remains constant. For instance, when a planet moves in an ellipse the potential energy increases as the distance from the Sun grows while the kinetic energy decreases accordingly. Thus, the velocity is smaller for the points lying far from the Sun and greater for the points nearer to the Sun. When the initial velocity at the point A increases and becomes greater than the circular velocity (79.1) the ellipse in which the planet moves elongates. The point A_1 of the orbit opposite to the point A moves away from the Sun as the initial velocity increases. If the distance from the Sun to the point A_1 is known the energy conservation law makes it possible to determine the velocity at the point A_1 as function of the initial velocity at the point A. Indeed, for the point A the total energy can be written as

$$E = \frac{mv_1^2}{2} - \gamma \frac{mM}{R_1} \tag{79.3}$$

It is equal to the same value for any other point of the trajectory, that is

$$E = \frac{mv^2}{2} - \gamma \frac{mM}{R} \tag{79.4}$$

where v and R are respectively the velocity of the planet and its distance from the Sun at an arbitrary point of the orbit. The comparison of formulas (79.3) and (79.4) enables us to establish the relationship between the velocity and the distance provided that the value of the energy E at the initial point A is known. For the elliptical orbits we have $E < 0$, that is the absolute value of the potential energy exceeds that of the kinetic energy.

For a parabolic orbit the velocity at the point at infinity is equal to zero, which corresponds to the zero value of the total energy, that is $E = 0$. From formula (79.3) it follows that the parabolic velocity v_{par} at the point A satisfies the equation

$$\frac{mv_{\text{par}}^2}{2} - \gamma \frac{mM}{R} = 0 \tag{79.5}$$

On solving the equation we arrive at the expression of v_{par} given by formula (79.2).

For the hyperbolic orbits passing through the point A we have $E > 0$, that is the kinetic energy exceeds the absolute value of the potential energy.

Thus, the shapes of all possible trajectories passing through a given point A are in a one-to-one correspondence with the values of the energy of the moving body.

Kepler's second law follows from the conservation of the angular momentum. Indeed, a planet rotating round the Sun is acted upon by the gravitational attractive force $\gamma \frac{Mm}{r^2}$ which is always directed towards the Sun; therefore, the angular momentum with respect to the centre of the Sun must be constant, that is

$$[\boldsymbol{r} \cdot m\boldsymbol{v}] = \text{const}$$

whence

$$[\boldsymbol{rv}] = \text{const} \tag{79.6}$$

Here (see Fig. 219) $\boldsymbol{r}$ is the radius vector and $\boldsymbol{v}$ is the velocity of the planet. The area swept over by the radius vector during time dt is equal to

$$dS = 1/2\ rv \sin \alpha\ dt$$

where α is the angle between r and v. Taking into account expression (79.6), we can write

$$2\frac{dS}{dt} = rv \sin \alpha = \text{const}, \text{ that is } \frac{dS}{dt} = \text{const} \tag{79.7}$$

As it follows from this law, the velocity of the planet in its orbital motion achieves its greatest value when the planet is in its nearest position relative to the Sun (such is the position at the point A in Fig. 217).

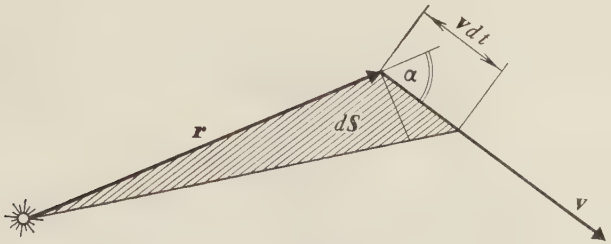

FIG. 219

Kepler's third law can be proved quite easily in the case when the orbit of a planet is assumed to be a circle. Such an assumption can serve as a good approximation to reality because, as is known, the eccentricities of the elliptical orbits of the planets in the Solar system are not very large. For instance, the eccentricity of the Earth's orbit is about 0.017 and for the orbit of the Mercury it is about 0.205. It should also be mentioned that in the case of real elliptical orbits a more intricate calculation gives the same result.

Let one planet have mass m_1 and move in a circular orbit of radius r_1 with period of revolution T_1 and let the same parameters for another planet be m_2, r_2 and T_2, respectively. The square of the linear velocity of the former planet is

$$v_1^2 = \frac{\gamma M}{r_1}$$

where M is the mass of the Sun. The orbital velocity of this planet is

$$v_1 = \frac{2\pi r_1}{T_1}$$

On substituting this expression in the foregoing formula we find

$$\frac{4\pi^2 r_1^2}{T_1^2} = \frac{\gamma M}{r_1}, \text{ that is } \frac{r_1^3}{T_1^2} = \frac{\gamma M}{4\pi^2} \tag{79.8}$$

The same relation can be derived for the other planet:

$$\frac{r_2^3}{T_2^2} = \frac{\gamma M}{4\pi^2} \tag{79.9}$$

The comparison of (79.8) and (79.9) now shows that

$$\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3}$$

which expresses what is asserted in Kepler's third law.

Thus, the Newtonian mechanics gives a complete explanation of the laws of motion of celestial bodies. At present these laws are still applied by scien-

tists in theoretical investigations of the motions of celestial bodies. The results of the investigations are confirmed by experimental measurements of the motions of the spaceships and artificial satellites.

80. Motion of Earth's Satellites and Spaceships

Cosmic flights have been rapidly developing since the historical launch of the first artificial Earth's satellite in the USSR in 1957 and after the round-Earth

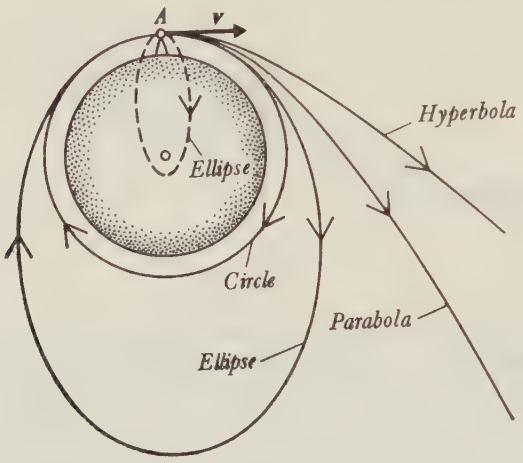

FIG. 220

flight of Yu. A. Gagarin on April 12, 1961. At present there are many artificial satellites rotating round the Earth and a number of spaceships launched from the Earth have become satellites of the Sun.

The laws governing the flight of satellites and spaceships are similar to those describing the rotation of the planets round the Sun. If we think of a satellite or a spaceship as a body thrown with an initial velocity v at a height h all the possible trajectories are obviously similar to those orbits in which the planets move provided that the influence of the atmosphere is absent (see Fig. 220).

For a small initial velocity v less than the critical velocity

$$v_{cr} = r_0 \sqrt{\frac{g_0}{r_0 + h}}$$

the trajectories are arcs of ellipses whose focuses coincide with the centre of the Earth. For very small velocities these arcs can be regarded approximately as arcs of parabolas.

When the initial velocity is equal to $v_{cr} \approx 7.93$ km/s the trajectory is a circle and the body becomes a satellite of the Earth. The velocity of motion of a satellite in a circular orbit can readily be found using the condition that the centripetal acceleration $v_{cr}^2/(r_0 + h)$ of the satellite must be equal to the acceleration of gravity g. Indeed, the acceleration of gravity at a height of h is given by the relation

$$g = g_0 \frac{r_0^2}{(r_0 + h)^2} \tag{80.1}$$

where g_0 is the acceleration of gravity on the surface of the Earth at a distance of r_0 (equal to the Earth's radius) from its centre. Then

$$\frac{v_{cr}^2}{r_0+h} = g = \frac{g_0 r_0^2}{(r_0+h)^2}$$

whence

$$v_{cr} \approx r_0 \sqrt{\frac{g_0}{r_0+h}} \tag{80.2}$$

If $h \ll r_0$ then

$$v_{cr} \approx \sqrt{r_0 g_0} \approx 7.93 \text{ km/s} \tag{80.3}$$

is the velocity of motion of the satellite in a circular orbit of radius equal to that of the Earth. This velocity is referred to as the *first cosmic velocity*.

When the initial velocity exceeds v_{cr} but is less than

$$v_{par} = r_0 \sqrt{\frac{2g_0}{r_0+h}}$$

the trajectory of the satellite is an ellipse, the centre of the Earth being at the focus of the ellipse which is nearer to the initial point. For $v = v_{par}$ the trajectory takes the form of a parabola and the spacecraft moving along it will never return to the Earth. The parabolic velocity (relative to the Earth) is specified by formula (79.5) in which only γM should be replaced by γM_e where M_e is the mass of the Earth. Taking into account formula (80.1) for the magnitude of the acceleration of gravity we can write

$$\frac{\gamma M_e}{(r_0+h)^2} = \frac{g_0 r_0^2}{(r_0+h)^2} \text{ whence } \gamma M_e = g_0 r_0^2 \tag{80.4}$$

By substituting this expression in (79.5) we find the parabolic velocity (for the Earth):

$$v_{par} = r_0 \sqrt{\frac{2g_0}{r_0+h}} \tag{80.5}$$

When $h \ll r_0$, that is when the spacecraft is launched along the tangent line to the Earth's surface, we have

$$v_{par} \approx \sqrt{2g_0 r_0} \approx 11.2 \text{ km/s} \tag{80.6}$$

This magnitude is referred to as the *second cosmic velocity* (the *escape velocity*).

Consequently, when the spacecraft is sent horizontally at the altitude h with a velocity exceeding $v_{par} = r_0 \sqrt{\frac{2g_0}{r_0+h}}$ it moves along a hyperbolic trajectory and leaves forever the spatial region where the Earth's gravitational attraction acts; it then becomes a satellite of the Sun, that is a small artificial planet.

All the calculations we have performed do not take into account the influence of the Sun and of the planets on the motion of the spacecraft. In other words, we have assumed that the reference system connected with the Earth is inertial and that the satellite moves relative to it; in reality the whole system "the Earth and the satellite" is in permanent accelerated motion relative to the Sun.

The mass of the spacecraft being negligibly small in comparison with that of the Earth, the centre of mass of the system "the Earth and the spacecraft" practically coincides with that of the Earth. Besides, when the distance between the spacecraft and the centre of the Earth is very small relative to the distance from the Earth to the Sun the influence of the Sun's attraction on the orbit of the spacecraft can be neglected. However, when the spacecraft is far

from the Earth we should take into account the forces of gravitational attraction exerted on the spacecraft by the Sun, by the Moon and by the other planets of the Solar system. Furthermore, even when an artificial satellite of the Earth moves in a circular orbit near the Earth the character of the motion depends on the non-homogeneity of the gravitational field of the Earth due to both the deflection of the shape of the Earth from a perfect sphere and the variation of the density of mass distribution over the Earth (particularly in its upper layers).

More complicated calculations show that the *third cosmic velocity*, that is the velocity that should be given to a spacecraft launched from the Earth for it to leave the Solar system (it is also called the *solar escape velocity*), is

$$v_{\text{s. e. v}} \approx 16.7 \text{ km/s} \tag{80.7}$$

This velocity can be calculated approximately as follows. First of all we note that $v_{\text{par}} = v_{\text{cr}} \sqrt{2}$, which follows from the comparison of (80.2) and (80.5). This means that the parabolic velocity is $\sqrt{2}$ times the circular velocity. The same obviously applies to the Earth or to a body moving in the Earth's orbit round the Sun (the orbit is considered circular). If the spacecraft were launched along an orbit round the Sun coinciding with the Earth's orbit it would move in this orbit with the velocity equal to the Earth's velocity relative to the Sun, that is with a velocity v_0 equal (approximately) to 30 km/s (more precisely, to 29.76 km/s). The spacecraft should therefore be given a velocity increase of magnitude

$$v_0 (\sqrt{2} - 1) \tag{80.8}$$

for it to leave the Solar system. In other words, the spacecraft must be given an amount of kinetic energy equal to

$$\frac{mv_0^2}{2} (\sqrt{2} - 1)^2 \tag{80.9}$$

(in a reference system moving in the orbit round the Sun like the Earth) where m is the mass of the spacecraft. Hence, for the spacecraft to leave the Earth's gravitational field it should be given the velocity

$$v_{\text{par}} = \sqrt{2g_0 r_0} \approx 11.2 \text{ km/s}$$

or, which is the same, it should receive an amount of kinetic energy equal to

$$\frac{mv_{\text{par}}^2}{2} \tag{80.10}$$

which must be expended on the work necessary to overcome the Earth's gravitational forces. Consequently, when the spacecraft is given the velocity $v_{\text{s.e.v}}$ relative to the Earth to which there corresponds an amount of kinetic energy equal to

$$\frac{mv_{\text{s. e. v}}^2}{2} = \frac{mv_0^2}{2} (\sqrt{2} - 1)^2 + \frac{mv_{\text{par}}^2}{2} \tag{80.11}$$

then the spacecraft leaves the Solar system. As follows from (80.11),

$$v_{\text{s. e. v}}^2 = v_0^2 (\sqrt{2} - 1)^2 + v_{\text{par}}^2 \tag{80.12}$$

On substituting in the last expression the corresponding numerical data we find $v_{\text{s.e.v}} \approx 16.75$ km/s.

Our considerations show that the laws of the planetary motion and, generally, of the motion of celestial bodies are the same as those governing the free fall of a body on the Earth. All these types of motion are produced by the action of the forces of gravitation solely.

PART TWO
Mechanics of Deformable Bodies

CHAPTER 10

Mechanics of Deformable Solids

81. The Concept of an Elastic Body. Forces and Deformations in Tension

Any change of the forces acting on a real body leads to a change of its shape or, as we say, to a *deformation* of the body. In the foregoing chapters, when studying the laws of mechanics of rigid bodies, we supposed that the deformations were sufficiently small and did not affect the motion of the bodies; therefore, we neglected them. However, there are many problems in which it is necessary to know the laws connecting the action of the forces applied to a body with the deformations produced by these forces; the study of these laws is the subject matter of the present chapter.

A body always undergoes deformations when it is acted upon by forces irrespective of whether the body is at rest (statics) or in a non-uniform motion (dynamics); (see Sec. 92). For instance, when two forces of equal magnitudes and opposite directions are applied to the ends of a (thin) homogeneous rod the latter elongates, the distances between its parts increase and the rod undergoes a deformation (tension). As the tensile force grows the distances between all the particles of the rod increase.

Now imagine that the same rod is acted upon by a single force applied to one of its ends. Under the action of the force the rod starts moving with an acceleration and the force produces the deformation of the rod. But the character of the deformation differs from that in the former case when the deformations of all parts of the homogeneous rod are the same while in the latter case different parts of the rod undergo different deformations: those parts which are closer to the end of the rod where the force is applied have a greater elongation than the parts lying farther from that end.

A schematic model of the deformation of the rod is shown in Fig. 221. The small blocks in the figure represent the particles of the rod and the springs between them schematize the internal forces acting between the particles. When the external force F is applied to the rightmost particle the whole chain starts moving with an acceleration. The forces acting on the springs decrease to the left from spring to spring. The force extending any of these springs gives an acceleration to all particles lying to the left of it. The mass of the remaining particles lying on the left of that spring depends on the position of the spring in the chain. That is why the deformations of different springs are different. The same explanation applies to the case of a real homogeneous rod subject to tension by a force applied to one of its ends. The interaction forces arising between the various parts of a deformed body are called *internal forces* in contrast to the *external forces* producing the deformation of the body.

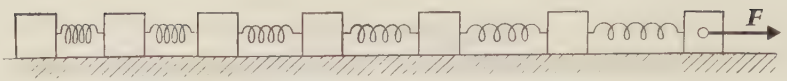

FIG. 221

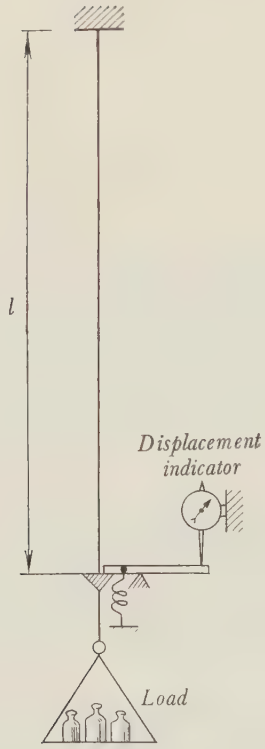

FIG. 222

Our discussion shows that when investigating deformations we are not allowed to transfer a force along its line of action as it was done for a rigid body. In the above example the character of the deformation is quite different when the force F is applied to the first particle, to the second particle, etc. (see Fig. 221).

The deformation varies depending on the variation of the external forces applied to the body; when the forces remain constant then, generally speaking, the deformations are also invariable.

In the general case the laws connecting the acting forces and the deformations produced by them are very complex. This is due to the fact that both the deformations and the internal forces are distributed over a deformed body in a complex way and also because usually the connection between the forces and the deformations is not one-to-one and depends on the magnitude and the character of variation of the forces applied to the body and on some other circumstances.

A *one-to-one* correspondence between the magnitudes of the forces (both external and internal) is characteristic only of elastic bodies within certain ranges of the variations of the forces. When this is the case the forces uniquely determine the deformations and vice versa.

To investigate the laws connecting the acting force and the deformation produced by it we begin with the simplest case of the deformation of a homogeneous rod (cylinder) subjected to tension (or compression) along its axis.

Let us discuss the results of an experiment in which a long steel rod is subject to tension; the scheme of the experiment is shown in Fig. 222. If the material of the rod is homogeneous all identical parts of the rod are extended similarly under the action of a given external force produced by the weight of the load. Thus, the deformation of tension is the same along the rod and it is natural to characterize it quantitatively by the *unit elongation* (or *longitudinal strain*):

$$\varepsilon = \frac{\Delta l_1}{l_1} \tag{81.1}$$

where Δl_1 is the elongation of a part of the rod whose original length is l_1. The magnitude of ε is the same for all the segments of the rod and depends solely on the magnitude of the tensile force F. Under the action of the force F there appear internal forces with which various parts of the rod act upon each other. Let us (mentally) separate a part of the rod (Fig. 223) and consider its equilibrium conditions. When this part is in equilibrium the forces exerted on its

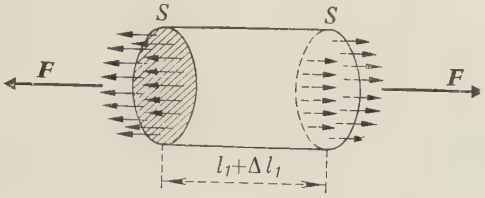

FIG. 223

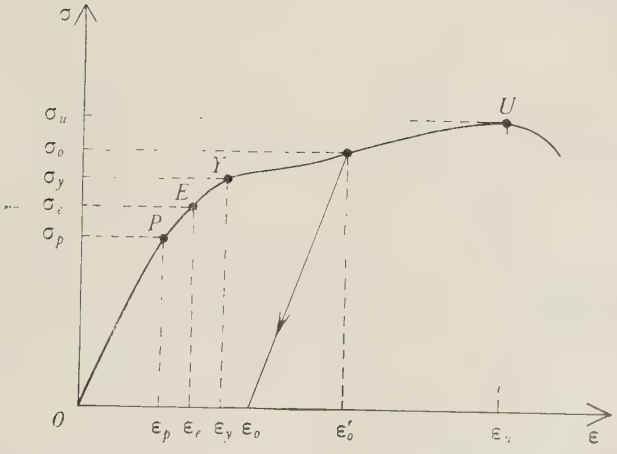

FIG. 224

ends by the adjoining parts of the rod are equal in magnitudes and opposite in directions. Since this applies to any segment of the rod, it follows that in every cross section of the rod there arises an internal force of magnitude F.

This force F can be thought of as a force applied to the given (plane) cross section of the rod. When the material of the rod is homogeneous we can assume that the force is *uniformly* distributed over the area of the cross section. The magnitude of the force acting on unit area of the cross section is called the *stress* and is denoted σ. The stress σ appearing in the extended rod is given by the formula

$$\sigma = \frac{F}{S} \tag{81.2}$$

where S is the cross-section area of the rod. Experiments show that the longitudinal strain (unit elongation) ε is specified by the stress σ.

In the experiment (Fig. 223) the tensile force F (together with the stress σ) is gradually increased and the corresponding elongation (longitudinal strain) ε is measured. The results of the experiment can be represented by the *stress-strain curve* showing the dependence between σ and ε; this curve is of the shape as in Fig. 224. When the tensile force is not large the stress σ and the strain ε are, approximately, *directly proportional* to each other. This *region of proportionality* occupies the part of the curve up to the point P. To the right of the point P the deformation increases faster and the curve bends towards the ε-axis. There is a region on the right of the point Y where the curve is almost parallel to the ε-axis and where stress does not practically increase as the deformations grow. This part of the curve and the corresponding intervals on the ε- and σ-axes are referred to as the *yield region* (or the *region of plastic deformations*). As the deformation ε is given a further increase the curve goes slightly upwards and the stress attains its maximum at a point U after which the curve goes downwards. The right end point of the curve corresponds to the rupture of the rod which takes place after the tensile force achieves the value $F = \sigma_u S$ corresponding to the maximum stress σ_u.

On plotting the stress-strain curve for the given rod $\sigma(\varepsilon)$ we take another specimen made of the same material and carry out another series of experiments in the following order: the rod is first extended gradually until the stress achieves a definite maximum value σ. Then the rod is gradually unloaded with simultaneous measurements of the values of the stress and of the strain. The results of the experiments indicate a *one-to-one dependence* between the force and the deformation produced by it as the force is decreased. If the curve $\sigma(\varepsilon)$ obtained when the specimen is loaded coincides with that resulting from the measurements when the specimen is unloaded we conclude that the deformation uniquely specifies the stress and vice versa. Let us carry out such experiments repeatedly varying the value of the maximum stress σ. It turns out that after a definite maximum value σ_e has been exceeded (see Fig. 224) the curves obtained in the experiments with the increase and decrease of the load no longer coincide. Namely, the unloading experiments give greater values of the deformation for the same values of the stress. Moreover, after the load is removed completely the deformation of the specimen does not return to the zero value: as we say, there appears a *residual strain* in the rod.

The part of the stress-strain curve from 0 to σ_e corresponds to the *elastic region* for the given material (for instance, steel). A steel rod behaves as an elastic body only for the strains not exceeding σ_e as the rod is subject to extension. Between σ_p and σ_y there is a point corresponding to the *elastic limit* for the extension of the specimen or, more precisely, for the material subject to the test (this is the point E $(\sigma_e,\ \varepsilon_e)$ on the diagram in Fig. 224).

Thus, a given body can only be considered elastic when the deformation it is subjected to does not exceed the elastic limit for the given material. Only in the elastic region are the stress and the strain in a one-to-one correspondence.

It should be noted that in the above method of the determination of the values of σ_e and ε_e corresponding to elastic limit the accuracy achieved essentially depends on the precision with which the forces and the deformations are measured in the experiments. When the precision is not very high greater values of σ_e and ε_e are obtained for the same material. That is why in the material testing laboratories for the elastic limit is taken the stress after whose removal the residual strain is of a certain percentage, for instance 0.1 per cent of the original strain.

The initial part of the curve $\sigma(\varepsilon)$ in Fig. 224 is a straight line. On this interval (extending approximately up to the point P) the dependence between the stress and the strain can be represented by a simple law of direct proportionality:

$$\sigma = E \cdot \varepsilon \tag{81.3}$$

Formula (81.3) expresses *Hooke's law*. The constant coefficient entering into it is called *Young's modulus* (the *modulus of elongation*). Its dimension is N/m^2 or N/mm^2. It is one of the most important characteristics of a given material (in many technical reference books Young's modulus is given in kgf/mm^2). The region within which Hooke's law applies is termed the *proportional region*; the values σ_p and ε_p of the stress and of the strain up to which Hooke's law is valid are called the *proportional limit*. For steel the proportional limit is very close to the elastic limit although, generally, they not necessarily coincide.

The part of the stress-strain curve lying on the right of the elastic limit corresponds to the *region of plastic deformations*. For such deformations the given body is inelastic.

If the specimen is extended up to a value ε_0' lying in the region of plastic deformations and then unloaded the magnitude of the strain decreases (approximately, as is shown in Fig. 224). When the specimen is completely unloaded the residual strain ε_0 is of a magnitude almost close to ε_0'. In the region of plastic deformations residual strains are almost equal to the initial strains. In this region we usually indicate two characteristic points: the *yield* (or *flow*) *limit* (the point Y on the diagram to which there corresponds σ_y) and the *ultimate strength* (the point U on the diagram to which there corresponds σ_u). On reaching the yield limit the material starts "flowing": the load does not increase while the deformation continues to grow. The ultimate strength σ_u is the stress under which the specimen does not yet rupture; it ruptures as this stress is exceeded.

The phenomenon of deformation in tension or compression is rather simple. Let us isolate (mentally) a cube from a rod subject to such a deformation. Under the deformation the cube goes into a parallelepiped and the cross section of the cube (together with the cross section of the whole rod) changes: the cross sections decrease under tension and increase under compression; this follows from the results of the experiments.

The decrease of the diameter can easily be observed if we stretch a rubber tube on which a metal ring is tightly fixed before the test. When the tube is placed vertically and stretched the ring falls down as the tension achieves a certain value (Fig. 225). The same experiment can be carried out with a metal rod using a direct stress testing machine (see Fig. 228 below).

Experiments show that the decrease of the cross section of a rod is proportional to the unit elongation ε. Denoting by ε_{tr} the relative contraction of the edge of the transversal face* of the cube (see Fig. 226) we can write

$$\varepsilon_{tr} = \mu\varepsilon \tag{81.4}$$

where μ is a constant factor called *Poisson's ratio*. The coefficient μ, like Young's modulus E, is one of the basic characteristics of the elastic properties of a material.

Simple considerations show that Poisson's ratio μ cannot exceed 1/2 for a homogeneous isotropic material.

Suppose that a cube with side a and volume a^3 is (mentally) separated out from the rod (before the latter is subjected to tension; see Fig. 226). If the

* This is the face to which the stresses appearing in tension are normal.

FIG. 225

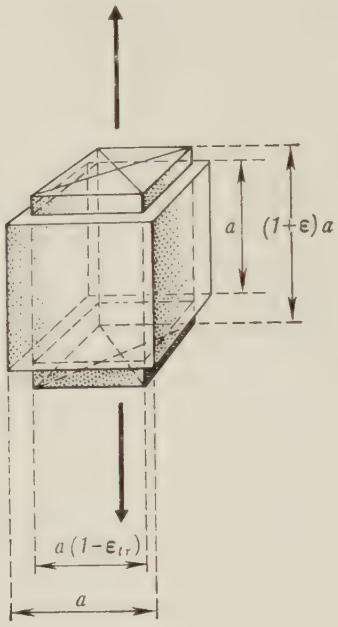

FIG. 226

edges of the cube are parallel to the axis of the rod the volume of the cube after the deformation of the rod becomes equal to

$$a^3(1+\varepsilon)(1-\varepsilon_{tr})^2 = a^3(1+\varepsilon)(1-\mu\varepsilon)^2 =$$
$$= a^3(1+\varepsilon-2\mu\varepsilon+\mu^2\varepsilon^2-2\mu\varepsilon^2+\mu^2\varepsilon^3) \tag{81.5}$$

Since the volume cannot decrease under the tension we must have

$$\varepsilon(1-2\mu) + \text{very small quantities} \geqslant 0 \tag{81.6}$$

whence, taking into account that $\varepsilon > 0$ and neglecting the very small quantities, we obtain

$$\mu \leqslant 1/2 \tag{81.7}$$

82. Physical Phenomena in a Deformable Body. Properties of Materials

Physical processes taking place as a body is subjected to a deformation are extremely complex and the explanation of the relationships between the forces and the deformations produced by them is rather intricate. There still remain many important aspects of this problem which have not yet been investigated thoroughly.

The phenomena described in Sec. 81 are characteristic of metals. As is known from the X-ray diffraction study, metals in the ordinary state are collections of small crystals distributed chaotically. It is known that the atoms in crystals form a certain configuration (a crystal lattice). For instance, the crystal lattice

of aluminium consists of a collection of similar cells adjoining each other. Every cell is a cube at whose vertices atoms are placed, the centre of each face of the lateral cubes containing one more atom. Such a structure of a crystal cell is called a body-centred (cubic) lattice.

It is evident that if the whole specimen consisted of one crystal (monocrystal) then, generally, its elastic properties are different for different directions. Bodies of that kind are said to be *anisotropic*. In reality, small crystals forming a piece of metal are distributed chaotically over the piece and are oriented

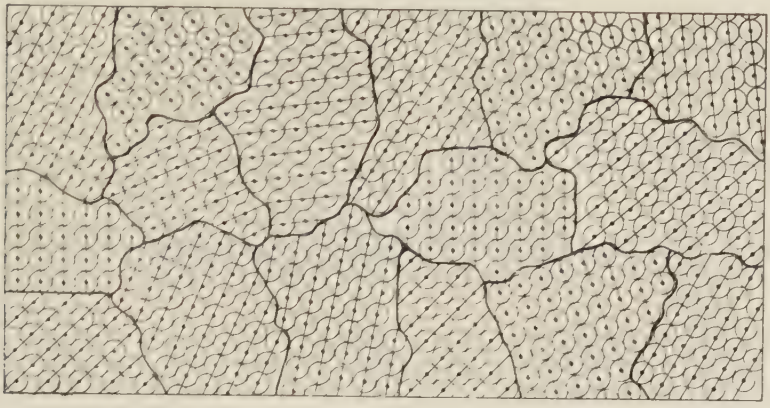

FIG. 227

in all possible ways relative to each other. Such a situation is shown schematically in Fig. 227. Therefore, the elastic properties of metals are the same in various directions and a piece of metal is an *isotropic* body. A plastic deformation of a monocrystal is accompanied by *gliding* along certain planes. Some parts of the crystal move easily relative to each other along these *gliding planes* and, after the load has been removed, they remain in the positions achieved during the deformation. It is likely that the same phenomena are involved in deformations of small crystals constituting a body subject to plastic deformations.

A rough scheme of a deformation in a piece of metal can be represented as follows. In the elastic (proportional) region the crystals change their shape without shifting and rupturing. After the load has been removed they return to the original positions. In the region of plastic deformations, besides changes of the shapes of the crystals, there take place gliding and displacements of the crystals relative to each other and partial rupture. These changes do not disappear completely after the load has been removed and the body remains in a state of deformation, that is residual strains appear in it.

Plastic deformations play an essential role in technology: such processes as stamping, bending and hammering metal work-pieces are only possible because of the presence of plastic deformations. It is evident that if metals could only be in a state of elastic deformation these technological processes would be impossible.

Mechanical properties of a metal or of some other material are determined experimentally by extending a specimen shaped as a rod of definite form having definite dimensions and processed in a definite way. The specimen is sub-

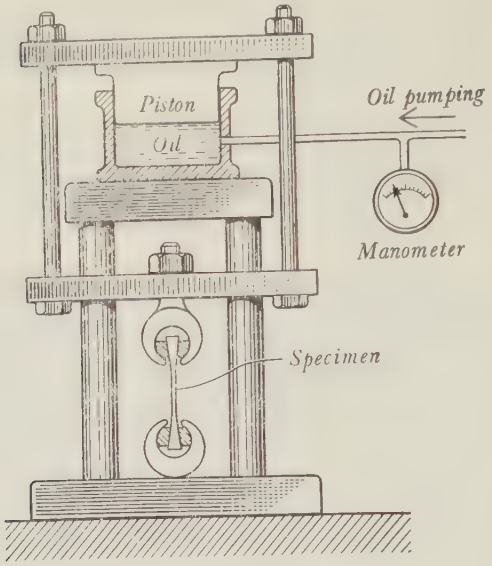

FIG. 228

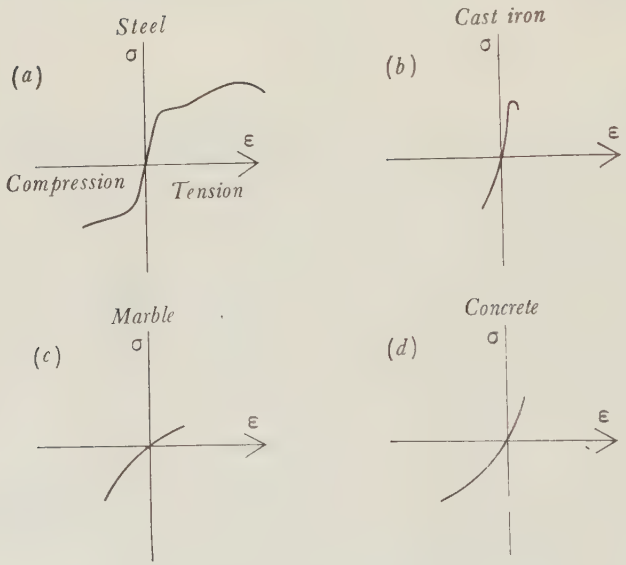

FIG. 229

jected to tension in special direct stress (testing) machines (Fig. 228) based on the principle of a hydraulic press. The measurement of the pressure in the cylinder with the piston (see Fig. 228) makes it possible to determine the tensile force; the displacement of the piston measured with a high precision enables us to find the elongation of the rod.

The connection between strains and stresses in a compression of rods is found, in principle, in the same manner. In compression tests shorter and wider rods are taken in order to avoid the bending of the rods as they are compressed. Metals have the same value of Young's modulus both for tension and for compression. In Fig. 229*a* a typical stress-strain curve for ordinary steel is shown. As is seen, the proportional limit has another value in the case of compression and the character of the shape of the curve in the region of plastic deformations is also slightly different.

For some other materials the stress-strain curves are of essentially different type. For example, for cast iron such a curve is shown in Fig. 229*b*. Practically, in the tension of a specimen of cast iron there is no region of plastic deformation. After the elastic limit in tension has been attained a very small flow region follows and then the specimen immediately starts rupturing. The materials whose stress-strain curve $\sigma(\varepsilon)$ is similar to that of cast iron are said to be *brittle* in contrast to the *ductile* materials having a comparatively wide region of plastic deformations. This distinction between the properties of brittle and ductile materials plays an important role in the choice of the material for practical applications. A component of a machine made of a ductile material subject to stresses which may sometimes slightly exceed the elastic limit is not ruptured while a component made of a brittle material can be ruptured in the same circumstances.

It should be noted that, practically cast iron has no proportional region; as is seen from the stress-strain curve, in Fig. 229*b*, the dependence of the stress on the strain is non-linear even for small values of ε. For comparison, in Fig. 229*c* and *d* the stress-strain curves for some other materials (for marble and concrete) are shown. The brittle materials, for instance such as marble and concrete, have greater ultimate strengths for compression than for tension.

Only a thorough knowledge of the elastic properties of the materials makes it possible to construct durable, reliable and not too cumbersome machines, buildings, etc.

83. Internal Forces and Stresses

Stresses appearing in a solid acted upon by external forces can easily be investigated by separating (mentally) a certain part from such a body. This separated part is acted upon by forces exerted on it by the other parts of the body, or, as we say, there are stresses on the surface bounding the separated part. The stresses always satisfy certain conditions implied by the fact that in the case of equilibrium (i.e. in the case of a state of rest or uniform motion) the resultant of the forces applied to the separated part is equal to zero and that in the case of a non-uniform motion the resultant is equal to the product of the mass of that part by its acceleration. Besides, quite analogous conditions must hold for the moments of the forces. Thus, if we take the projections of the forces and of the moments on three coordinate axes we obtain *six* equations which should be satisfied by the forces acting on the given part: three equations for the projections of the forces and three equations for the moments of the forces with respect to the three axes. These conditions do not obviously depend on the character of the deformations and have the same form for both the elastic region and the region of plastic deformations.

Let us consider the stresses appearing in the simplest case of a homogeneous rod subjected to tension. Imagine that a small prism is (mentally) cut from the rod as it is shown in Fig. 230*a* where the right base of the prism is shaded. Then,

obviously, the stresses on the four lateral faces of the prism are equal to zero and only the stresses on its two bases having magnitudes σ_0 and σ and opposite directions normal to the axis of the rod are different from zero. The stress σ_0 is given by the formula

$$\sigma_0 = \frac{F}{S} \tag{83.1}$$

where F is the force applied to the corresponding cross section of the rod normal

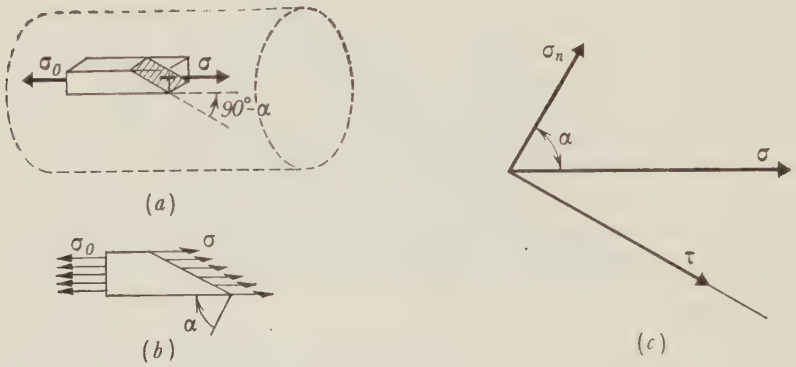

FIG. 230

to its axis and S is the cross-section area. If the rod is homogeneous the stresses are the same across the whole section and have the same value for all the cross sections provided that the rod is at rest (or in a uniform motion).

The stress σ in the oblique section of the rod in which the other base of the prism lies (the normal to the section forms an angle α with the axis of the rod) is at an angle of $90° - \alpha$ to the plane of that section. The magnitude of the stress σ can be found from the equilibrium condition for the prism we have separated. It is evident that the resultants of the forces applied to the faces of the prism must be horizontal and their magnitudes are equal and the directions are opposite. It follows that

$$\sigma_0 S = \sigma \frac{S}{\cos \alpha}$$

that is

$$\sigma = \sigma_0 \cos \alpha \tag{83.2}$$

where S is the area of the normal section of the prism.

The stress σ can be resolved into a (normal) component σ_n perpendicular to the plane of the section and a (tangential) component τ tangent to the section (Fig. 230*c*). The normal component is given by the formula

$$\sigma_n = \sigma \cos \alpha = \sigma_0 \cos^2 \alpha \tag{83.3}$$

and the tangential component is determined by the formula

$$\tau = \sigma \sin \alpha = \sigma_0 \cos \alpha \sin \alpha \tag{83.4}$$

Both components depend on the angle α between the normal to the plane of the section and the axis of the rod. Changing the angle α between the normal to the section and the axis of the rod we obtain, according to formulas (83.2), (83.3), and (83.4), different values of the stress σ_n and τ. The tangential component attains its maximum magnitude equal to $\sigma_0/2$ when the normal to the section forms an angle of 45° with the axis of the rod, in this case the magnitude of the normal component obviously coincides with that of the tangential component (see Fig. 231).

Our analysis of a very simple state of strain of a solid (arising in tension) clearly indicates that in the general case the distribution of stresses in a solid can be very complex.

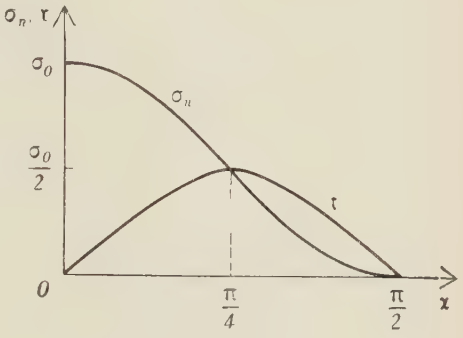

FIG. 231

84. Shearing Stresses and Strains

(Pure) shearing takes place when, for instance, a homogeneous rod of circular cross section undergoes torsion as one of its bases is turned about the axis of the rod through an angle φ. A deformation of this kind can be visualized if we take a rubber rod (or tube) with a set of orthogonal lines on its lateral surface (see Fig. 232*a*) and twist it (Fig. 232*b*). The circular lines perpendicular to the axis of the rod do not change under the deformation while the straight lines parallel to the axis go into spirals.

The rod can be mentally divided into sufficiently thin disks such that the corresponding parts of the screw lines can be thought of as straight line segments within the given disk when the rod is under the torsion. Next, let us cut a ring from the disk and then a small cube from that ring (as shown in Fig. 235*b* below). The disk being very thin, the upper face of the cube receives a small displacement relative to the lower face under that deformation (the lateral faces become oblique) and the angle between the lateral faces and the lower face differs from the right angle. The cube is in a state of a (*pure*) *non-dilatational strain*. It is characteristic of this kind of deformation that a parallelepiped cut from the body (subject to this deformation) in a corresponding manner undergoes a change of its angles solely. Shearing strain only appears when there are tangential stresses on the faces of the cube. Imagine that we have a body of the form of a cube with edge of 1 cm to whose four faces tangential forces are applied (see Fig. 233*a*) and that these forces are uniformly distributed over the faces. It is evident that for the cube to be in equilibrium it is necessary that all the tangential stresses should be equal:

$$\tau_1 = \tau_2 = \tau_3 = \tau_4 = \tau$$

Under the action of the tangential forces the right angles between the corresponding faces of the cube will decrease by a small angle γ (Fig. 233*b*). The magnitude of the deformation of the cube can be characterized by the angle γ which is in a definite relationship with the magnitude of the tangential stresses τ on the corresponding faces of the cube.

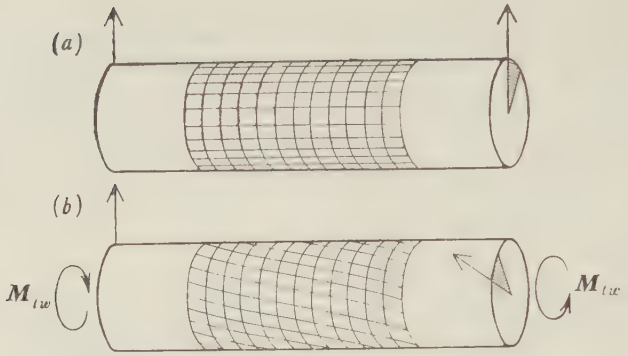

FIG. 232

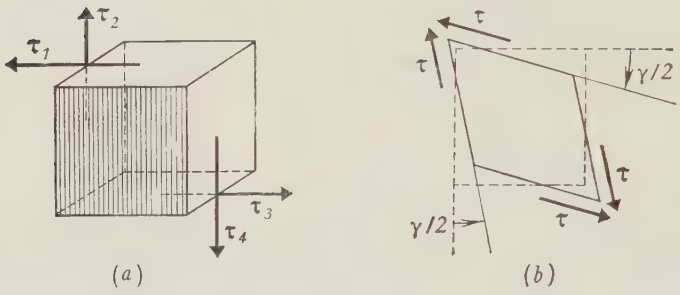

FIG. 233

Experiments show that for a given material the relationship between γ and τ is approximately of the same type as the connection between ε and σ for the same material (Fig. 234). Namely, in the elastic region there also exists a linear part for which

$$\tau = G\gamma \tag{84.1}$$

The coefficient G whose dimension is N/m^2 is called the *modulus of rigidity* (or the *modulus of elasticity in shear*).

Now let us determine the strains and the stresses appearing in a homogeneous rod of circular cross section subject to torsion (this deformation has already been discussed above). Let a rod of diameter D and length l_0 be made of a material whose modulus of elasticity in shear is equal to G and let a twisting moment M_{tw} produce a torsion through an angle of φ_0 (that is let the bases of the rod turn relative to each other through the angle φ_0).

First of all we note that in any cross section of the rod (perpendicular to its axis) the moment of the internal forces with respect to the axis is equal to M_{tw}, that is to the twisting moment producing the torsion of the rod. For, if we (mentally) separate a part B (see Fig. 235*a*) from the twisted rod then, since this part B is in equilibrium, the sum of the moments of all the forces acting on it must be equal to zero. One of the ends of this part is acted upon by the

moment M_{tw} of the external forces producing the torsion while on the other end there are tangential internal forces whose resultant moment is M_{int}. It follows that the magnitude of M_{int} is equal to that of M_{tw}, the directions of the moments being opposite.

Next we determine the distribution of the tangent stresses in a cross section of the rod and their connection with the strain. To this end we cut out of the rod a disk (Fig. 235*b*) of sufficiently small altitude dl at a distance of l from the fixed base of the rod and assume that in the torsion the lower base of the disk turns through an angle φ and the upper base through an angle $\varphi + d\varphi$. Further, from this disk we separate a ring with internal radius r and external radius $r + dr$. All the small cubes cut out from that ring are in the same state of shearing strain characterized by the angle $d\alpha$ common for all the cubes. The upper base of the disk being turned relative to the lower one by a small angle $d\varphi$, it is obvious that the angle $d\alpha$ must be proportional to the radius r of the ring. The (linear) displacement of the surface of the upper ring relative to the surface of the lower ring is

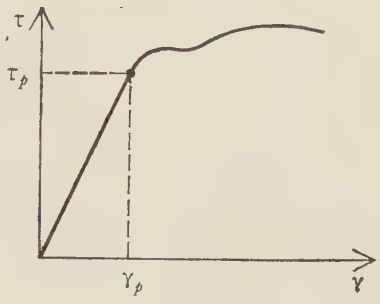

FIG. 234

$$da = dl\, d\alpha = r\, d\varphi \qquad (84.2)$$

Therefore the angle $d\alpha$ characterizing the shearing strain is given by the relation

$$d\alpha = r \frac{d\varphi}{dl} \qquad (84.3)$$

Thus, this angle is equal to the radius of the ring times the derivative $\frac{d\varphi}{dl}$ of the angle of torsion of the rod with respect to its length. We can now determine the tangential force acting on the surface of the ring whose area is $2\pi r\, dr$. By formulas (84.1) and (84.3), the stress τ is expressed as

$$\tau = G\, d\alpha = Gr \frac{d\varphi}{dl} \qquad (84.4)$$

We thus find the tangential force on the surface of the ring: it is equal to

$$\tau \cdot 2\pi r\, dr = 2\pi r^2 G \frac{d\varphi}{dl} dr \qquad (84.5)$$

The moment of this force with respect to the axis of the rod is

$$dM = 2\pi r^3 G \frac{d\varphi}{dl} dr \qquad (84.6)$$

On adding together the moments of all forces over the whole surface of the disk (to this end we integrate (84.6) with respect to r) we find

$$M = 2\pi G \frac{d\varphi}{dl} \int_0^{D/2} r^3\, dr = \frac{\pi D^4}{2^5} G \frac{d\varphi}{dl} \qquad (84.7)$$

This moment must be equal to the twisting moment M_{tw} because the moments applied to any two adjoining disks are equal to each other.

Equation (84.7) shows that for a homogeneous rod the derivative $\frac{d\varphi}{dl}$ of the angle of twist with respect to the length is constant along the whole rod. The angle of torsion of the ends of the rod which are at a distance of l_0 from each other is equal to

$$\varphi_0 = l_0 \frac{d\varphi}{dl} \quad \text{whence} \quad \frac{d\varphi}{dl} = \frac{\varphi_0}{l_0} \tag{84.8}$$

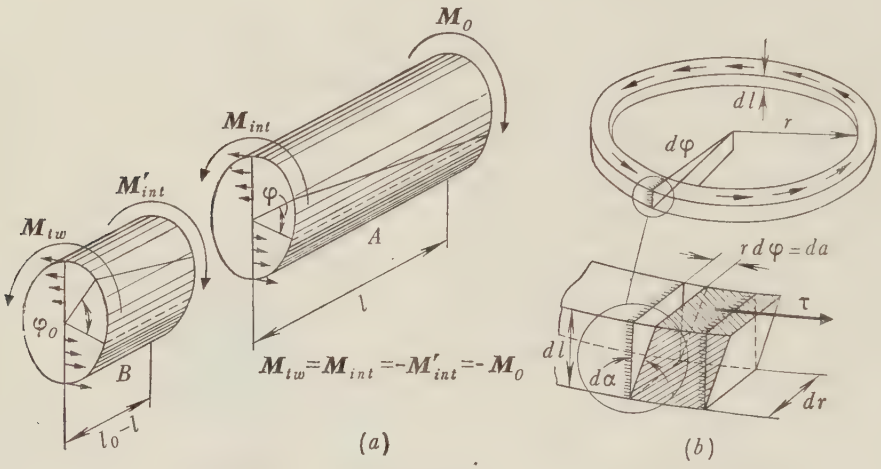

FIG. 235

The substitution of expression (84.8) into formula (84.7) results in the formula for dependence of the angle of twist φ_0 on the moment M_{tw}:

$$M_{tw} = M_0 = \frac{\pi D^4}{2^5} G \frac{\varphi_0}{l_0} \tag{84.9}$$

The coefficient $\frac{\pi D^4 G}{32 l_0}$ is called the *torsional rigidity* (*stiffness*) of the rod.

This formula should be taken into account in the construction of a shaft transmitting a given rotational moment in order to avoid the undesirable twist of the shaft.

85. Stresses in an Elastic Body. General Case

The examples we considered in the foregoing sections show that on different plane areas passing through a definite point of a loaded deformable body there are different stresses which however are in a certain relation determined by definite laws. It turns out that in the most general case all the stresses in the vicinity of an arbitrary point (on all the plane sections passing through that point and for any state of strain of a deformable body) are determined by six quantities (numbers) which are the components of a symmetric tensor of the second rank. Tensors prove convenient for describing stresses and some other physical quantities.

To find the relationship between the stresses on various areas near a given point it is convenient to consider the equilibrium conditions for a small tetra-

hedron cut out of the body in the vicinity of this point. Suppose that we have chosen a coordinate system with origin at that point whose axes are numbered as *1*, *2*, and *3* (see Fig. 236). The projections of vectors on these coordinate axes will be supplied with the corresponding indices. The plane area with unit normal vector $\mathbf{\nu}$ passes near the point *O*; the faces of the tetrahedron *ABCO* lie in this plane and in the coordinate planes.

The faces of the tetrahedron are acted upon by certain forces exerted by other parts of the body. Let us write down the equilibrium conditions for the tetrahedron *ABCO* under the action of these forces. We shall use the relations between the area dS of the face *ABC* and the areas dS_1, dS_2, and dS_3 of the faces perpendicular to the axes *1*, *2*, and *3* respectively: $dS_1 = \nu_1 dS$, $dS_2 = \nu_2 dS$, and $dS_3 = \nu_3 dS$ where ν_1, ν_2, and ν_3 are the direction cosines of the unit vector $\mathbf{\nu}$ (these direction cosines are equal to the projections of the vector $\mathbf{\nu}$ on the corresponding coordinate axes).

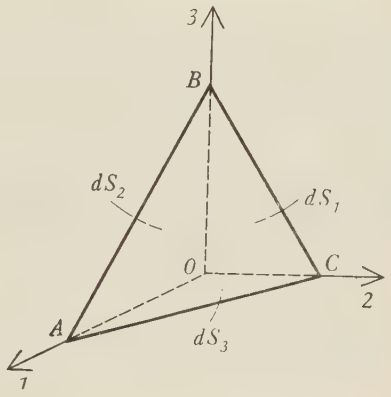

FIG. 236

Let us denote by $\mathbf{\sigma}_\nu\, dS$ the force acting on the area dS where $\mathbf{\sigma}_\nu$ is the stress on that area (the stress is a vector). For each area we can consider two opposite forces, namely a force exerted by the particle lying on one side of that area and a force exerted by the particles on the other side of the area. An example of such forces are the force $\mathbf{\sigma} dS$ and the reaction force $-\mathbf{\sigma}\, dS$ shown in Fig. 237*a*. We shall consider the "positive" side and the "negative" side of each area between which we shall distinguish according to a certain rule stated below.

The equilibrium conditions written below involve forces of the type of $\mathbf{\sigma}\, dS$ acting in the direction from the positive side to the negative side of the corresponding area. It should also be noted that among the forces and stresses under consideration there can be "tensile" and "compressive" ones. When using these terms we mean of course the components of the forces normal to the area in question (the corresponding cases are shown in Fig. 237*a* and *b*, respectively).

A tensile stress is usually considered positive while a compressive one is considered negative. The positive side of the area dS is the one seen from the outside of the tetrahedron. The positive side of an area lying in a plane perpendicular to the axis i ($i = 1, 2, 3$) is the one which is seen from the terminus of the unit vector of that axis issued from the origin (see Fig. 238 where the section of the tetrahedron by a plane *abc* normal to the axis *1* is shown).

Taking into account our convention concerning the signs of the stresses, we can write the condition that the resultant of the forces applied to the tetrahedron is equal to zero (the equilibrium condition):

$$\mathbf{\sigma}_\nu\, dS - \mathbf{\sigma}_1\, dS_1 - \mathbf{\sigma}_2\, dS_2 - \mathbf{\sigma}_3\, dS_3 = 0 \qquad (85.1)$$

For the sake of visuality, the vectors representing the acting forces are shown in Fig. 239 for a simpler "plane" case when all the forces are parallel to the coordinate plane (*1*, *2*). In this case, instead of the tetrahedron, it is sufficient

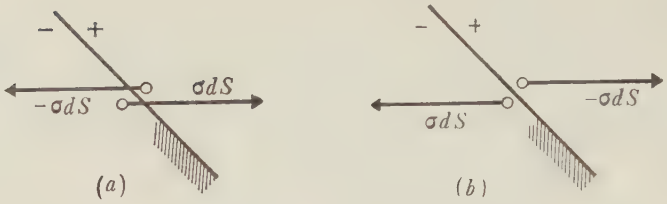

FIG. 237

to consider a small prism (whose section is shown in the figure), and all the vectors lie in one plane. It is evident that for this case the general equation (85.1) degenerates into

$$\sigma_\nu \, dS - \sigma_1 \, dS_1 - \sigma_2 \, dS_2 = 0$$

The mass forces acting on the particles lying within the tetrahedron can be neglected because the volume of the tetrahedron is an infinitesimal of the third

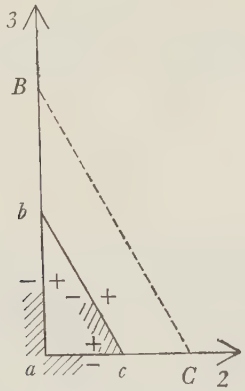

FIG. 238

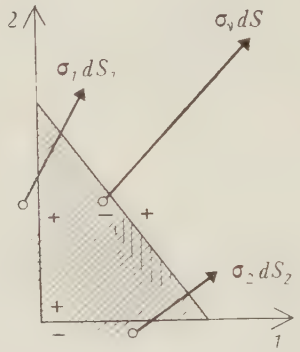

FIG. 239

order of smallness relative to its linear dimensions (while the areas of the faces are of the second order of smallness).

On dividing (85.1) by dS we arrive at the equation

$$\sigma_\nu = \nu_1\sigma_1 + \nu_2\sigma_2 + \nu_3\sigma_3 \qquad (85.2)$$

where ν_1, ν_2, and ν_3 are the direction cosines of the normal ν to area dS.

Expression (85.2) is analogous to the formula

$$a_\nu = \nu_1 a_1 + \nu_2 a_2 + \nu_3 a_3 = \nu a$$

expressing the projection a_ν of a vector a on an arbitrary direction specified by a unit vector ν in terms of the projections a_1, a_2, and a_3 of the vector and the direction cosines of ν.

Formula (85.2) determines the vector σ_ν (the stress on an arbitrary area with normal ν) in terms of the three vectors σ_1, σ_2, and σ_3. Given these three

vectors (they are the stresses on the three areas normal to the coordinate axes), the stress $\boldsymbol{\sigma}_\nu$ on any plane area with an arbitrary normal $\boldsymbol{\nu}$ can readily be found. Each vector is representable by a triple of numbers (which are its projections on the coordinate axes) and therefore three vectors are specified by nine numbers. Let us express the vectors $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$, and $\boldsymbol{\sigma}_3$ in terms of their projections:

$$\left.\begin{aligned}\boldsymbol{\sigma}_1 &= \sigma_{11}\boldsymbol{e}_1 + \sigma_{21}\boldsymbol{e}_2 + \sigma_{31}\boldsymbol{e}_3\\ \boldsymbol{\sigma}_2 &= \sigma_{12}\boldsymbol{e}_1 + \sigma_{22}\boldsymbol{e}_2 + \sigma_{32}\boldsymbol{e}_3\\ \boldsymbol{\sigma}_3 &= \sigma_{13}\boldsymbol{e}_1 + \sigma_{23}\boldsymbol{e}_2 + \sigma_{33}\boldsymbol{e}_3\end{aligned}\right\} \tag{85.3}$$

Here $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ are unit vectors in the directions of the coordinate axes *1*, *2*, and *3* respectively and σ_{ik} (i, $k = 1, 2, 3$) is the projection of the kth vector $\boldsymbol{\sigma}_k$ ($k = 1, 2, 3$) (this vector is the stress on the area dS_k perpendicular to the kth coordinate axis) on the ith coordinate axis. The nine numbers σ_{ik} are the *components* of a *second-rank tensor*. The components σ_{11}, σ_{22}, and σ_{33} with equal indices are the *normal stresses* on the areas dS_1, dS_2, and dS_3, respectively. The components with different indices (σ_{12}, σ_{31}, etc.) are the projections on the coordinate axes of the tangential stresses acting on the corresponding areas. The quantities σ_{ik} with $i \neq k$ are referred to as *shearing* tangential *stresses*.

Let us substitute the expressions of the vectors $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$, and $\boldsymbol{\sigma}_3$ given by (85.3) into equality (85.2) and then successively multiply scalarly the equality thus obtained by $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$. This results in the expressions of the projections of the vector $\boldsymbol{\sigma}_\nu$ on the axes *1*, *2*, and *3*:

$$\left.\begin{aligned}\sigma_{1\nu} &= \sigma_{11}\nu_1 + \sigma_{12}\nu_2 + \sigma_{13}\nu_3\\ \sigma_{2\nu} &= \sigma_{21}\nu_1 + \sigma_{22}\nu_2 + \sigma_{23}\nu_3\\ \sigma_{3\nu} &= \sigma_{31}\nu_1 + \sigma_{32}\nu_2 + \sigma_{33}\nu_3\end{aligned}\right\} \tag{85.4}$$

because the axes are mutually orthogonal and therefore $\boldsymbol{e}_i\boldsymbol{e}_k = 1$ for $i = k$ and $\boldsymbol{e}_i\boldsymbol{e}_k = 0$ for $i \neq k$. System of relations (85.4) can be written (symbolically) as

$$\boldsymbol{\sigma}_\nu = \mathscr{T}\boldsymbol{\nu} \tag{85.5}$$

where $\mathscr{T}$ is a tensor of rank two called the *stress tensor*. The tensor $\mathscr{T}$ is represented by the matrix

$$\mathscr{T} = \begin{pmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{pmatrix} \tag{85.6}$$

Its components are the projections of the vectors $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$, and $\boldsymbol{\sigma}_3$ on the coordinate axes. Thus, the vector representing the stress on an area with normal $\boldsymbol{\nu}$ is equal to the product of the tensor $\mathscr{T}$ by the normal vector $\boldsymbol{\nu}$.

It can be shown that when the coordinate system is rotated the components σ_{ik} of the tensor are transformed like the products x_ix_k of the corresponding coordinates of the point. This property agrees with the general mathematical definition of a second-rank tensor and indicates that the collection of the numbers σ_{ik} is in fact a tensor.

Condition (85.1) is necessary for the equilibrium of the tetrahedron $ABCO$ but it is not sufficient. Besides this condition, it should be required that the sums of the moments of all forces applied to the surface of the tetrahedron with

respect to any axis should be equal to zero*. The moment of a force can be represented by a vector and it is therefore sufficient that the sums of the moments of the forces acting on the tetrahedron with respect to any three axes parallel to the three coordinate axes should be zero. For the sake of simplicity, let us consider the condition that the sum of the moments with respect to the axis parallel to the axis *3* and passing through the centre of gravity of the area dS is equal to zero. The area dS being considered infinitesimal, the stress in that area can be regarded as being constant over the area. Under this assumption the resultant of all the forces acting on each area dS is applied at its centre of gravity. In Fig. 240 we see the top view of the tetrahedron from the direction of the axis *3*. The axis parallel to axis *3* and passing through the centre of gravity of the area dS_3 is seen as the point c in Fig. 240. It is evident that the forces $\sigma_\nu\, dS$ and $\sigma_3\, dS_3$ intersect that axis. Hence, the moments of these forces are equal to zero. Now it only remains to consider the moments of the forces $\sigma_1\, dS_1$ and $\sigma_2\, dS_2$. The equilibrium condition for these moments reduces to

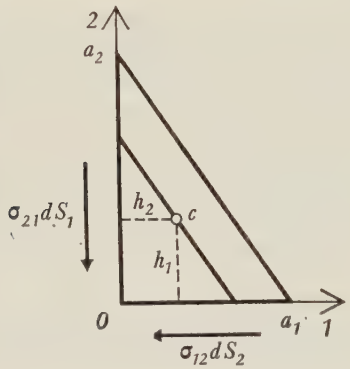

FIG. 240

$$\sigma_{21}\, dS_1 h_1 - \sigma_{12}\, dS_2 h_2 = 0$$

where h_1 and h_2 are the distances from the axis c to the areas dS_1 and dS_2, respectively. Denoting the edges of the tetrahedron directed along the corresponding coordinate axes as a_1, a_2, and a_3 (Fig. 240) we can write: $dS_1 = 1/2 a_2 a_3$ and $dS_2 = 1/2\ a_1 a_3$. Taking into account these expressions and the fact that $a_2 = 3h_2$ and $a_1 = 3h_1$, we obtain

$$\sigma_{12} = \sigma_{21} \tag{85.7}$$

We see that the tangential stresses on the areas dS_1 and dS_2 are normal to the axis *3*, are directed towards the edge of the tetrahedron lying on that axis or from it and *are equal to each other*.

It can similarly be shown that

$$\sigma_{31} = \sigma_{13} \quad \text{and} \quad \sigma_{32} = \sigma_{23} \tag{85.8}$$

Thus, to determine the stresses in the vicinity of a point it is in fact necessary to know not nine but only six numbers. The stress tensor $\mathcal{T}$ is *symmetric*: its components with different indices are pairwise equal (see (85.7) and (85.8)). This property is known as the *law of equal shearing stresses.*

The indicated equality of shearing stresses is a very important fact. Since the coordinate system with origin at a given point can be chosen quite arbitrarily, it follows that on any two small plane areas intersecting at a right angle

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* By condition (85.1), the resultant force acting on the tetrahedron is equal to zero and therefore the system of the external forces applied to the tetrahedron reduces to a couple of forces. The moment of a couple being the same about any point, the moments of the couple with respect to any parallel axes coincide.

at a point the components of the shearing stresses normal to the intersection line of these planes are always equal in their magnitudes and are simultaneously directed towards that line or from it. This means that the shearing stresses on the faces of any sufficiently small cube cut out from a deformable body are connected with each other by a definite relationship. For instance, the components of the shearing stresses parallel to the plane (*1*, *2*) applied to the faces normal to that plane are equal and are directed as shown in Fig. 241. In the case represented in Fig. 241*a* the tangential stresses are considered positive and in

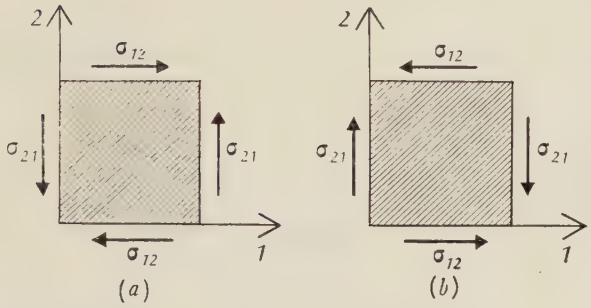

FIG. 241

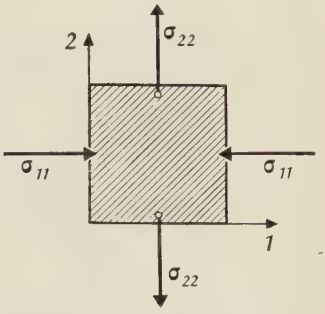

FIG. 242

the case shown in Fig. 241*b* they are considered negative because in the former case the stresses on the positive sides of the areas are in the directions of the coordinate axes and in the latter case they are in opposite directions. The same applies to the tangential components of the stresses parallel to the plane (*2*, *3*) and to the plane (*3*, *1*).

The properties of the stresses we have established thus imply that the sum of the shearing forces acting on any small cube is equal to zero. Therefore, the normal forces acting on the faces of a small cube must mutually balance irrespective of the tangential forces. For example, in Fig. 242 the normal stresses on the faces parallel to the axis *3* are shown. Here the stresses σ_{22} are *positive* (*tensile*) while the stresses σ_{11} are *negative* (*compressive*).

An important consequence of this analysis is that for any point of a body in a state of strain among all the possible directions of unit vector $\mathbf{v}$ issued from the point there are always at least three mutually perpendicular directions for which the vectors representing the stresses (on the planes with these three vectors $\mathbf{v}$ as normals) coincide with the normals $\mathbf{v}$ in their directions. Or, equivalently, the shearing stresses on these plane areas are equal to zero. Such three directions specify the so-called *principal axes of stress*. The corresponding normal stresses are referred to as *principal stresses*. When the axes of principal stress are taken as coordinate axes the stress tensor is represented by only three numbers (by the numerical values of the principal stresses).

The axes of principal stress can be found in the same way as the principal axes of inertia which were determined for the inertia tensor in Sec. 64. The only distinction between these cases is that for the inertia tensor the moments of inertia with respect to the principal axes are always positive quantities while

the principal stresses can be positive (tensile) or negative (compressive). Therefore, in the general case, if the surface analogous to the ellipsoid of inertia constructed for the stress tensor is a central surface of the second order it can be either an ellipsoid or a hyperboloid.

86. Small Deformations of a Body

The particles of an elastic body change their mutual positions under the action of external forces, that is the body or its parts change their form. This

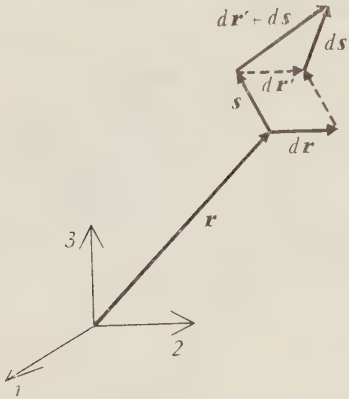

FIG. 243

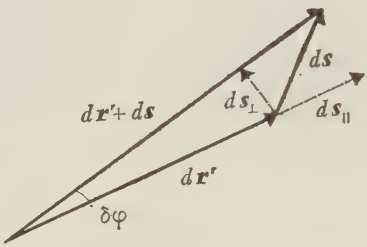

FIG. 244

is the phenomenon which we call a *deformation* of the body. In this section we shall study small deformations.

Under the action of the forces every particle receives a displacement represented by a vector $\boldsymbol{s}$. If all the particles have the same displacement $\boldsymbol{s}$ the body undergoes no deformation; in this case it simply gains, as a whole, the displacement $\boldsymbol{s}$. A deformation only appears when some (or all) particles of the body have different displacement vectors $\boldsymbol{s}$, that is when the vector $\boldsymbol{s}$ is a function of the position vector $\boldsymbol{r}$ of the particle characterizing the location of the particle before the deformation. A particle located originally at a point with radius vector $\boldsymbol{r}$ gains a displacement $\boldsymbol{s}$ while another particle with radius vector $\boldsymbol{r} + d\boldsymbol{r}$ has a displacement $\boldsymbol{s} + d\boldsymbol{s}$ (Fig. 243). The deformation is characterized by the quantity $d\boldsymbol{s}$ or, more precisely, by the relationship between $d\boldsymbol{s}$ and $d\boldsymbol{r}$. When $d\boldsymbol{s}$ depends linearly on $d\boldsymbol{r}$ (which is the case for small deformations) the relationship between $d\boldsymbol{s}$ and $d\boldsymbol{r}$ is determined by a tensor $\mathscr{S}$. Our immediate task is to determine the components of the tensor $\mathscr{S}$.

The fact that the relations between $d\boldsymbol{s}$ and $d\boldsymbol{r}$ are linear implies that the particles which are on the line of the vector $d\boldsymbol{r}$ before the deformation occupy a new position on the line of the vector $d\boldsymbol{r}' + d\boldsymbol{s}$ where $d\boldsymbol{r}' = d\boldsymbol{r}$ as is shown in Fig. 244. Since we deal with small deformations there must be $ds \ll dr$. This means that the line segment dr undergoes an extension or a contraction by a small quantity $ds_{\parallel}$ and turns through a small angle $\delta\varphi = \frac{ds_{\perp}}{dr} \ll 1$. The ratio

$\varepsilon = \frac{ds_{\|}}{dr}$ is termed *unit elongation* (it may of course be negative and then it can be called "unit shortening").

Let us consider the positions of the particles of the body characterized by their radius vectors $\boldsymbol{r}$ relative to a Cartesian coordinate system $Ox_1x_2x_3$, directions of the coordinate axes being specified by unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ respectively. Then

$$\left.\begin{aligned} \boldsymbol{s} &= s_1\boldsymbol{e}_1 + s_2\boldsymbol{e}_2 + s_3\boldsymbol{e}_3 \\ d\boldsymbol{s} &= ds_1\boldsymbol{e}_1 + ds_2\boldsymbol{e}_2 + ds_3\boldsymbol{e}_3 \\ d\boldsymbol{r} &= dx_1\boldsymbol{e}_1 + dx_2\boldsymbol{e}_2 + dx_3\boldsymbol{e}_3 \end{aligned}\right\} \tag{86.1}$$

where each of the quantities s_1, s_2, and s_3 depends on x_1, x_2, and x_3 (x_1, x_2, and x_3 are the projections of the vector $\boldsymbol{r}$). Therefore, we can write

$$\left.\begin{aligned} ds_1 &= \frac{\partial s_1}{\partial x_1}dx_1 + \frac{\partial s_1}{\partial x_2}dx_2 + \frac{\partial s_1}{\partial x_3}dx_3 \\ ds_2 &= \frac{\partial s_2}{\partial x_1}dx_1 + \frac{\partial s_2}{\partial x_2}dx_2 + \frac{\partial s_2}{\partial x_3}dx_3 \\ ds_3 &= \frac{\partial s_3}{\partial x_1}dx_1 + \frac{\partial s_3}{\partial x_2}dx_2 + \frac{\partial s_3}{\partial x_3}dx_3 \end{aligned}\right\} \tag{86.2}$$

These equalities express the linear dependence between $d\boldsymbol{s}$ and $d\boldsymbol{r}$ in a small neighbourhood of the point $\boldsymbol{r}$. The magnitudes of the derivatives $\frac{\partial s_1}{\partial x_1}, \ldots$ remain constant within that small region*.

Equalities (86.2) indicate that the relationship between $d\boldsymbol{s}$ and $d\boldsymbol{r}$ is specified by the tensor

$$\mathscr{S} = \begin{pmatrix} \frac{\partial s_1}{\partial x_1} & \frac{\partial s_1}{\partial x_2} & \frac{\partial s_1}{\partial x_3} \\ \frac{\partial s_2}{\partial x_1} & \frac{\partial s_2}{\partial x_2} & \frac{\partial s_2}{\partial x_3} \\ \frac{\partial s_3}{\partial x_1} & \frac{\partial s_3}{\partial x_2} & \frac{\partial s_3}{\partial x_3} \end{pmatrix} \tag{86.3}$$

With the aid of the tensor $\mathscr{S}$ equalities (86.2) can be written as

$$d\boldsymbol{s} = \mathscr{S}\, d\boldsymbol{r} \tag{86.4}$$

As will be shown, the tensor $\mathscr{S}$ determines not only the state of strain of an elementary volume separated (mentally) from a deformable body near an arbitrary point $\boldsymbol{r}$ but also a rotation of that volume as a whole without deformation. Let us represent $\mathscr{S}$ as a sum of two tensors $\mathscr{S}_s$ and $\mathscr{S}_a$, the former being symmetric and the latter antisymmetric:

$$\mathscr{S} = \mathscr{S}_s + \mathscr{S}_a \tag{86.5}$$

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* The partial derivative $\frac{\partial s_k}{\partial x_i}$ is the ratio of the increment ∂s_k of the kth component (projection) of the vector $\boldsymbol{s}$ ($k = 1, 2, 3$) received only due to the increment ∂x_i of the ith ($i = 1, 2, 3$) component of the vector $\boldsymbol{r}$.

where

$$\mathscr{S}_s = \begin{pmatrix} \frac{\partial s_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial s_1}{\partial x_2}+\frac{\partial s_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial s_1}{\partial x_3}+\frac{\partial s_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial s_2}{\partial x_1}+\frac{\partial s_1}{\partial x_2}\right) & \frac{\partial s_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial s_2}{\partial x_3}+\frac{\partial s_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial s_3}{\partial x_1}+\frac{\partial s_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial s_3}{\partial x_2}+\frac{\partial s_2}{\partial x_3}\right) & \frac{\partial s_3}{\partial x_3} \end{pmatrix} \qquad (86.6)$$

$$\mathscr{S}_a = \begin{pmatrix} 0 & \frac{1}{2}\left(\frac{\partial s_1}{\partial x_2}-\frac{\partial s_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial s_1}{\partial x_3}-\frac{\partial s_2}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial s_2}{\partial x_1}-\frac{\partial s_1}{\partial x_2}\right) & 0 & \frac{1}{2}\left(\frac{\partial s_2}{\partial x_3}-\frac{\partial s_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial s_3}{\partial x_1}-\frac{\partial s_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial s_3}{\partial x_2}-\frac{\partial s_2}{\partial x_3}\right) & 0 \end{pmatrix}$$

It can easily be verified that $\mathscr{S}_a\, d\boldsymbol{r} = [\delta\boldsymbol{\varphi}\, d\boldsymbol{r}]$ where the vector $\delta\boldsymbol{\varphi}$ represents a very small angular displacement with very small components

$$\delta\varphi_1 = \frac{1}{2}\left(\frac{\partial s_3}{\partial x_2}-\frac{\partial s_2}{\partial x_3}\right), \quad \delta\varphi_2 = \frac{1}{2}\left(\frac{\partial s_1}{\partial x_3}-\frac{\partial s_3}{\partial x_1}\right),$$

$$\text{and } \delta\varphi_3 = \frac{1}{2}\left(\frac{\partial s_3}{\partial x_1}-\frac{\partial s_1}{\partial x_2}\right)$$

These components are equal to the small angles of rotation about the axes *1*, *2*, and *3*, respectively.

Indeed, we have

$$\mathscr{S}_a\, d\boldsymbol{r} = \begin{pmatrix} 0 & -\delta\varphi_3 & \delta\varphi_2 \\ \delta\varphi_3 & 0 & -\delta\varphi_1 \\ -\delta\varphi_2 & \delta\varphi_1 & 0 \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}$$

whence $(\mathscr{S}_a\, d\boldsymbol{r})_1 = -\delta\varphi_3\, dx_2 + \delta\varphi_2 dx_3 = [\delta\boldsymbol{\varphi}\, d\boldsymbol{r}]_1$, etc.

It can readily be shown that the vectors $[\delta\boldsymbol{\varphi}\, d\boldsymbol{r}]$ represent the displacements of the points of a rigid body undergoing a revolution by an angle of $\delta\varphi$ about an axis coinciding with the direction of the vector $\delta\boldsymbol{\varphi}$. That is why the deformation in a small region adjoining a point $\boldsymbol{r}$ is determined solely by the symmetric part $\mathscr{S}_s$ of the tensor $\mathscr{S}$.

Let us discuss the physical meaning of the components of the tensor $\mathscr{S}_s$. The diagonal components $\frac{\partial s_1}{\partial x_1}$, $\frac{\partial s_2}{\partial x_2}$ and $\frac{\partial s^3}{\partial x_3}$ are unit elongations (or shortenings) for the distances between the particles lying on line segments parallel to the unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ of the coordinate axes *1*, *2*, and *3*, respectively. The non-diagonal components specify the rotations of the line segments which are parallel to the coordinate axes before the deformation. These lines turn through some small angles; the projections of the lines corresponding to the axes *1* and *2* onto the plane (*1*, *2*) are shown in Fig. 245. As is seen from the figure, the sum

$$\frac{\partial s_1}{\partial x_2}+\frac{ds_2}{\partial x_1}$$

is equal to the increments (gained under the deformation) by the right angles in the plane normal to the axis *3*. These angular increments are denoted as $\gamma_{12} = \gamma_{21}$. Similarly, the sum

$$\frac{\partial s_1}{\partial x_3}+\frac{\partial s_3}{\partial x_1} = \gamma_{13} = \gamma_{31} \qquad (86.7)$$

is equal to analogous increments of the right angles in the plane normal to axis *2*. Further, for the plane normal to axis *1* an analogous role is played by

$$\frac{\partial s_2}{\partial x_3} + \frac{\partial s_3}{\partial x_2} = \gamma_{23} = \gamma_{32} \tag{86.8}$$

All that has been said applies to a small neighbourhood of the point r. The unit elongations are denoted

$$\varepsilon_1 = \frac{\partial s_1}{\partial x_1}, \quad \varepsilon_2 = \frac{\partial s_2}{\partial x_2} \quad \text{and} \quad \varepsilon_3 = \frac{\partial s_3}{\partial x_3} \tag{86.9}$$

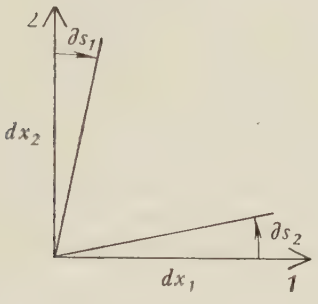

FIG. 245

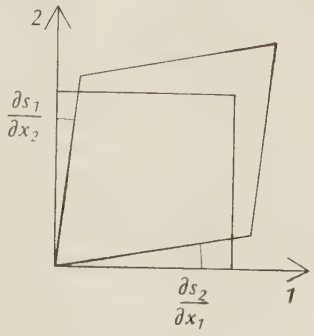

FIG. 246

A positive value of ε_i ($i = 1, 2, 3$) corresponds to a real elongation while its negative value corresponds to unit shortening. However, we usually speak of unit elongation in both cases bearing in mind that a negative value of ε_i ($i = 1, 2, 3$) means that in reality a shortening takes place.

Using this notation we can write the symmetric part $\mathscr{S}_s$ of the tensor $\mathscr{S}$ in the form

$$\mathscr{S}_s = \begin{pmatrix} \varepsilon_1 & \frac{1}{2}\gamma_{12} & \frac{1}{2}\gamma_{13} \\ \frac{1}{2}\gamma_{21} & \varepsilon_2 & \frac{1}{2}\gamma_{23} \\ \frac{1}{2}\gamma_{31} & \frac{1}{2}\gamma_{32} & \varepsilon_3 \end{pmatrix} \tag{86.10}$$

The symmetric tensor (86.10) is called a *strain tensor*.

Let us consider a sufficiently small part of a body having the shape of a cube before the deformation. Then after the deformation the edges of the cube increase or decrease while the right angles of its faces undergo distortions by the quantities γ_{ik}. In Fig. 246 we see the section of such a cube by a plane normal to axis *3*. One pair of the right angles decreased by angles of magnitude γ_{12} and the other pair increased by the same angle $\gamma_{12} = \gamma_{21}$.

Every symmetric tensor (such is the tensor $\mathscr{S}_s$) has at least three mutually orthogonal *principal directions*. A small cube whose faces are perpendicular to the principal directions undergoes only elongations (or shortenings) along its edges while the angles of the faces of such a cube do not vary. There can also be a special case when the edges of the cube retain their lengths and only the angles between the edges change. Such a deformation is called a non-dilatational strain (this type of deformation was mentioned in Sec. 84).

87. Relationship Between Stresses and Strains

In the elastic (proportional) region the dependence between the strain and stress tensors is linear. This is the general form of *Hooke's law* whose special case we used in the simplest situation of a longitudinal stress. General theoretical considerations indicate that for an anisotropic (crystalline) body whose elastic properties are different in different directions there is a dependence of each component of the strain tensor on all the components of the stress tensor. The corresponding calculations show that, since the tensors are symmetric, the number of independent coefficients specifying these linear relations is equal to 21. Thus, the elastic properties of an anisotropic material are characterized by twenty one parameters.

In the simpler case of an isotropic body it turns out that only two independent coefficients are needed for establishing the relationship between the strain and stress tensors. This relationship can be demonstrated in the simplest way if the principal axes of the stress tensor are taken as coordinate axes. The same axes are obviously the principal axes of the strain tensor. For a sufficiently small cube with faces perpendicular to the principal directions of the stress tensor only the normal stresses can be different from zero and the deformation of the cube is such that its faces receive displacements along the normals without distortion of the right angles between them.

If $\sigma_{11} \neq 0$ while $\sigma_{22} = \sigma_{33} = 0$ then the stresses σ_{11} on the faces normal to axis *1* produce unit elongation ε_1 along that axis. By Hooke's law, $\sigma_{11} = E\varepsilon_1$ where E is Young's modulus. Such a simple relation holds only in the simple case when there are no stresses on the faces normal to axes *2* and *3*. In the general case the normal stresses appear on all the faces and therefore the unit elongation along axis *1* is not equal to $\varepsilon_1 = \sigma_{11}/E$ because the stresses σ_{22} and σ_{33}, when different from zero, produce contractions (in the direction of axis *1*) proportional to the elongations under the stresses σ_{22} and σ_{33}.

The corresponding proportionality factor is equal to Poisson's ratio μ (Sec. 81). Therefore, the unit elongation along axis *1* is less than σ_{11}/E by the magnitude of the resultant contraction due to the positive (tensile) stresses on the other two pairs of the faces. The contraction produced by the stress σ_{22} is equal to

$$\mu\varepsilon_2 = \mu\frac{\sigma_{22}}{E}$$

and the contraction due to the stress σ_{33} is equal to

$$\mu\varepsilon_3 = \mu\frac{\sigma_{33}}{E}$$

Assuming that the results of the actions of the forces are mutually independent and adding together the deformation inputs of the forces on all pairs of the faces we obtain the following expression for unit elongation along axis *1*:

$$\varepsilon_1 = \frac{\sigma_{11}}{E} - \mu\left(\frac{\sigma_{22}}{E} + \frac{\sigma_{33}}{E}\right) = \frac{1}{E}\{\sigma_{11} - \mu(\sigma_{22} + \sigma_{33})\} \qquad (87.1)$$

The proportionality of one component ε_1 of the strain tensor to the components of the stress tensor is thus determined only by the two coefficients $1/E$ and μ/E, that is by the two parameters E and μ characterizing the elastic properties of the isotropic material. As we know, the quantities E and μ are found in simple experiments; the corresponding data are compiled into tables that can be found in reference books.

It can similarly be proved that

$$\left.\begin{aligned}\varepsilon_2 &= \frac{1}{E}\{\sigma_{22} - \mu(\sigma_{33} + \sigma_{11})\}\\ \varepsilon_3 &= \frac{1}{E}\{\sigma_{33} - \mu(\sigma_{11} + \sigma_{22})\}\end{aligned}\right\} \tag{87.2}$$

In the derivation of formulas (87.1) and (87.2) we assume that, due to the proportionality law, the deformations produced by different forces can simply be added together. Namely, if a force a produces a deformation b and a force c a deformation d then, by the linearity, the simultaneous action of both forces a and c determined by the force $a + c$ generates a deformation of magnitude $b + d$.

Denoting

$$\sigma_{11} + \sigma_{22} + \sigma_{33} = 3\sigma \tag{87.3}$$

we can rewrite the formulas obtained above in the form

$$\left.\begin{aligned}\varepsilon_1 &= \frac{1}{E}\{(1+\mu)\sigma_{11} - 3\mu\sigma\}\\ \varepsilon_2 &= \frac{1}{E}\{(1+\mu)\sigma_{22} - 3\mu\sigma\}\\ \varepsilon_3 &= \frac{1}{E}\{(1+\mu)\sigma_{33} - 3\mu\sigma\}\end{aligned}\right\} \tag{87.4}$$

Equalities (87.4) have been derived for the case when the coordinate axes coincide with the principal directions. If the coordinate axes (and the corresponding directions of the edges of a small cube) are chosen in a diffferent way the cube can undergo shearing strains as well. However, the linear character of the dependence between components of the strain and stress tensors makes it possible to use the same formulas in the most general case. To prove what has been said, let us represent the stress tensor as a sum of two tensors one of which involves only the normal stresses and the other only the shearing (tangential) stresses:

$$\mathscr{T} = \mathscr{T}_{\mathrm{n}} + \mathscr{T}_{\mathrm{t}} = \begin{pmatrix}\sigma_{11} & 0 & 0\\ 0 & \sigma_{22} & 0\\ 0 & 0 & \sigma_{33}\end{pmatrix} + \begin{pmatrix}0 & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & 0 & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & 0\end{pmatrix} \tag{87.5}$$

Then the deformations produced by the stresses entering into the tensor $\mathscr{T}_{\mathrm{n}}$ can be found by formulas (87.4) because in this case only the normal stresses are taken into account. To these deformations we must then add those produced by the shearing stresses entering into the tensor $\mathscr{T}_{\mathrm{t}}$.

Experiments show that the shearing stresses on the faces of the cube produce distortions of the corresponding right angles and that the increments of the angles are proportional to the stresses. The corresponding proportionality coefficient G (called the rigidity modulus or the modulus of elasticity in shear; see Sec. 81) is constant for a given isotropic material in proportional region. For instance, the stresses σ_{12} (Fig. 247*a*) generate distortions by angles γ_{12} of the right angles in the plane normal to axis *3* (Fig. 247*b*) and

$$\sigma_{12} = G\gamma_{12} \tag{87.6}$$

Similarly, for the tangential stresses on the faces parallel to axes *1* and *2* we have

$$\sigma_{23} = G\gamma_{23} \quad \text{and} \quad \sigma_{31} = G\gamma_{31} \tag{87.7}$$

The distortion of the right angles in a given coordinate plane by a quantity γ depends solely on the tangential stresses (equal in their magnitudes) on the four faces normal to that plane and are completely independent of the other tangential components of the stresses. There is therefore a very simple connec-

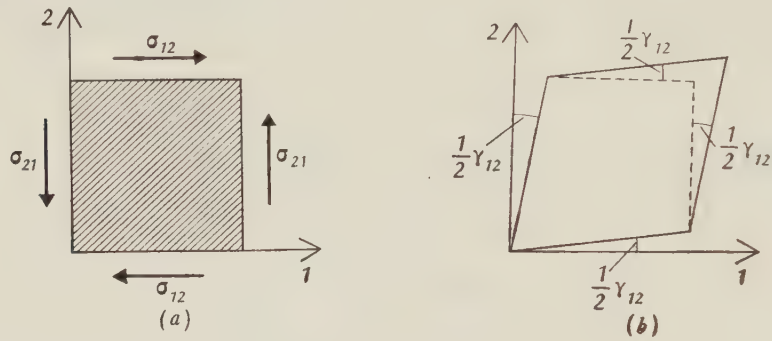

FIG. 247

tion between the tensor $\mathscr{T}_t$ and the tensor $\mathscr{S}''_s$ whose components are the shearing strains γ_{ik}. The tensor $\mathscr{S}''_s$ is a part of the strain tensor $\mathscr{S}_s$ which can be represented as the sum

$$\mathscr{S}_s = \mathscr{S}'_s + \mathscr{S}''_s$$

where

$$\mathscr{S}'_s = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} \quad \text{and} \quad \mathscr{S}''_s = \frac{1}{2}\begin{pmatrix} 0 & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & 0 & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & 0 \end{pmatrix} \tag{87.8}$$

Taking into account (87.6) and (87.7) we find that the sought-for connection is of the form

$$\mathscr{T}_t = \begin{pmatrix} 0 & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & 0 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & 0 \end{pmatrix} = G\begin{pmatrix} 0 & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & 0 & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & 0 \end{pmatrix} = 2G\mathscr{S}''_s \tag{87.9}$$

As to the tensor $\mathscr{T}_n$, according to (87.4) it can be written as

$$\mathscr{T}_n = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} = \frac{E}{1+\mu}\begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} + \frac{3\mu\sigma}{1+\mu}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{87.10}$$

The quantity 3σ can be expressed in terms of the quantity $3\varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$. Indeed, on adding together all the three equalities (87.4) we obtain

$$3\varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \frac{1+\mu}{E} 3\sigma - \frac{9\mu}{E} \sigma = \frac{3\sigma}{E} (1 + \mu - 3\mu)$$

whence

$$\sigma = \frac{E}{1-2\mu} \varepsilon \tag{87.11}$$

The substitution of this expression in (87.10) results in

$$\mathscr{T}_n = \frac{E}{1+\mu} \left(\mathscr{S}_s' + \frac{3\mu\varepsilon}{1-2\mu} \mathscr{E} \right) \tag{87.12}$$

where

$$\mathscr{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the *unit tensor*.

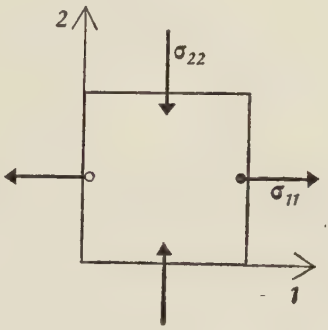

FIG. 248

Formula (87.12) expresses the connection between $\mathscr{T}_n$ consisting of the normal stresses σ_{jj} ($j = 1, 2, 3$), the tensor $\mathscr{S}_s$ whose components are the longitudinal strains ε_i ($i = 1, 2, 3$), and the quantity ε (or σ). Later we will show that ε and σ are scalars, that is they do not depend on the choice of the coordinate system although each of the quantities $\varepsilon_1, \varepsilon_2, \ldots$ and $\sigma_{11}, \sigma_{22}, \ldots$ depends on that choice.

We now establish a relation between the rigidity modulus G, Young's modulus E and Poisson's ratio μ. To this end we consider a simple case when the deformation is only due to the action of normal stresses (for properly chosen coordinates) and then show that for some other coordinates the same deformation is non-dilatational. Let us suppose that $\sigma_{33} = 0$; then the state of strain is only due to the action of the stresses σ_{11} and σ_{22}. Let $\sigma_{11} = \sigma_0$, $\sigma_{22} = -\sigma_0$ ($\sigma_{33} = 0$) and let there be no shearing stresses (see Fig. 248). Under the action of such stresses there appears tension along the axis *1* and compression (of the same magnitude) along the axis *2*. For, by (87.1) and (87.2), we have

$$\left.\begin{aligned} E\varepsilon_1 &= \sigma_{11} - \mu\sigma_{22} = (1+\mu)\sigma_0 \\ E\varepsilon_2 &= \sigma_{22} - \mu\sigma_{11} = -(1+\mu))\,\sigma_0 \end{aligned}\right\} \tag{87.13}$$

whence

$$\varepsilon_1 = -\varepsilon_2 = \varepsilon_0 = \frac{1+\mu}{E} \sigma_0$$

Next we will prove that this deformation s (pure) non-dilatational when considered relative to coordinates with axes forming an angle of $\pi/4$ with the former axes.

Let us cut out of the small cube in question a prism with a square base as shown in Fig. 249*a*. If the edge of the original cube is a then the small prism with the shaded triangle (Fig. 249*b*) as base is acted upon by the forces indica-

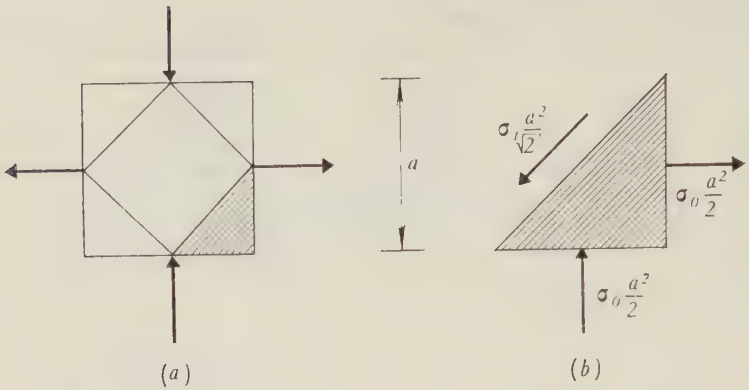

FIG. 249

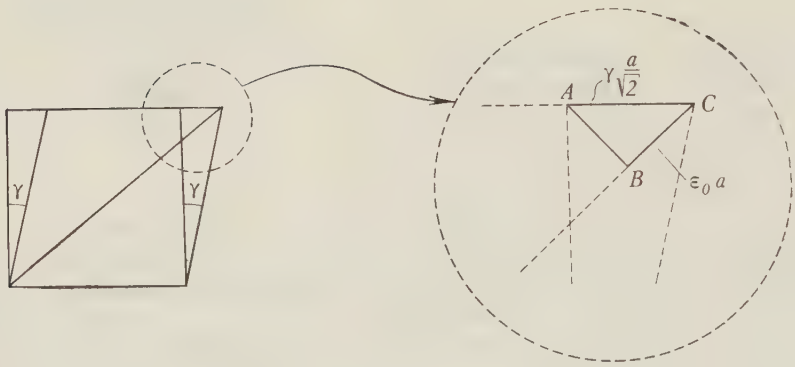

FIG. 250

ted in the figure. For the resultant of these forces to be zero there must be fulfilled the condition $\sigma_t = \sigma_0$ for the tangential stress σ_t. The same applies to the other three small prisms. Therefore, the inner prism (with the square base) is only under the action of tangential stresses and its angles receive increments (positive and negative) of magnitude

$$\gamma = \frac{\sigma_0}{G}$$

The corresponding unit elongation of the diagonal of the normal section of the inner prism is equal to

$$\varepsilon_0 = \frac{1+\mu}{E}\sigma_0$$

The relationship between the parameters μ, E, and G can now be found purely geometrically. The sections of the prism before and after the deformation are shown in Fig. 250. The angular increment γ is very small and therefore, taking into account that the triangle ABC is right and isosceles we receive the rela-

tion

$$\frac{\gamma a}{\sqrt{2}} = \sqrt{2}\,\varepsilon_0 a, \quad \text{that is} \quad \gamma = 2\varepsilon_0$$

The substitution of the two foregoing expressions in the last relation results in the sought-for connection between μ, E, and G:

$$G = \frac{E}{2(1+\mu)}. \tag{87.14}$$

Furmula (87.14) enables us to establish the general relation between the stress and strain tensors using (87.9) and (87.12):

$$\mathscr{T} = \mathscr{T}_n + \mathscr{T}_t = \frac{E}{1+\mu}\left(\mathscr{S}'_s + \frac{3\mu\varepsilon}{1-2\mu}\mathscr{E}\right) + \frac{E}{1+\mu}\mathscr{S}''_s$$

Now, by virtue of (87.8), we obtain

$$\mathscr{T} = \frac{E}{1+\mu}\left(\mathscr{S}_s + \frac{3\mu\varepsilon}{1-2\mu}\mathscr{E}\right) \tag{87.15}$$

Given the components of the strain tensor $\mathscr{S}_s$, we can use (87.15) to find the components of the stress tensor $\mathscr{T}$ for the given isotropic material provided that the parameters E and μ are known.

It is also possible to establish the inverse dependence of $\mathscr{S}_s$ on $\mathscr{T}$. Indeed, from (87.15) we get

$$\frac{1+\mu}{E}\mathscr{T} = \mathscr{S}_s + \frac{3\mu\varepsilon}{1-2\mu}\mathscr{E}$$

Now, taking into account (87.11), we arrive at the sought-for dependence:

$$\mathscr{S}_s = \frac{1+\mu}{E}\mathscr{T} - \frac{3\mu\sigma}{E}\mathscr{E} \tag{87.16}$$

Now it only remains to show that ε and σ are scalars, that is quantities whose numerical values are independent of the choice of the coordinate system. For any tensor it is possible to find its eigenvalues λ_1, λ_2 and λ_3 corresponding to the principal directions. For the tensors $\mathscr{T}$ and $\mathscr{S}_s$ the eigenvalues correspond to the principal stresses along the principal axes of these tensors. The determination of the eigenvalues of a tensor was discussed in detail in Sec. 64 in connection with the inertia tensor. The eigenvalues λ_1, λ_2 and λ_3 do not depend on the choice of the coordinate system. In the case of the tensor $\mathscr{T}$ the eigenvalues are found from the equation

$$\begin{vmatrix} \sigma_{11}-\lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\lambda \end{vmatrix} = 0$$

(see (64.11)). This equation can be rewritten as

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

The roots of this equation are scalars and therefore, obviously, the coefficients a_1, a_2, and a_3 are also scalars, that is they are independent of the choice of the coordinate axes. As can easily be shown, we have

$$-a_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = 3\sigma \tag{87.17}$$

whence it follows that σ is a scalar quantity. The fact that ε is also a scalar is proved in like manner.

The quantity σ can be interpreted as the average normal stress:

$$\sigma = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3} \tag{87.18}$$

In case σ is negative it is analogous to fluid pressure. The quantity $\varepsilon = \frac{1+\mu}{E}\sigma$ can be interpreted as the measure of the omnidirectional extension under the action of the stress σ, or $-\varepsilon$ can be thought of as the measure of the omnidirectional compression under the action of the pressure $-\sigma$. The tensor $\mathscr{T}_n$ can be written as the following sum of two tensors:

$$\mathscr{T}_n = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix} + \begin{pmatrix} \sigma_{11}-\sigma & 0 & 0 \\ 0 & \sigma_{22}-\sigma & 0 \\ 0 & 0 & \sigma_{33}-\sigma \end{pmatrix} \tag{87.19}$$

The first of the tensors on the right-hand side of (87.19) is equal to $\sigma\mathscr{E}$ and represents an omnidirectional stress σ or pressure $-\sigma$; the other tensor serves as a measure of the deviation from the average stress. It is evident that the tensor $\sigma\mathscr{E}$ is invariable with respect to any transformation of coordinates.

88. Strain Energy

Let us find the work that should be expended on a deformation of a volume $dv = dx_1\, dx_2\, dx_3$ of an elastic material. This work is equal to the potential energy of strain which is briefly called the *strain energy*.

The normal stresses are not connected with the shearing strains and are only dependent on the tensile (or compressive) strains. The normal stress along a given axis is dependent on the strains along all the three axes. Therefore, the work performed by the normal forces arising in a state of strain can be calculated independently of the work of the forces connected with the shearing strains.

Taking into account (87.12) and the relation $\frac{E}{1+\mu} = 2G$ we can find the elementary amounts of work performed by the normal forces:

$$\left.\begin{aligned} \sigma_{11}\, dx_2\, dx_3\, d\varepsilon_1\, dx_1 &= 2G\left(\varepsilon_1 + \frac{\mu}{1-2\mu}\,3\varepsilon\right) d\varepsilon_1\, dv \\ \sigma_{22}\, dx_3\, dx_1\, d\varepsilon_2\, dx_2 &= 2G\left(\varepsilon_2 + \frac{\mu}{1-2\mu}\,3\varepsilon\right) d\varepsilon_2\, dv \\ \sigma_{33}\, dx_1\, dx_2\, d\varepsilon_3\, dx_3 &= 2G\left(\varepsilon_3 + \frac{\mu}{1-2\mu}\,3\varepsilon\right) d\varepsilon_3\, dv \end{aligned}\right\} \tag{88.1}$$

Adding together these three equations we obtain

$$2G\left\{\varepsilon_1\, d\varepsilon_1 + \varepsilon_2\, d\varepsilon_2 + \varepsilon_3\, d\varepsilon_3 + \frac{\mu}{1-2\mu}\,3\,\varepsilon d\,(3\varepsilon)\right\} dv$$

since $3\varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$ and $d\varepsilon_1 + d\varepsilon_2 + d\varepsilon_3 = d\,(3\varepsilon)$.

The whole work expended on bringing the corresponding strains from zero to the magnitudes ε_1, ε_2, and ε_3 is equal to

$$dA_1 = 2G\left(\int_0^{\varepsilon_1} x\,dx + \int_0^{\varepsilon_2} x\,dx + \int_0^{\varepsilon_3} x\,dx + \frac{\mu}{1-2\mu}\int_0^{3\varepsilon} x\,dx\right) dv =$$

$$= G\left(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \frac{\mu}{1-2\mu}(3\varepsilon)^2\right) dv$$

The density of the potential energy appearing because of the work of the normal stresses is expressed in terms of the strains ε_1, ε_2, and ε_3 as

$$u_1 = \frac{dA_1}{dv} = G\left(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \frac{\mu}{1-2\mu}(3\varepsilon)^2\right) \tag{88.2}$$

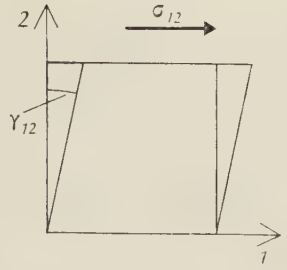

FIG. 251

The work performed by the shearing stresses is found in a still simpler manner because the components of the tangential stresses normal to a given axis depend solely on the shearing strains in the plane perpendicular to that axis, for instance $\sigma_{12} = G\gamma_{12}$, etc. The elementary work of the components of the tangential forces on the faces parallel to axis *3* (Fig. 251) can be written as

$$\sigma_{12}\,dx_1\,dx_3\,d\gamma_{12}\,dx_2 = \sigma_{12}\,d\gamma_{12}\,dv$$

The forces on the other three faces belonging to this group (see Fig. 251) perform no work: the tangential forces on the two faces of areas $dx_2\,dx_3$ are normal to the displacement of this area and the displacement of the lower face $dx_1 dx_3$ is equal to zero. Therefore, the whole work performed by the tangential stresses acting on the volume under consideration is

$$dA_2 = \left(\int_0^{\gamma_{12}} \sigma_{12}\,d\gamma_{12} + \int_0^{\gamma_{13}} \sigma_{13}\,d\gamma_{13} + \int_0^{\gamma_{23}} \sigma_{23}\,d\gamma_{23}\right) dv =$$

$$= G\left(\int_0^{\gamma_{12}} x\,dx + \int_0^{\gamma_{13}} x\,dx + \int_0^{\gamma_{23}} x\,dx\right) dv = \frac{1}{2} G\left(\gamma_{12}^2 + \gamma_{13}^2 + \gamma_{23}^2\right) dv \tag{88.3}$$

Consequently, the total density of the strain energy is given by the formula

$$u = u_1 + \frac{dA_2}{dv} = G\left(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \frac{1}{2}\gamma_{12}^2 + \frac{1}{2}\gamma_{13}^2 + \frac{1}{2}\gamma_{23}^2 + \frac{\mu}{1-2\mu}(3\varepsilon)^2\right) \tag{88.4}$$

expressing the density in terms of the magnitudes of the strains.

The density of the potential energy of strain can also be written in terms of the stresses in the form

$$u = \frac{1}{4G}\left(\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 + 2\sigma_{12}^2 + 2\sigma_{13}^2 + 2\sigma_{23}^2 - \frac{\mu}{1+\mu}(3\sigma)^2\right) \tag{88.5}$$

As an exercise, let the reader derive this formula (*Hint*: the differential $d\varepsilon_1$ entering into the product $\sigma_{11}\, d\varepsilon_1$ should be replaced by the expression $\frac{1}{2G}\left(d\sigma_{11} - \frac{\mu}{1+\mu}\, d\,(3\sigma)\right)$ according to formula (87.16), etc.).

It can easily be shown that the stresses and the strains can be expressed as

$$\sigma_{11} = \frac{\partial u}{\partial e_1}, \ldots, \sigma_{12} = \frac{\partial u}{\partial \gamma_{12}}, \ldots \tag{88.6}$$

and

$$\varepsilon_1 = \frac{\partial u}{\partial \sigma_{11}}, \ldots, \gamma_{12} = \frac{\partial u}{\partial \sigma_{12}}, \ldots \tag{88.7}$$

89. Forces and Deformations in the Bending of Beams

Bending of a beam is an important example of a deformation of an elastic body*. Such a deformation is produced by external forces normal to the axis of the beam. The bending phenomenon can be visualized if we bend a rubber beam with a network of parallel lines marked on its surface (Fig. 252). As shown in the figure, the upper layers of the beam are stretched, the lower layers are compressed and the length of the layer adjoining the axis OO' is almost invariable.

An exhaustive analysis of the strains and stresses in an elastic beam subject to bending is a rather intricate problem. However, there is an approximate approach to the problem based on *Bernoulli's hypothesis of plane sections* suggesting that under bending the cross sections of the beam remain plane. This approximate approach to the solution of the problem is much simpler and here we present some basic results of this kind.

Let us consider a horizontal beam (Fig. 252) whose one end has a rigid fixing and whose other end is acted upon by a vertical force $\boldsymbol{P}$. The force $\boldsymbol{P}$ bends the beam. Every cross section of the beam perpendicular to its axis before the application of the load turns under the bending but remains plane when the deformations are small.

In the study of the bending phenomenon the basic questions to answer are whether the beam can endure a given load, what strains appear in the beam, what stresses arise in the material of the beam, and how the deflection of the end of the beam produced by the load can be found. The answers to these questions can be obtained by using the methods developed in the engineering science called *strength of materials*. Here we discuss some fundamental ideas and results related to these questions.

For simplicity, let us begin with the deformation of a beam with a rigid** fixing of one of its ends and a couple of forces applied to the other end (see Fig. 253). In practice such a case is ecountered more rarely than the one shown in Fig. 253 but its theoretical study is simpler.

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* By a beam is meant a bar (a rod) subjected to a lateral load. A bar (a rod) of a constant cross section is a cylindrical body whose length is much greater than the dimensions of the cross section.

** We say that there is a rigid fixing of an end of a beam when that end is supported in such a way that the corresponding terminal cross section of the beam does not change its position in space. In reality the terminal section does not remain plane and always undergoes a deformation but the comparison of the experimental data with the results of the theoretical analysis shows that the assumption that the terminal section is nondeformable is practically quite admissible.

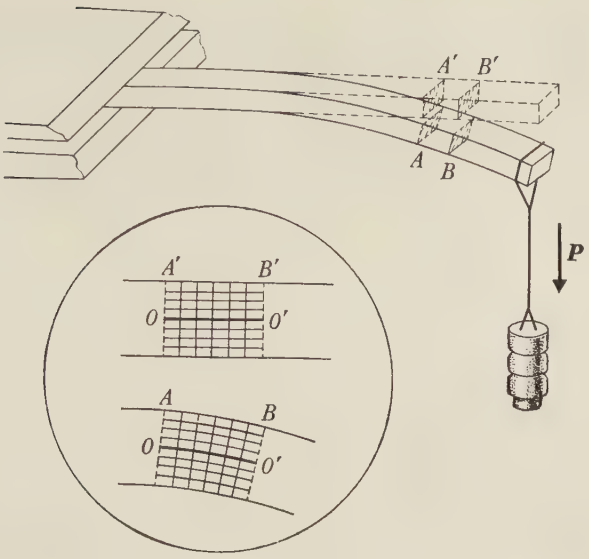

FIG. 252

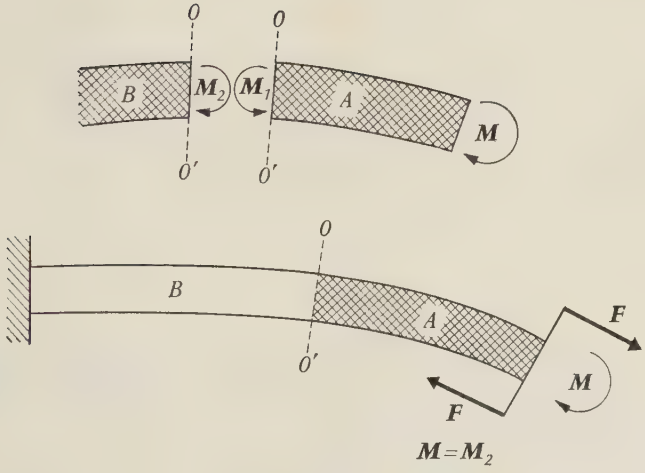

FIG. 253

In order to determine the magnitudes and the directions of the internal forces arising in the beam we will use the method based on a (mental) cutting out of a part of the body. The part of the body isolated in this way is considered in conditions corresponding to the deformation of the beam after the equilibrium of state sets in. For this part we write down the equilibrium conditions both for the external forces applied to that part and for the internal forces acting on the interface separating the part in question from the remaining part of the body. The internal forces are usually unknown but the equilibrium conditions

for the isolated part make it possible to obtain the necessary information about the internal forces provided that the external forces are known.

In the case under consideration (see Fig. 253) we isolate the part of the beam lying on the right from an arbitrary plane section OO' normal to the axis of the beam (this part is marked by the letter A in the figure). The equilibrium condition for the part A implies that the tangential internal force in the section OO' is zero. Indeed, if a non-zero force acting along the line OO' were applied to the part A this part could not be in equilibrium because the couple of forces

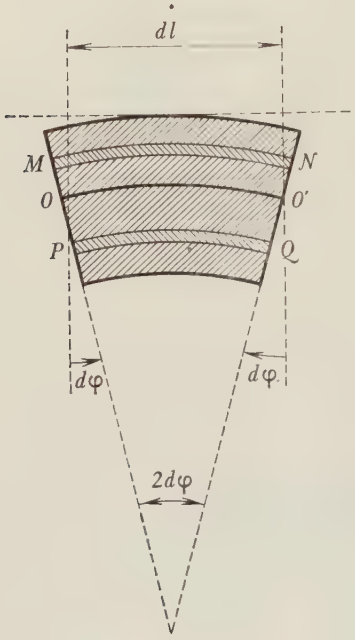

FIG. 254

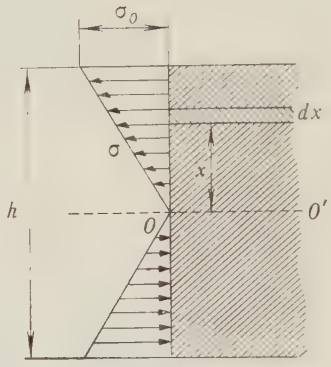

FIG. 255

($\boldsymbol{F}$) with moment $\boldsymbol{M}$ can only be balanced by another couple or by a system of couples whose resultant moment is $-\boldsymbol{M}$. It follows that the internal forces in the section OO' reduce to a couple of forces or to a system of couples with the resultant *bending moment* $\boldsymbol{M}_1 = -\boldsymbol{M}$.

The length of the part A being taken quite arbitrarily, we conclude that in any cross section of the beam the internal forces reduce to a system of couples of forces whose resultant moment has a magnitude equal to that of the moment of the couple of external forces applied to the beam and whose direction is opposite to that of the external couple. By the third law of dynamics, the moment of forces applied to the remaining part B of the beam (see Fig. 253) is

$$\boldsymbol{M}_2 = -\boldsymbol{M}_1$$

whence

$$\boldsymbol{M}_2 = \boldsymbol{M}$$

Thus, this moment is equal to the moment of the couple applied to the right end of the beam.

It should be noted that although the resultant internal force in any cross section, for instance, in the section OO', is equal to zero this does not mean that the normal and the tangential (shearing) stresses in the section are equal to zero when the beam is in the state of strain under consideration. The bending moment M_1 of the internal forces in the section can only be produced by a certain distribution of the normal stresses because the tangential stresses cannot form a couple with a non-zero moment (because the corresponding arms are equal to zero). In the approximate theory the influence of the tangential stresses in the case under consideration is neglected since these stresses turn out to be very small. The comparison of experimental data with the results of the theoretical investigation confirms the validity of such an approximation.

Using the indicated approximation we can find the relationship between the strains and the stresses in the following way. Let us cut from the beam a part of a sufficiently small length dl (Fig. 254). Under the bending of the beam this part assumes (approximately) the shape shown in the figure. Both cross sections of the isolated part turn to form an angle $d\varphi$ between them. Let us also divide (mentally) the beam into sufficiently thin horizontal layers parallel to the middle line of the beam. It is evident that the layer adjoining the middle line OO' (Fig. 254) does not change its length; that is why it is referred to as the *neutral layer* of the beam and the line OO' is called the *neutral axis* of the beam. The layers lying above the neutral layer are stretched (for instance, this is the case for the layer MN in Fig. 254) while the layers lying below the neutral layer are compressed (for instance, the layer PQ). Since the cross sections are assumed to remain plane under the deformation it is obvious that the elongations and the shortenings of the layers are proportional to their distances from the neutral layer.

If the deformations are sufficiently small so that the state of strain of the beam corresponds to the elastic (proportional) region the magnitude of the normal stress in each layer is proportional to its elongation or shortening. Then the distribution of the normal stresses in the face planes of the part of length dl we have cut out from the beam (that is in the cross section of the beam) is as shown in Fig. 255. Denoting by x the distance from a given layer to the neutral layer we can write the expression

$$\sigma = \sigma_0 \frac{x}{b} \tag{89.1}$$

for the stress in that layer where σ_0 is the stress in the layer lying at the greatest distance b from the neutral layer.

For the sake of simplicity, we suppose that all the cross sections of the beam are identical and are of the shape of a rectangular parallelepiped. Therefore, the neutral layer of the beam under consideration adjoins the middle line of the beam (we have already tacitly implied this fact) and $b = h/2$ where h is the altitude of the cross section of the beam. Consequently, denoting by a the horizontal dimension of the cross section we obtain the following expression for the internal force acting on the layer of width dx lying at a distance of x from the neutral layer:

$$dF = \sigma a\, dx = \frac{\sigma_0}{b} xa\, dx = \frac{2\sigma_0 a}{h} x\, dx \tag{89.2}$$

The layers symmetric with respect to the neutral layer are acted upon by forces of equal magnitudes and opposite directions and therefore the resultant of the internal forces in any section is equal to zero while the magnitude of the sum

of the moments of the internal forces must be equal to that of the moment M of the forces forming the external couple applied to the beam.

We can now find the bending moment in the cross section. To this end we integrate expression (89.2) over the whole cross section to obtain

$$M = \int_{-h/2}^{h/2} x\, dF = \frac{\sigma_0}{b} \int_{-h/2}^{h/2} ax^2\, dx = \frac{2\sigma_0}{h} I \tag{89.3}$$

The quantity

$$I = \int_{-h/2}^{h/2} ax^2\, dx \tag{89.4}$$

whose numerical value is

$$I = ah^3/12$$

in the case under consideration is called the *moment of inertia of the cross section* of the beam with respect to the axis passing through the neutral layer. The application of the term "moment of inertia" in this case is accounted for by the fact that formula (89.4) resembles the formula for the moment of inertia of a plane (thin) body with unit areal mass density: here the area $a\,dx$ is multiplied by x^2 which is the square of the distance from the neutral axis and the product is integrated over the whole area of the cross section. However, this is only a formal analogy and expression (89.4) has nothing in common with the moment of inertia of a body. The dimension of I is (length)^4 and therefore this quantity has only a geometrical sense.

If a beam has a cross section S different from a rectangular parallelepiped and the beam is bent in the plane perpendicular to the moment M of the couple applied to it (i.e. in the plane of the load) the neutral axis may no longer coincide with the middle line. However, it always passes through the centre of gravity O of the section perpendicularly to the plane of the load (Fig. 256). In this more general case the bending moment in the cross section of the beam is expressed by the formula

$$M = \frac{\sigma_0}{b} \int_S ax^2\, dx = \frac{\sigma_0}{b} I \tag{89.5}$$

where $I = \int ax^2\, dx$, b is the greatest distance from a point of the section to the neutral layer (as is seen from Fig. 256,b is the greatest of the distances b_1 and b_2 and σ_0 is the maximum normal stress). The derivation of formula (89.5) is also based on the assumption that the cross sections of the beam remain plane under bending.

The magnitude of the bending moment M in the cross section and the maximum stress σ_0 in the section are connected by a simple relation implied by formula (89.5):

$$M = \sigma_0 \frac{I}{b} = \sigma_0 w \quad \text{where} \quad w = \frac{I}{b} \tag{89.6}$$

The maximum stress arises in the layer which is at the greatest distance from the neutral layer. The quantity w is determined by the shape of the cross section solely. For a given section it is always possible to find the quantity w called the *section modulus* which is equal to the ratio of the moment of inertia of the section to the maximum distance of a layer from the neutral layer. In technical

problems it is often sufficient to determine the value of the maximum stress σ_0, which makes it possible to judge upon the strength of the given beam by comparing σ_0 with the ultimate strength σ_u (the body ruptures when the stress achieves the value σ_u).

In the case we have considered when the end of a beam is acted upon by a couple with moment M (see Fig. 253) the bending moment is the same (equal to M) for all the cross sections. However, for some other types of a load this may not be the case. It nevertheless turns out that in many problems it is possible

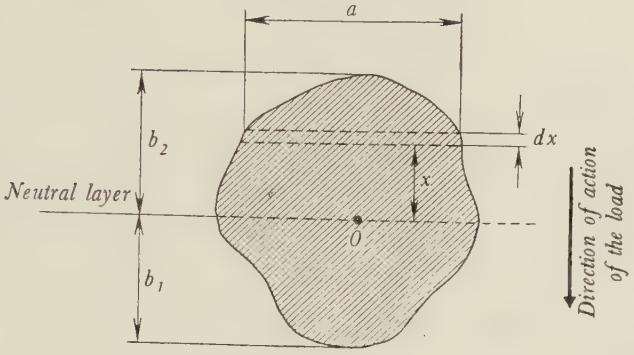

FIG. 256

to determine the bending moments M in all the cross sections from a given load. On dividing M by the section modulus w we find the maximum stress in every section for the given load. That is why the primary problem in testing a given beam for strength is to find (by calculations) from a given load, the *critical* (*dangerous*) section in which the stress has a maximum value.

90. Determination of Deflection of a Beam

Bending of a beam is characterized by the shape of the *deflection line* by which is usually meant a curve passing through the centres of gravity of the cross sections of the beam. We will confine ourselves to the case when the deflection line is in the plane of action of the external forces or, which is the same, when the plane of the bending and the plane of the action of the external forces coincide.

Let us determine the deflection line for a beam with a rigid fixing of one of its ends and with the other end loaded by a couple of forces (see Fig. 253). We denote by $d\varphi$ the angle formed by the two cross sections of the beam (subject to bending) bounding its part of length dl (see Sec. 89).

Let $y = f(l)$ be the equation of the sought-for deflection line where y is the deflection of the point with the coordinate l from the axis on which the particles of this line lay before the deformation sets in. The quantity $\alpha = \arctan \frac{dy}{dl}$ is the angle between the tangent to the deflection line with the l-axis at the point with coordinate l (Fig. 257). The angle α being very small, we can write $\alpha \approx \approx \frac{dy}{dl}$. Therefore, the variation of the angle of inclination of the tangent of the

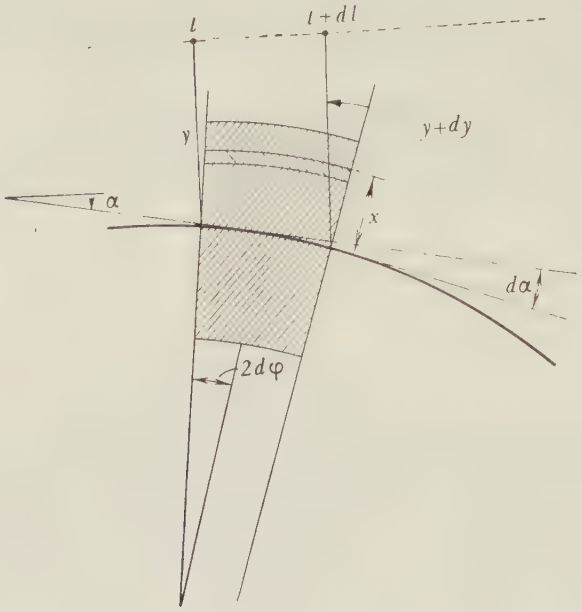

FIG. 257

deflection line to the l-axis corresponding to the passage from a point with the coordinate l to another point with the coordinate $l + dl$ is

$$d\alpha \approx \frac{d^2y}{dl^2}\, dl$$

Since the cross sections of the beam are always perpendicular to the deflection line we have $d\alpha = 2d\varphi$ (see Figs. 254 and 257). Consequently,

$$2d\varphi = d\alpha \approx \frac{d^2y}{dl^2}\, dl, \quad \text{that is} \quad \frac{d^2y}{dl^2} \approx \frac{2d\varphi}{dl} \tag{90.1}$$

By formula (89.5), the moment of the internal forces in any cross section is $M = \frac{\sigma_0}{b} I$. The maximum stress σ_0 (which arises in the layer lying at the greatest distance from the neutral layer) is in a relation with the strain of that layer specified by Hooke's law. As it follows from Figs. 254 and 257, the unit elongation of this layer lying at a distance of b from the neutral layer is expressed by the formula

$$\varepsilon = \frac{d\alpha}{dl} b = 2 \frac{d\varphi}{dl} b \tag{90.2}$$

Accordingly, the stress in this layer is

$$\sigma_0 = E\varepsilon = 2E \frac{d\varphi}{dl} b$$

Taking into account (90.1) we obtain

$$\sigma_0 \approx Eb \frac{d^2y}{dl^2} \tag{90.3}$$

On substituting the expression of σ_0 given by formula (90.3) in equation (89.5) we arrive at the following differential equation specifying the deflection line:

$$M = EI \frac{d^2y}{dl^2} \tag{90.4}$$

In the case under investigation the bending moment M is the same for all the cross sections and is equal to the moment $M = M_{ext}$ of the couple of the external forces applied to the beam. The two-fold integration of equation (90.4) results in the sought-for equation of the deflection line. The first integration from the point $l = 0$ (the fixing) to an arbitrary point $l = \eta$ gives us the first derivative of the function $y = f(l)$ at the point η:

$$\left(\frac{dy}{dl}\right)_\eta - \left(\frac{dy}{dl}\right)_0 = \int_0^\eta \frac{d^2y}{dl^2}\, dl = \int_0^\eta \frac{M_{ext}}{EI}\, dl = \frac{M_{ext}}{EI}\, \eta \tag{90.5}$$

Taking into account that the direction of the tangent to the deflection line at the point of the rigid fixing is horizontal, that is $\left(\frac{dy}{dl}\right)_0 = 0$, we see that the second integration (the integration of equation (90.5)) from $l = 0$ to an arbitrary value of l yields

$$y(l) - y(0) = \int_0^l \left(\frac{dy}{dl}\right)_\eta d\eta = \int_0^l \frac{M_{ext}}{EI}\, \eta\, d\eta = \frac{M_{ext}}{EI}\, \frac{l^2}{2} \tag{90.6}$$

The deflection is equal to zero at the rigid fixing, that is $y(0) = 0$, whence it follows that the deflection line is a parabola described by the equation

$$y(l) = \frac{M_{ext}}{2EI}\, l^2 \tag{90.7}$$

Let the length of the beam be L. Then the maximum deflection of the beam (at its end point $l = L$) is equal to

$$\frac{M_{ext}}{2EI}\, L^2 \tag{90.8}$$

We now investigate the deformation of the same beam of length L when its end $l = 0$ is rigidly fixed while the other end is loaded by a vertical force $\boldsymbol{P}$ (see Fig. 252). To this end we again cut the beam at a distance l from the fixing and determine the internal forces in the cross section of the beam bounding the part of the beam lying on the right of the section (Fig. 258). In this case the internal forces differ from those in the case treated in Sec. 89 because the stresses must have a non-zero tangential (shearing) component since for the part cut from the beam to be in equilibrium the resultant of the internal forces in the section l must have a component $\boldsymbol{Q}'$ of the same magnitude as the force $\boldsymbol{P}$ and of opposite direction. The component $\boldsymbol{Q}'$ is called the *transverse (shearing) force*. It is obvious that the transverse force has the same magnitude equal to P for all the sections. The transverse force $\boldsymbol{Q}$ acting on the remaining part of the beam (lying on the left from the section l) has the same magnitude (P) and the opposite (with respect to $\boldsymbol{Q}'$) direction.

We see that the part of the beam of length $L - l$ is acted upon by the forces $\boldsymbol{P}$ and $\boldsymbol{Q}'$ forming a couple with a moment $\boldsymbol{M}_1$ whose magnitude is

$$P(L - l) = M_1 \tag{90.9}$$

Consequently, for the part cut from the beam to be in equilibrium it is necessary that this part should be acted upon by one more couple whose moment M' is equal to M_1 in its magnitude and opposite in the direction. The forces forming such a couple can only act in the cross section bounding the part in question. Hence, the deformation of the beam must be such that the normal stresses in the cross section result in the bending moment M' (Fig. 258).

As has been said, the magnitude M' of the moment M' is equal to that of the moment M_1, its sense being opposite. Thus, the bending moment M in the

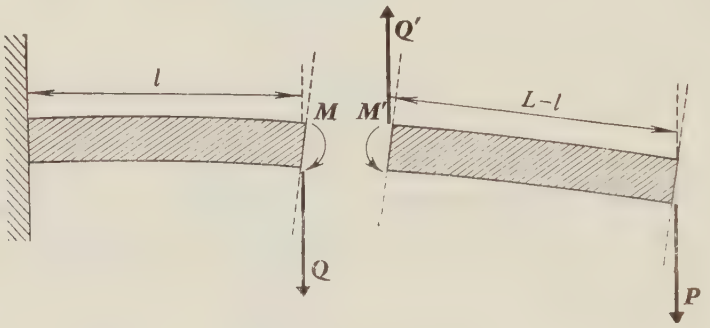

FIG. 258

section of the remaining part of the beam is equal to M_1. We see that in the case under investigation the bending moment M in the cross sections is different for different sections and is no longer constant as it was in the example considered in Sec. 89. Indeed, the magnitude M_1 of the moment depends on the coordinate l:

$$M = M_1 = P(L - l) \tag{90.10}$$

Thus, the internal forces in the cross section result from the normal stresses producing elongation and compression of the layers and from the tangential (shearing) stresses creating the transverse force Q. Therefore, in contrast to the previous case, the shearing stresses play a certain role in the deformation of the beam. But the comparison of the experimental data with the results of the theoretical calculations shows that the influence of the tangential stresses is not very strong and therefore, practically, when testing the beam for strength, we can neglect the shearing stresses just as it was done in Sec. 89.

It follows that the determination of the deflection line of the beam again reduces to the solution of the same differential equation (90.4):

$$M = EI \frac{d^2y}{dl^2} \tag{90.11}$$

But in the case we are dealing with the bending moment depends on the coordinate l and is no longer constant (see formula (90.10)). The bending moment diagram is shown for this case in Fig. 259.

If the cross section of the beam does not vary along its length the dangerous section in which the normal stress attains the maximum value is at the fixing, that is at the end $l = 0$ of the beam. The maximum normal stress is

$$\sigma_{0\,\max} = \frac{M_{\max}}{w} = \frac{PL}{I} b \tag{90.12}$$

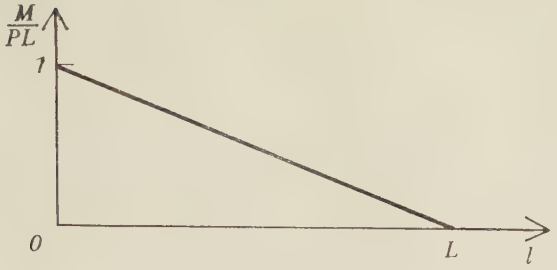

FIG. 259

where b is the maximum distance from a layer to the neutral layer and I is the moment of inertia of the cross section of the beam. If the stresses in the section $l = 0$ are sufficiently small so that the beam can endure the load, the stresses in the other sections are still smaller. If the beam ruptures when the load exceeds the admissible value the rupture takes place at the rigid fixing.

As another example, let us consider a beam which is freely supported at its ends as shown in Fig. 260*a*. The beam is loaded by a force P applied at a distance of a from the right-hand support A. Our immediate task is to determine the reactions F_B and F_A of the supports, that is the forces exerted on the beam by the supports. (For our further aims it is sufficient to find the reaction force F_A alone.) The reaction F_A is readily found from the equilibrium conditions for the whole beam. In the equilibrium the sum of the moments of all the acting forces about any point, say about the point B, must be equal to zero. This condition gives us the relation

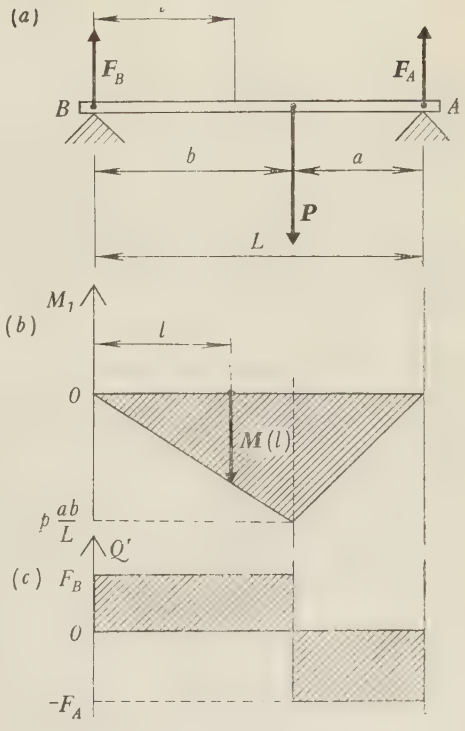

FIG. 260

$$F_A L = Pb \quad \text{whence} \quad F_A = \frac{Pb}{L} \qquad (90.13)$$

As in the foregoing example, from the equilibrium condition for the part of the beam of length $L - l$ we find that the bending moment in a cross section with coordinate $l < b$ is equal to the sum of the moments of the forces applied to the part cut from the beam:

$$M_1 = P(b-l) - F_A(L-l) = -P\frac{la}{L} \qquad (90.14)$$

Similarly, for the sections with $l > b$ we obtain the expression

$$M_1 = -F_A (L-l) = -Pb \left(\frac{L-l}{L} \right) \tag{90.15}$$

The bending moment diagram (i.e. the graph of the function $M_1 = M(l)$) is shown in Fig. 260*b*. The shearing (transverse) force diagram is shown in Fig. 260*c*. At the point of application of the load P the shearing force Q' has a jump of magnitude P.

If the section modulus w is known for every cross section of the beam the bending moment diagram makes it possible to find the critical (dangerous) section.

To determine the deflection line of the beam it is necessary to integrate twice the right-hand side of the equation

$$\frac{d^2y}{dl^2} = \frac{1}{EI} M(l) \tag{90.16}$$

where $M(l)$ is the known function of l, and the parameters E and I are Young's modulus and the moment of inertia of the cross section respectively (these parameters are supposed to be known).

In the examples we have considered the bending moment diagrams are found from given loads and are independent of the elastic properties of the supports. In more intricate cases it may be necessary to have some information about the deformation of the supports. Some examples of this kind are discussed in the following section.

91. On Deformation of Supports

When studying the deformation of a freely supported beam in Sec. 90 we first determined the reactions of the supports and then proceeded to the determination of the deformation of the beam under the action of the external forces including the known load and the reactions of the supports. However, such an approach is only possible when the reactions of the supports do not depend on the deformation of the beam and the deformation of the beam itself is completely independent of the deformation of supports.

In some more complex situations involving a certain connection between the deformations of the beam and the deformations of the supports this simple method is inapplicable and the solution of the problem requires a more intricate technique making it possible to determine simultaneously the reactions of the supports and the deformation of the beam. To demonstrate characteristic features of problems of this kind we begin with the discussion of two simple examples below in which the reactions of supports can simply be measured.

Let us consider an (easily deformable) wooden beam suspended from two dynamometers with the aid of two springs A and B as is shown in Fig. 261. When a load P is applied to the beam the dynamometers measure the magnitudes of the reactions of the supports of the beam. When the stiffness of the spring A is changed, say increased, it turns out that, practically, the reactions of the supports remain invariable. The same occurs when the flexural rigidity of the beam is varied.

This can be accounted for purely theoretically. Indeed, in the case under consideration the equilibrium conditions for the beam imply that the tensile forces

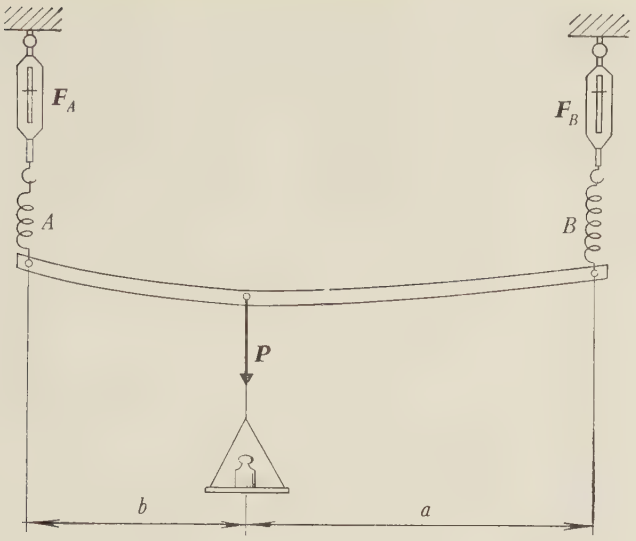

FIG. 261

of the springs A and B are given by the formulas

$$\left.\begin{aligned} F_A &= \frac{a}{a+b}P \\ F_B &= \frac{b}{a+b}P \end{aligned}\right\} \qquad (91.1)$$

The lengths a and b being almost invariable, the data shown by the dynamometers do not practically depend on the deformations of the supports and on the

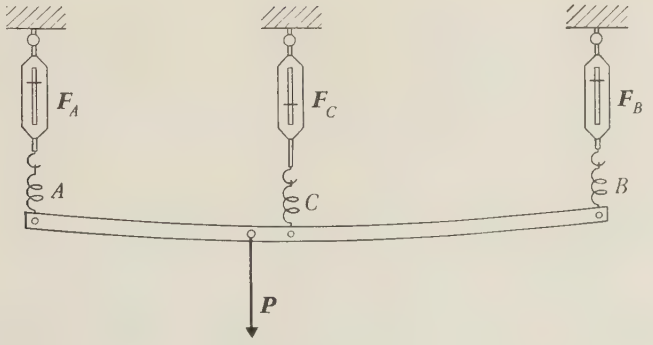

FIG. 262

deformation of the beam itself. Relations (91.1) are a consequence of the equilibrium conditions for the beam and are independent of the deformations of the supporting springs A and B and of the deformation of the beam. We dealt with a situation of this kind in Sec. 90 where the analysis of the deformation of

the beam did not involve the determination of the magnitude of the deformations of the supports.

The situation becomes quite different when the same wooden beam is suspended from three springs as shown in Fig. 262. It turns out that in this case the data shown by the dynamometers are dependent on both the stiffness of the springs and the flexural rigidity of the beam. To gain a better understanding of the phenomenon let us suppose that the stiffness factor of the spring C is much greater than the stiffness factors of the springs A and B. Then the dynamome-

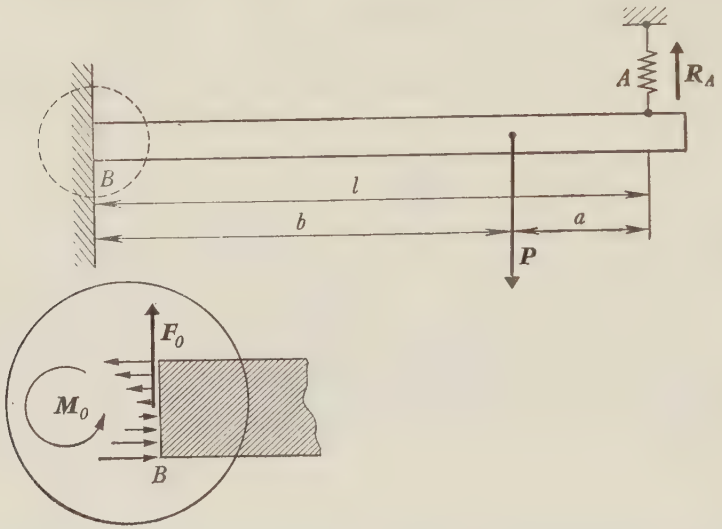

FIG. 263

ter C measures the reaction force which is almost equal to the load P (the forces measured by the dynamometers A and B are very small). On the contrary, if the stiffness of the spring C is very small in comparison with the springs A and B the reaction forces applied to the ends of the beam are of almost the same magnitudes as those given by formulas (91.1). If the flexural rigidity of the beam is increased, the magnitude of the force F_C measured by the middle dynamometer also increases.

It follows that the deformation of a beam freely supported at three points cannot be determined without calculating the deformations of the beam and of the supports. For, we have two conditions specifying the equilibrium of the beam, namely the resultant of all the external forces (including the reactions of the supports) must be equal to zero and the sum of the moments of all these forces must be equal to zero, while there are three unknowns, that is the three reaction forces F_A, F_B, and F_C. The three unknowns cannot be found from the two equations. That is why we say that in problems of this kind we deal with *statically indeterminate structures.* For such structures the reactions of supports cannot be found without the determination of the deformation of the bodies involved (including the supports). Thus, in these cases we deal with a more complex problem of a simultaneous determination of the reaction forces and of the deformations. What has been said is demonstrated by two typical examples below.

(1) *Let a beam be rigidly fixed at one end and supported by a spring at the other end.* This structure is shown schematically in Fig. 263. The right-hand end of the beam is acted upon by a force of tension $\boldsymbol{R}_A$ (the support reaction) of the spring A while at the rigid fixing on the left-hand end of the beam there is a complex system of reaction forces acting in the cross section B; this system reduces to a force $\boldsymbol{F}_0$ and a moment $\boldsymbol{M}_0$ (since the section B can neither gain a linear displacement nor turn through an angle).

In order to determine the bending moment and the shearing force in every section of the beam we must know the quantities P and R_A. The former is given while the latter cannot be found without knowing F_0 and M_0. Thus, we have the three unknown quantities R_A, M_0, and F_0; they cannot be found from two equilibrium equations of statics. The magnitudes of the forces $\boldsymbol{R}_A$ and $\boldsymbol{F}_0$ and the moment $\boldsymbol{M}_0$ are essentially dependent on the flexural rigidity of the beam and on the stiffness of the supporting spring A.

The problem is solved by taking into account the deformations of the supports in the following way. Let us first imagine that there is no spring and that, besides the given force $\boldsymbol{P}$, the beam is acted upon by an arbitrary force $\boldsymbol{R}$ applied at the point A and having the direction opposite to that of the force $\boldsymbol{P}$. Then we can compute the deformation of the beam at the point A as function of the magnitude R of the force $\boldsymbol{R}$, that is the function $y_A = f(R)$ where y_A is the deflection of the end A of the beam. The deflection y_A can simply be found from the basic equation for the deflection line of a beam (see equation (90.4)). Let the beam under consideration have a constant cross section and let its moment of inertia be I. Then

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI} \tag{91.2}$$

where $M(x)$ is the bending moment in the cross section of the beam at a distance of x from the rigid fixing and E is Young's modulus of the material of the beam. We obviously have

$$M(x) = P(b - x) - R(l - x) \quad \text{for} \quad 0 \leqslant x \leqslant b$$

and

$$M(x) = -R(l - x) \quad \text{for} \quad b \leqslant x \leqslant l$$

On performing two successive integrations of the expression $M(x)/EI$ for $0 \leqslant x \leqslant b$ we find the quantities $\frac{dy}{dx}$ and y as functions of x on the interval $0 \leqslant x \leqslant b$. This makes it possible to find $\frac{dy}{dx}(b)$ and $y(b)$. For the interval $b \leqslant x \leqslant l$ the function $y(x)$ is constructed in like manner. The calculations result in

$$y(l) = y_A = \frac{Pb^2}{3EI}\left(b + \frac{3}{2}a\right) - \frac{Rl^3}{3EI} \quad \text{where} \quad a = l - b \tag{91.3}$$

We have thus determined the deformation of the beam under the action of the forces P and R. Next we find the deformation of the supporting spring A under the (imaginable) action of the force R. It is given by the expression

$$y_A = \frac{1}{k}R \tag{91.4}$$

where k is the stiffness factor of the spring. Now, equating the deflection of the beam at its end A and the elongation of the spring we obtain the value $R = R_A$ of the (reaction) force with which the spring acts on the beam.

The equality of expressions (91.3) and (91.4) gives

$$R_A = P \frac{b^2 (b + 3/2\, a)}{\left(1 + \frac{3EI}{kl^3}\right) l^3}$$

Since the force R_A with which the spring acts on the beam at the point of support A has already been found we can determine in an ordinary way the deformation of the beam under the action of the forces P and R_A. For instance, the deflection at the point l is readily found by substituting the value $R = R_A$ found above in (91.3).

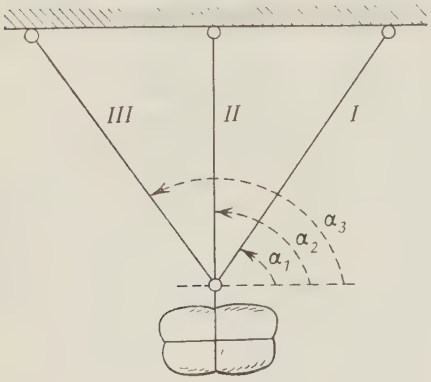

FIG. 264

It should be noted that if the spring is perfectly rigid, that is $k \to \infty$, the magnitude of the reaction force R_A is independent of the elastic properties of the beam and is equal to

$$R_A = \left(\frac{b}{l}\right)^3 \left(1 + \frac{3}{2}\frac{a}{b}\right) P \qquad (91.5)$$

As is seen, in this case

$$R_A < P \frac{b}{l}$$

The quantity $P \frac{b}{l}$ is equal to the reaction force of the support A for a beam freely supported at two points A and B (see Fig. 260*a*). Since $R_A < P \frac{b}{l}$, the bending moment in the rigid fixing cross section of the beam shown in Fig. 263 is equal to $Pb - R_A l$ and thus is greater than zero.

For $k \to 0$ when the stiffness of the spring is very small its reaction force R_A also tends to zero. In the limiting case ($k = 0$) the moment M (0) balances completely the moment Pb of the external force, that is M (0) $= Pb$ (this is also quite clear from equilibrium conditions of statics which apply to this case).

Many more complex problems can also be solved in an analogous manner. The general scheme of the method can be characterized as follows. At the initial stage all the unknown support reactions are replaced by arbitrary forces and the deformation of the structure in question under the action of all the forces (including the arbitrary forces substituting for the unknown reactions) is determined. Then the deformation of the supports under the action of the same forces is found. Finally, equating to each other the expressions of the corresponding deformations obtained in these two ways we arrive at a system of equations from which the unknown reactions are found. After the magnitudes of the reaction forces have been determined, the deformation of the structure is found in the ordinary way.

One more example below involves a statically indeterminate structure (not involving beams) whose investigation is carried out in a different way although the general idea remains in fact the same.

(2) *A load suspended from three ropes.*

Let a load be suspended from three ropes lying in one plane as shown in Fig. 264. If only two ropes supported the load the tensile force in each of the

ropes would be independent of the elastic properties of the ropes and would be specified only by the angles the ropes form with the vertical. Indeed, the force of weight $\boldsymbol{P}$ of the load can always be uniquely resolved into two components along two given directions (along the directions of the two ropes in this case).

For a load supported by three ropes such a simple method proves inapplicable even when the angles which the ropes form with the vertical are known. For, a given force cannot be resolved uniquely into components along three given directions (see Fig. 265). The dotted lines in the figure are parallel to the ropes

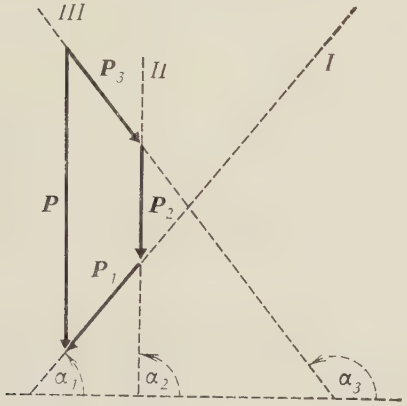

FIG. 265

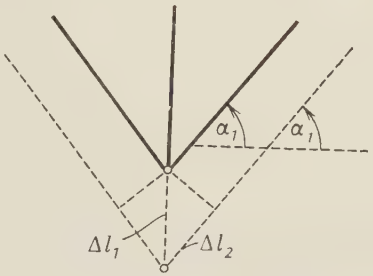

FIG. 266

and form the angles α_1, α_2, and α_3 (see Fig. 264) with the horizontal line. It is quite clear that any (closed) vectorial quadrangle consisting of three vectors parallel to the indicated dotted lines and of a vector equal to $\boldsymbol{P}$ gives a formal resolution of that type which is not unique.

To find the tensile forces in the ropes we must know their elastic properties. For the sake of simplicity, let us assume that the lateral ropes are identical and that the middle rope occupies a vertical position. Then $\alpha_2 = \pi/2$ and $\alpha_1 + \alpha_3 = \pi$. Let l_1, E_1 and S_1 denote, respectively, the length, Young's modulus, and the cross-section area of the middle rope. Let E_2 and S_2 denote Young's modulus and the cross-section area of the lateral ropes. The elongation Δl_1 of the middle rope is connected with the elongations Δl_2 of the lateral ropes by the equality (see Fig. 266):

$$\Delta l_2 = \Delta l_1 \sin \alpha_1 \tag{91.6}$$

The tensile forces in the middle rope and in the lateral ropes are

$$P_1 = \frac{E_1 S_1}{l_1} \Delta l_1 \tag{91.7}$$

and

$$P_2 = \frac{E_2 S_2}{l_2} \Delta l_2$$

respectively. Taking into account (91.6) we obtain

$$P_2 = \frac{E_2 S_2}{l_2} \Delta l_1 \sin \alpha_1 \tag{91.8}$$

Since the resultant force of the tensile forces in the ropes must balance the force of weight P of the load we have

$$P = P_1 + 2P_2 \sin \alpha_1 \tag{91.9}$$

The substitution of expressions of the forces given by formulas (91.7) and (91.8) in equality (91.9) yields

$$P = \left(\frac{E_1 S_1}{l_1} + \frac{2E_2 S_2}{l_2} \sin^2 \alpha_1 \right) \Delta l_1 \tag{91.10}$$

whence we find the elongation of the middle rope:

$$\Delta l_1 = \frac{P}{\frac{E_1 S_1}{l_1} + \frac{2E_2 S_2}{l_2} \sin^2 \alpha_1} \tag{91.11}$$

Finally, on substituting the expression we have found in formulas (91.7) and (91.8) we arrive at the sought-for solution:

$$P_1 = \frac{P}{1 + 2 \frac{E_2 S_2 l_1}{E_1 S_1 l_2} \sin^2 \alpha_1} \quad \text{and} \quad P_2 = \frac{P \sin \alpha_1}{2 \sin^2 \alpha_1 + \frac{E_1 S_1 l_2}{E_2 S_2 l_1}} \tag{91.12}$$

It should be noted that in the examples we have considered and in all analogous problems it is supposed that the deformations are small in comparison with the dimensions of the bodies involved. That is why in the first example it was legitimate to assume that the length of the beam was invariable and that the curvature of the deflection line was very small. By similar reason we supposed that in the second example the angles between the ropes remained invariable while the ropes elongated under the action of the load. For very large deformations these assumptions do not agree with reality; in such cases the computations become much more complex.

Thus, for small deformations there are two different cases: a simple one when the elastic deformations of the bodies involved do not depend on the rigidity of the supports and when all the forces acting on the system are uniquely determined by the given external forces and by the geometrical configuration and a more complicated case when the elastic deformations in the system depend on the rigidity of the supports. In the former case we speak of statically determinate structures and in the latter case of statically indeterminate ones. For a statically indeterminate structure the deformations of the supports play a very important role since they specify the magnitudes of the reactions of the supports and therefore specify the elastic deformations of the whole system. This group of problems admits of an essentially wider class of physical conditions for the systems in question. In conclusion we once again stress that the methods we have considered are only applicable when the deformations are very small.

92. Stresses in Overload and Weightlessness

In a physical body there appear stresses under the action of its own gravity. When a body lies on a horizontal plane (Fig. 267) the forces of gravitation act on its every particle while the reaction force Q' of the supporting plane balancing the resultant gravity force P is applied only to the surface of contact between the body and the support. Such a body is acted upon by the following forces: its every particle of mass Δm_i is under the action of the gravity force ΔP_i and

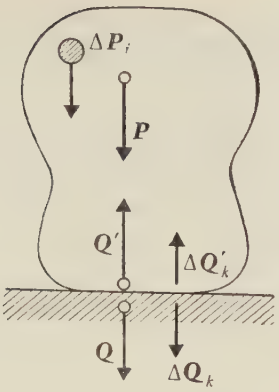

FIG. 267

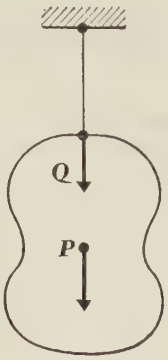

FIG. 268

a part of the surface of each particle adjoining the plane of contact is acted upon by a force $\Delta Q'_k$. It is evident that

$$P = \sum \Delta P_i \quad \text{and} \quad Q' = \sum \Delta Q'_k$$

The action of these forces produces a state of strain and there arise stresses in the body. The distribution of the stresses over the volume of the body is rather complex and is specified by the structure of the body and by its elastic properties. It is obvious that the stresses in the lower part of the body are greater than

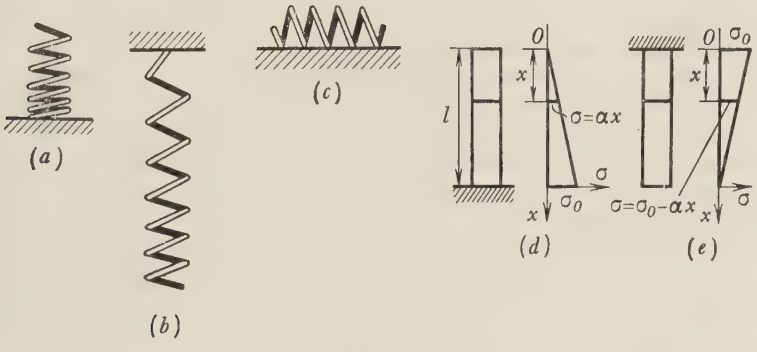

FIG. 269

in the upper part, the maximum stresses being near the plane of support. The force of weight Q (see Fig. 267) which can be written as

$$Q = \Sigma \Delta Q_k$$

where $\Delta Q_k = -\Delta Q'_k$. The force Q is of different physical nature (in comparison with the gravity force P which is equal to Q) and is applied not to the body but to the support. If the same body is suspended from a thread attached to it (Fig. 268) then the force Q stretching the thread is again equal to P but the dis-

tribution of the stresses within the body is quite different. Namely, in the first place, in the latter case the stresses are of opposite sign and, in the second place, the maximum stresses appear near the point at which the thread is attached to the body.

A deformation under the action of gravity forces can be demonstrated by the example of a spring with small stiffness factor (see Fig. 269). In the case shown in Fig. 269*a* the spring is in the vertical position on the table, in Fig. 269*b* it is suspended from its upper end and in the third case (Fig. 269*c*) it lies on the table

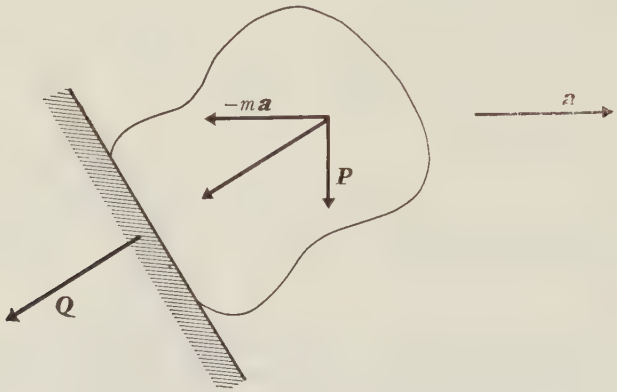

FIG. 270

in a horizontal position. As is seen, the shape of the spring and the stresses arising in it are quite different in these three cases.

Fig. 269*d* and *e* demonstrates the normal stresses σ in a homogeneous cylinder of altitude l and cross-section area S for two different cases when the cylinder is supported from below and from above, respectively. In the former case (Fig. 269*d*) the normal stress in the cross section at a distance x from the upper end of the cylinder is compressive and is equal to $\sigma = \alpha x$ where $\alpha = \sigma_0/l$ ($\sigma_0 = Q/S$) is the stress in the lowermost section adjoining the support (Q is the force of weight of the cylinder). For the cylinder suspended from its upper end (Fig. 269*e*) the normal stress in the same cross section x is tensile and is equal to $\sigma = \sigma_0 - \alpha x$. Accordingly, in the former case the magnitude of the stress increases in the downward directions while in the latter case the stress decreases in this direction. In both cases the deformations result in an increase of the cross-section area of the cylinder in the downward direction; this change of the cross section area can be calculated by formula (84.1).

The situation is similar for any other body although in some cases the deformations produced by the gravity forces may be negligibly small (for instance, this is the case for a standard weight of 1 kg).

In the case of the state of weightlessness all these stresses become equal to zero in all sections, that is for the weightlessness we have $\sigma(x) = 0$ for any x.

The stresses appearing under the action of the gravity forces play an especially important role and have a significant magnitude when the weight of the body increases (e.g. in a spaceship) owing to the phenomenon of *overload*. The overload of a body is characterized by the ratio of the magnitude of the mass force

acting on the body in an accelerated reference system to the force of gravitation. If the given body is at rest relative to a reference system which is in an accelerated translatory motion then each particle of mass Δm is under the action of two mass forces: the force of gravitation $\Delta \boldsymbol{P}$ and the inertia force $-\Delta m\boldsymbol{a}$ where $\boldsymbol{a}$ is the acceleration of the reference system (see Fig. 270). In such a reference system the force of weight of the body is equal to

$$\boldsymbol{Q} = \boldsymbol{P} - m\boldsymbol{a} \tag{92.1}$$

This force is applied to the support making the body to be at rest relative to the accelerated system of reference. Formula (92.1) expresses the most general definition of the force of weight which is applicable to any reference system. We can also consider the force of weight for a particle Δm of the body:

$$\Delta \boldsymbol{Q} = \Delta \boldsymbol{P} - \Delta m\boldsymbol{a} \tag{92.2}$$

In connection with formula (92.2) the following remark should be made: the mass forces $\Delta \boldsymbol{P}$ and $-\Delta m\boldsymbol{a}$ are applied to the particle while the force $\Delta \boldsymbol{Q}$ is applied to a part of the surface bounding that particle.

As has been mentioned, the overload of a body can be characterized by a parameter (*overload factor*) equal to the ratio*

$$n = \frac{|\Delta \boldsymbol{P} - \Delta m\boldsymbol{a}|}{|\Delta \boldsymbol{P}|} \tag{92.3}$$

The definition of the overload factor n expressed by (92.3) indicates that the absolute value of the force of weight is n times that of gravitational force in the given reference system. The gravity forces depend solely on the mutual position of the bodies. A person feels overload in a car of a train, in an automobile, in an airplane, etc. when the motion is accelerated. The overload becomes particularly noticeable when the acceleration $\boldsymbol{a}$ of the motion varies or when its magnitude is sufficiently large (for instance, when it is several times the magnitude of the acceleration of gravity g). Very large accelerations (of the order of 10 g) are painful since the weight of all the organs of the man increases about 10 times. Such an overload can only be endured by a well trained person.

For the state of weightlessness we obviously have $\Delta \boldsymbol{Q} = 0$, that is the force of gravitation is balanced by the inertia force. In this situation there is no overload, no stresses arise in the body due to the force of weight and any particle of the body exerts no action on the other particles. In such a state no physical body has any ordinary stresses (of course there may appear stresses resulting from a temperature gradient, stresses arising due to the structure of the body and the like but here we mean only the stresses produced by the action of the forces of weight). As to the bodies on the Earth, all of them, except those in a free fall, both resting and moving, are acted upon by forces arising due to the weight and all of them have the corresponding stresses. When the parameter characterizing the overload assumes a value of n all such stresses increase n times.

Let us also discuss the change of the force of weight in an accelerated reference system which is in a compound motion involving a translatory motion and a rotation. Apparently, the notion of a force of weight of a body in a reference system only makes sense when the body is at rest or in a uniform rectilinear motion relative to that system. Therefore, the changes of the force of weight correspond-

* In engineering the quantity $n - 1$ is sometimes taken as a parameter characterizing the overload.

ing to the rotational part of the motion of the reference system can only appear under the action of the centrifugal inertia forces.

Besides the gravitational force $\Delta \boldsymbol{P}$ and the inertia force $-\Delta m\boldsymbol{a}$, appearing because of the acceleration of the translatory part of the motion of the reference system, every particle of a body in the reference system in question is acted upon by the centrifugal inertia force $-\Delta m\omega^2\boldsymbol{\rho}$ where $\boldsymbol{\rho}$ is the radius vector drawn from the axis of rotation (perpendicularly to it) to the particle. Generally speaking, the latter force is different for different particles. When $\boldsymbol{\rho}$ is much larger than the dimensions of the body in question (for instance, this is the case for an ordinary body on the Earth's surface) the resultant centrifugal force can be regarded as applied to the centre of mass of the body and is equal to $m\omega^2\boldsymbol{\rho}_0$ where $\boldsymbol{\rho}_0$ is the radius vector of the centre of mass.

Hence

$$\boldsymbol{Q} = \boldsymbol{P} - m\boldsymbol{a} + m\omega^2\boldsymbol{\rho}_0$$

Consequently, the weight of a body on the surface of the rotating Earth differs from $\boldsymbol{P}$ only by the centrifugal inertia force since in this case $\boldsymbol{a} = 0$.

We distinguish between the notions of a force of weight and of a gravitational force because, for instance, in the state of weightlessness the body is acted upon only by the gravitational forces (when its motion is considered relative to the corresponding inertial reference system) while the force of weight is equal to zero and produces no stresses.

CHAPTER 11

Equilibrium of Fluids and Gases

93. Solids, Fluids, and Gases

All physical bodies consist of molecules which are in permanent motion. In solids the molecules oscillate about certain equilibrium positions and the displacements of the molecules are so small that they do not affect the character of motion of solid bodies or its parts studied in mechanics. The equilibrium positions of the molecules of a solid, that is their average positions, are quite definite and fixed. In mechanics by a particle of a solid is meant its small part * containing a very large number of molecules; the particles themselves are considered continuous and non-deformable in the mechanical motion or in the deformation of the body.

Every solid has a definite shape. To change this shape, that is to produce a deformation of the body, some forces must be applied to the body or to some of its parts. Solid bodies, in contrast to fluids and gases, retain a definite geometrical form. Fluids and gases are physical bodies that have no definite shape and assume the shape of the vessel they fill.

However, it should be noted that the indicated distinction between solids and fluids (or gases) is to a certain extent conditional and makes sense only within the limits of mechanics. The matter is that every body may behave as a solid or a fluid depending on the physical conditions and on the phenomenon in which it takes part. For instance a rotating top made of asphalt behaves as a rigid body while the same top, when subject to heating, behaves as a fluid. A more definite approach making it possible to distinguish between a solid and a fluid (or a gas) is elaborated in solid-state physics. For the aims of mechanics the distinction we have mentioned is quite sufficient.

In a gas the molecules are in a chaotic motion and collide with each other like small balls. The molecules are not connected with one another when they move; because of permanent collisions they tend to move in all the possible directions so that the gas fills uniformly the volume of the vessel in which it is placed. That is why an amount of a gas is a physical body which does not possess its own definite shape and volume. Its volume is determined by the volume of the vessel which it fills. In the analysis of mechanical phenomena in a gas the latter can be thought of as a continuous body which tends to expand and to fill the entire volume in which it is placed. Such an understanding is correct only in case the small particles of the gas consist of a very large number of molecules. For instance, when the air is in ordinary conditions the number of its molecules in a volume of 1 mm^3 is of the order of 10^{16}.

* More precisely, a particle of a body is a sufficiently small part separated out of the body such that its dimensions are very small in comparison with the dimensions of the body.

In a fluid or gas the molecules are not invariably connected with each other and the molecules can move quite arbitrarily relative to each other in their chaotic motion. In a fluid, in contrast to a gas, the average distance between the molecules remains almost invariable. This means that an amount of a fluid is a physical body which, like a gas, has no definite shape but, unlike a gas, possesses an almost invariable volume. The volume of a fluid changes only when it is acted upon by very large external forces.

A given amount of a fluid is always separated by an interface from other solids or gases; in the case of a gas medium adjoining a fluid we speak of a *free surface* of the fluid.

An amount of gas is usually bounded either by a fluid surface or by a surface of some solids; there can also be such circumstances when there are no bounding surfaces, for instance, this is the case for the upper atmosphere.

In mechanics solids, fluids, and gases are considered, to a sufficiently accurate approximation, as continuous bodies. A solid possesses a definite shape and a definite volume under given invariable conditions. An amount of fluid only possesses a definite volume and has no definite shape and an amount of gas has neither a definite shape nor volume.

94. The Concept of Pressure

A fluid or a gas placed in a closed volume can be subject to a constant external action. For example, let a fluid (or a gas) be in a cylinder with a piston (Fig. 271). Suppose that a constant force F is applied to the piston and that the piston and the fluid are in equilibrium. Then, obviously, the fluid (gas) acts on the piston with a force F' of the same magnitude and opposite direction. Considering the equilibrium condition for a volume of the fluid adjoining the piston (see Fig. 271) we readily conclude that this volume is acted upon by some forces exerted on it by the remaining part of the fluid; this means that, as in the case of a solid, there are some internal forces within the fluid.

When in a fluid (or in a gas) there arise invariable internal (static) forces their character essentially differs from that of the stresses in a solid because in fluids and gases these forces have no tangential components. Thus, static (time invariable) internal forces in a fluid or in a gas are always normal to the bounding surface of a (mentally) separated volume. In a state of equilibrium a fluid or a gas cannot transmit tangential forces from one part to another. We have already mentioned this ptoperty when we studied friction in Sec. 38 where it was said that there was no friction of rest in a fluid or gas. This can be confirmed by a number of experiments. As a simple example, let us consider a body floating on the surface of a fluid (Fig. 95); in this case any force f applied in the horizontal direction produces a motion of the body.

This property is also common for the gases. For instance, a balloon freely flying in the air can be given a displacement by any force, however small. The air thus does not prevent the balloon from being displaced.

These and other simple experiments indicate that in fluids and gases in the state of equilibrium there can only appear normal internal forces; besides, these forces always compress the given volume of a fluid or gas. That is why when considering for the internal forces in fluids or gases we speak about *pressure*. It follows that a pressure in a gas or a fluid is a scalar quantity measured by the magnitude of the compressive force acting on unit area of the surface of a (mentally) isolated volume and that this force is always normal to the surface.

The dimension of the pressure is equal to the ratio of the dimensions of the force and of the area. In the SI system as the unit of pressure is taken 1 *pascal* (1 Pa):

$$1 \text{ Pa} = \frac{1 \text{ N}}{1 \text{ m}^2}$$

In the cgs system the unit of pressure is 1 *bar*: 1 bar = 1 dyn/cm².

In engineering pressure is usually measured in the units equal to 1 kgf/cm² or 1 kgf/m². The former unit is called the *technical atmosphere* (denoted 1 at).

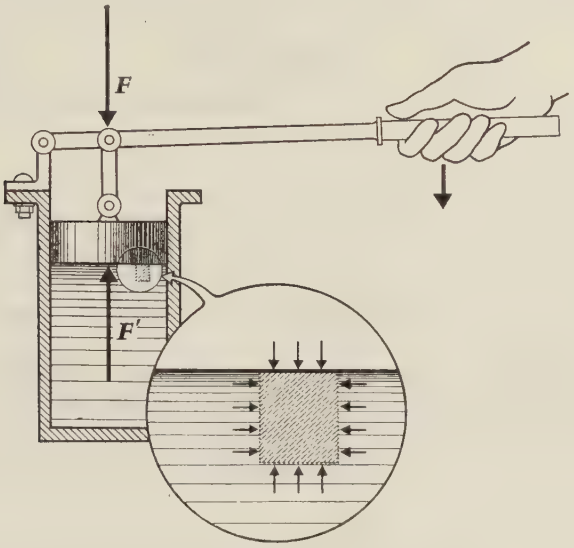

FIG. 271

Besides, in physics pressure is often measured in millimetres of mercury column in a mercury pressure gauge. This unit of pressure (1 mm Hg) is connected with another unit called a *physical* (or *standard*) *atmosphere* (denoted 1 atm):

$$1 \text{ atm} = 760 \text{ mm Hg} = 1.033 \text{ at} = 1.013 \times 10^5 \text{ N/m}^2$$

In the state of equilibrium the pressure in fluids and gases obeys *Pascal's law* which reads: *in a state of equilibrium the pressure is the same at all the points inside a fluid or gas; it is the same for all directions and is transmitted throughout the volume occupied by the fluid or gas.*

Pascal's law can easily be deduced from the equilibrium conditions for a volume of an arbitrary size and shape separated (mentally) within the fluid (or gas).

The first part of this law can be proved theoretically using the method by means of which the stresses in a deformable body were analysed (see Sec. 85).

This law can also be confirmed experimentally. Let us consider the system shown in Fig. 272*a*. It involves three pipes soldered together whose openings adjoin one point from different directions. Each of the pipes is connected with a pressure gauge. These pipes are submerged into the fluid (Fig. 272*b*) subject to pressure. It turns out that irrespective of the orientation of the openings of the pipes and for any force acting on the piston all the three pressure gauges show

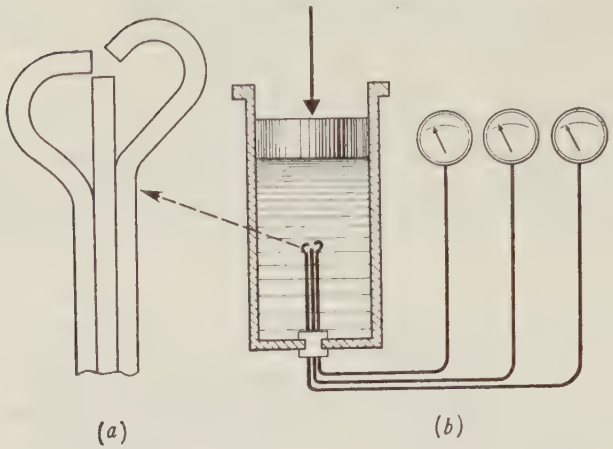

FIG. 272

the same data. When the force applied to the piston is varied the pressure in the fluid also varies and all the three pressure gauges simultaneously indicate the same pressure variation.

Since for a sufficiently small volume isolated from the fluid the force of weight is negligibly small in comparison with the forces applied to the surface of the volume, the forces of gravity do not affect the pressure inside the fluid and the pressure is the same for any small area passing through a given point within the fluid.

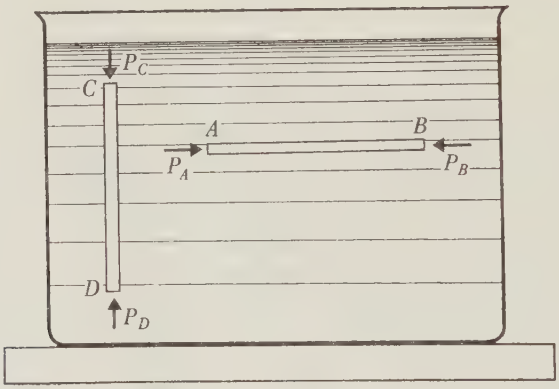

FIG. 273

The second part of Pascal's law follows from similar considerations.

Let us show that in a fluid or gas in the state of equilibrium the pressure is the same at all the points lying in one horizontal plane. To this end let us consider a volume of the fluid of the shape of a thin prism with the ends at two points A and B lying in one horizontal (see Fig. 273) and compare the pressures at these points. The equilibrium condition for the prism implies that the pressure at

one of its ends must be equal to that at the other end; we have thus proved that the pressure is equal at all the points lying in one horizontal plane.

We now find the connection between the values of the pressure at two points C and D lying in one vertical inside the fluid (or gas). Let us consider a thin prism with the ends at the points C and D (see Fig. 273). The equilibrium condition for the volume of the fluid within the latter prism can be written in the form

$$S_0 p_C + P = S_0 p_D \tag{94.1}$$

where P is the force of gravity acting on the volume of the fluid under consideration and S_0 is the cross-section area of the prism. It follows that

$$p_C + \frac{P}{S_0} = p_D$$

whence

$$p_D - p_C = p_g = \frac{P}{S_0} \tag{94.2}$$

The quantity $p_g = P/S_0$ is the additional pressure appearing because of the weight of the fluid column between the points C and D. When this force of weight is negligibly small we can approximately write $p_C = p_D$.

95. Relationship Between the Density of a Gas and the Pressure

The density of a fluid is only slightly dependent on the pressure. For example, when the pressure in a volume of water is increased by 1000 atm the variation of the volume is only about 5 per cent. In ordinary hydrostatical experiments the pressure variations are of the order of 10 atm and therefore we can always neglect the variations of the volume in the calculations and consider the fluid to be *incompressible.*

The density ρ is measured in various units whose dimension formula is m/V where m is the mass and V is the volume. In the SI system the unit of density is 1 kg/m^3, in the cgs system it is 1 g/cm^3 and in the technical system 1 kgf·s^2/m^4. Instead of the density, in experiments and calculations is often used the *specific weight* which is the weight of unit volume of the substance in question. The specific weight is usually measured in N/m^3, gf/cm^3 and kgf/m^3. In ordinary conditions the specific weight of water is 9800 N/m^3 in the SI system, 1 gf/cm^3 in the cgs system, and about 1000 kgf/m^3 in the technical system.

The density of a gas is essentially dependent on the pressure it is subject to. *For constant temperature the density of a gas* (*or its specific weight*) *is proportional to the pressure* (this follows from the well-known *Boyle-Mariotte law*).

Let p_{in} and γ_{in} denote, respectively, some initial pressure and specific weight of a gas and let p and γ denote the same quantities for some other state of the gas. The Boyle-Mariotte law is written in the form

$$\frac{\gamma_{\text{in}}}{\gamma} = \frac{p_{\text{in}}}{p} \tag{95.1}$$

This law is derived from the results of some simple experiments known to the reader from the elementary course of physics. Given the volume and the weight of an amount of gas, it is easy to determine its density or the specific weight.

As is also known, when the temperature is variable for the *perfect* (*ideal*) *gases* the following relation holds:

$$p = \frac{R}{\mu} \gamma T \tag{95.2}$$

Here T is the temperature of the gas (in the absolute, i.e. Kelvin scale), R is a quantity constant for all perfect gases (R is called the *universal gas constant*) and μ is the molecular weight of the gas. The Kelvin scale is connected with the Celsius scale by the relation $T = t + 273$.

For ordinary temperatures most gases satisfy equation (95.2) known as the *Clapeyron equation.*

96. Distribution of Pressure in a Fluid in Equilibrium State

When formulating and deriving Pascal's law we did not take into account the weight of the fluid (or gas). Now let us estimate the influence of the weight on the distribution of pressure within an incompressible fluid in the state of rest.

It is obvious that the distribution of the pressure is uniform in the horizontal direction because, if otherwise, there can be no equilibrium. It follows that the free surface of an immovable fluid is always horizontal except near the walls of the vessel where the surface tension may curve it. It should be noted that this conclusion also applies to a non-homogeneous fluid. As is seen from (94.2), the pressure varies along the vertical. It increases together with the depth as we pass from the point C to the point D (see Fig. 273); this increase is due to the force of gravity acting on the fluid column (the prism) with vertical lateral surface and horizontal bases at the points C and D.

If the fluid is incompressible or, more precisely, if its compressibility is negligibly small, the specific weight γ does not depend on the pressure. Then the weight of the fluid column is equal to

$$P = \gamma S_0 l \tag{96.1}$$

where S_0 is the cross-section area of the prism and l is its altitude. Consequently, the pressure on the upper base exceeds that on the lower base by

$$p_g = \frac{P}{S_0} = \gamma l \tag{96.2}$$

that is the pressure is a linear function of altitude.

The increase of the pressure with the depth causes the buoyancy forces acting on the bodies immersed into the fluid or floating on its surface (see Sec. 98).

The distribution law for pressure we have established gives an explanation of the hydrostatic paradox. Consider the two vessels filled with the same fluid shown in Fig. 274. It turns out that the forces of pressure acting on the bottoms of the vessels are not equal to the weights of the amounts of fluid filling the vessel. The force of pressure can exceed that weight (Fig. 274*a*) or be less than the weight (Fig. 274*b*) since the pressure at the bottom depends solely on the altitude of the fluid column and on the specific weight and therefore in both cases the force of pressure exerted on the bottom is equal to the product of the pressure by the bottom area S:

$$P = \gamma h S \gamma l$$

The pressure difference in communicating vessels is used in the construction of manometers (Fig. 275*a*). It is evident that the fluid is in equilibrium only when the difference in fluid levels in the vessels produces a counterpressure γh balancing the pressure difference $p_2 - p_1$ above the free surfaces of the fluid in the communicating vessels. The equilibrium condition implies

$$p_2 - p_1 = \gamma h$$

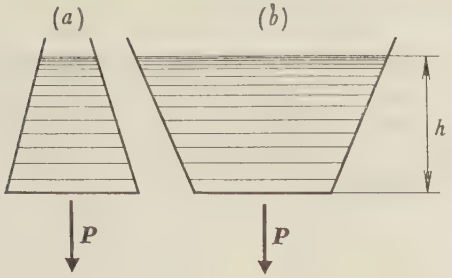

FIG. 274

where h is the level difference of the fluid in the communicating vessels. (If the cross sections of the pipes are equal the forces of surface tension can be neglected.) In pressure gauges of this type the pressure is measured by the altitude difference of the columns of the fluid filling the vessels. Therefore, we speak of pressure differences measured in mm of water (H_2O) or in mm Hg and the like. For the sake of convenience in real manometers are used communicating vessels of different diameters (Fig. 275*b*) because in this case the variation of the fluid level in the vessel whose cross section is much greater than that of

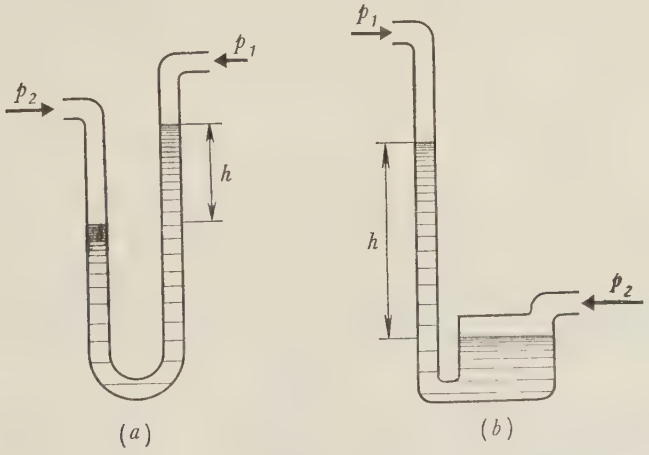

FIG. 275

the other vessel can be neglected and therefore it suffices to measure the altitude of the fluid column in the narrower pipe.* For instance, it can readily be checked that when the ratio of the diameters is equal to 50 the error arising in connection with the neglect of the variation of the fluid level in the wider pipe is less than 0.05 per cent.

* It is clear that in such a case the forces of surface tension can no longer be neglected. But if we measure only the variations of the pressure and not the pressure itself then the additional constant pressure difference caused by the surface tension does not affect the results of the measurements.

97. Distribution of Pressure in a Gas

When an amount of gas is in equilibrium the pressure increases only in the downward direction because of the variation of the weight of the gas column lying above the level at which the pressure is measured. It is apparent that in any horizontal plane the pressure is the same at all the points. In order to determine the variation of pressure in the vertical direction it is necessary to take into account the change of the density (or of the specific weight) caused by the variation of the pressure.

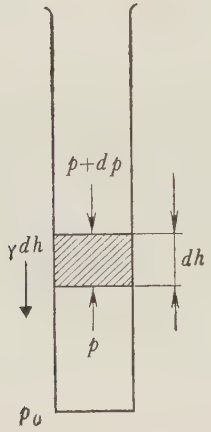

FIG. 276

Let us write down the equilibrium condition for a gas column of the shape of a cylinder with cross-section area of 1 cm² (Fig. 276) and a very small altitude dh:

$$p + dp + \gamma\, dh - p = 0$$

that is

$$dp = -\gamma\, dh \tag{97.1}$$

where dp is the pressure difference between the upper and the lower bases of the cylinder. Consequently, the variation of the pressure corresponding to the variation of the altitude by h is equal to

$$p_h - p_0 = \int_0^h dp = -\int_0^h \gamma\, dh$$

that is

$$p_0 - p_h = \int_0^h \gamma\, dh \tag{97.2}$$

When a gas has a constant temperature its specific weight γ and pressure p obey the Boyle-Mariotte law (95.1):

$$p = \gamma \frac{p_0}{\gamma_0} \tag{97.3}$$

In this case the dependence of the pressure on the altitude is found as follows. The substitution of (97.1) in (97.3) yields

$$dp = -p \frac{\gamma_0}{p_0} dh \quad \text{whence} \quad \frac{dp}{p} = -\frac{\gamma_0}{p_0} dh$$

On integrating the last relation from 0 to h we find

$$\int_{p_0}^{p_h} \frac{dp}{p} = -\frac{\gamma_0}{p_0} \int_0^h dh, \quad \text{or, equivalently,} \quad \ln \frac{p_h}{p_0} = -\frac{\gamma_0 h}{p_0}$$

The relation we have obtained is finally transformed to the formula

$$p_h = p_0 e^{-\frac{\gamma_0 h}{p_0}} \tag{97.4}$$

This is the *barometric* (*height*) *formula* showing that, as h grows, the pressure decreases according to an exponential law. Using formula (97.4) we can calculate the air pressure in the static atmosphere on condition that the temperature of the air remains constant at every altitude.

For the sake of visuality, let us draw the dotted line which represents the dependence between p and h corresponding to the formula

$$p = p_0 - \gamma_0 h \tag{97.5}$$

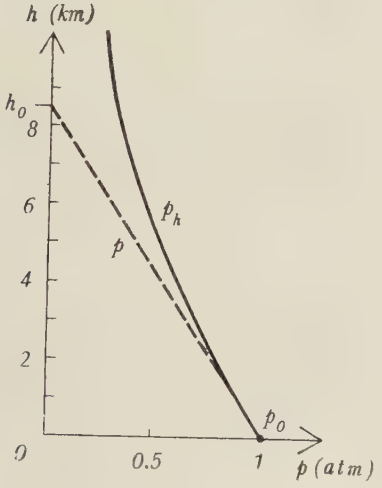

FIG. 277

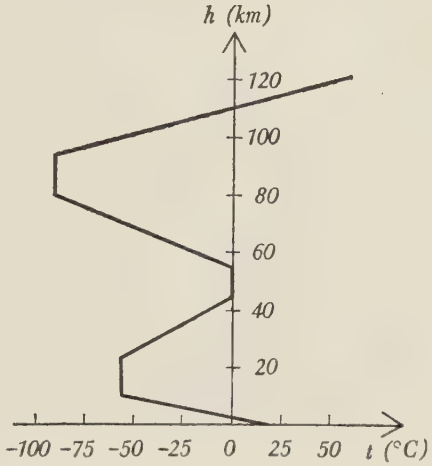

FIG. 278

(see Fig. 277) where γ_0 is the specific weight of the air near the Earth's surface at the level $h = 0$. This dotted line represents the dependence of the air pressure on the altitude for the imaginable atmosphere consisting of an incompressible gas with constant specific weight γ_0. In this case the altitude h_0 for which the pressure is equal to zero is referred to as the height of the "homogeneous atmosphere". By (97.5), the height h_0 is given by the formula

$$h_0 = \frac{p_0}{\gamma_0} \tag{97.6}$$

For the temperature of 15 °C and for the normal atmospheric pressure at the sea level we have $p_0 = 1.013 \times 10^5$ N/m^2 and the specific weight of the air is $\gamma_0 \approx 12$ N/m^3. Hence, approximately,

$$h_0 \approx 8400 \text{ m} = 8.4 \text{ km}$$

If the air had the same specific weight in higher levels as near the Earth's surface then the Earth would be surrounded by an air layer of the width of about 8.4 km and this layer would produce the same pressure near the Earth's surface as the real atmosphere. Using formula (97.6) we can rewrite formula (97.4) in the form

$$p_h = p_0 e^{-h/h_0} \tag{97.7}$$

The continuous line in Fig. 277 represents the pressure reduction with altitude corresponding to the law expressed by formula (97.7).

In a thin layer of air near the surface of the Earth the pressure variation is approximately the same as in the homogeneous atmosphere: as is seen the continuous line and the dotted line in Fig. 277 are close to each other in the vicinity of $h = 0$. Therefore, in approximate calculations for a not very large height (in comparison with 8.4 km) formula (97.5) can be used.

Formula (97.5) makes it possible to estimate the error arising in experiments with a gas when the pressure is assumed to be constant throughout the volume occupied by the gas. Suppose that this gas is the air. In ordinary laboratory experiments we deal with air columns of height not exceeding, approximately, 10 m and the corresponding error is therefore less than

$$\frac{10}{8400} \approx \frac{1}{8}\%$$

In reality the air temperature does not remain constant with the altitude variation. For technical calculations not requiring high accuracy some conditional mean value of the temperature corresponding to the given altitude can be taken. This mean value is found from the data obtained in a large number of experiments (see Fig. 278). It can be assumed, approximately, that the temperature decreases with the altitude according to a linear law up to the height of 11 km and remains constant (approximately —55 °C) above this level. Recent measurements made with the aid of rockets have shown that from the altitude of 25 km up to ≈45 km there is a temperature rise and the temperature at the latter level is about 0 °C. Beginning with 55 km the temperature again reduces and reaches almost —90 °C at the altitudes of 80-95 km. Then again there is an increase of temperature which reaches almost 1000 °C for the altitude of about 230 km.

98. Equilibrium of Bodies Floating on the Surface of a Fluid

A body immersed partly or wholly in a fluid or gas is acted upon by an expulsive (buoyancy) force produced by the surrounding medium. As early as the 3rd century B.C. the great Greek scientist Archimedes stated the fundamental law for this phenomenon: *every body immersed in a fluid (or gas) is acted upon by a supporting force exerted by the surrounding medium whose magnitude is equal to the weight of the amount of the fluid (or gas) displaced by the body; this force is directed vertically upwards and its line of action passes through the centre of mass of the displaced amount of the fluid (or gas).*

The existence of such a force and its magnitude can easily be explained on the basis of the pressure distribution in fluid which is under the action of the gravity forces. To derive Archimedes' law we can divide (mentally) the body immersed in the fluid into cylinders of small cross-section area having vertical generatrices, determine the magnitude of the force acting on each of the cylinders and then find the resultant of all the forces applied to these cylinders.

In the case when a body is only partly immersed in a fluid the argument is the same with the distinction that only the part of the body immersed in the fluid should be divided into small cylinders.

The weight of a body floating on the surface of a fluid is equal to the weight of the volume of the fluid displaced by the body. When the centre of gravity of the floating body lies below the centre of gravity of the volume of the fluid displaced by the body the equilibrium of the body is obviously stable.

The equilibrium can also be stable when the centre of gravity of the body lies above the centre of gravity of the displaced fluid; for instance, this is the case

for ships. In this connection, let us consider a wooden bar shaped as a rectangular parallelepiped floating on the water surface as shown in Fig. 279. The density of wood is approximately 0.5 that of the water and therefore the centre of mass A of the bar lies above the centre of mass B of the volume of water displaced by the bar independently of the position it occupies when floating. However, experiments show that in case the longest edge of the parallelepiped is hori-

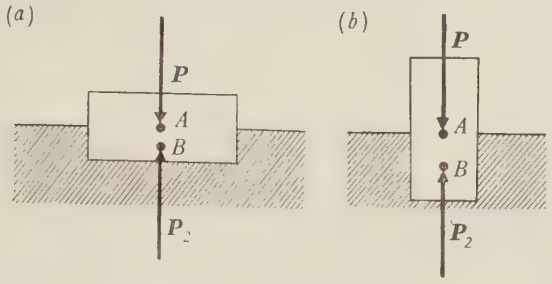

FIG. 279

zontal the floating bar is in the state of a stable equilibrium (see Fig. 279*a*). When placed in the position shown in Fig. 279*b*, the bar immediately turns over and, after several swings, occupies the stable position.

99. Equilibrium Conditions for a Body Immersed in a Fluid or Gas

A solid (whose volume, within certain limits, is independent of the pressure) when put in a vessel with fluid, either floats on the surface of the fluid or lies on the bottom of the vessel. If the weight of the body is exactly equal to that of the volume of the fluid displaced by the body the latter is in a state of neutral equilibrium in any part of the vessel.

What has been said can be demonstrated in simple experiments. For instance, we can put an egg in a glass vessel with water and make it float at any depth (Fig. 280) by gradually dissolving salt in the water. The egg always occupies such a position in which its centre of gravity lies in one vertical line with the centre of gravity of the volume of water displaced by the egg and whose centre of gravity is lower than the centre of gravity of that volume of water. On being shifted from such a stable position the egg turns over and returns to the state of the stable equilibrium.

The volume of a real physical body usually decreases as the pressure grows. If such a change of the volume is considerably large the body is always in a state of unstable equilibrium when immersed in a fluid of constant density. Indeed, suppose that the weight of such a body is equal to that of the volume of the fluid it displaces when the body is at a certain depth where the pressure has a definite value. If the body is given a small downward shift from this equilibrium position the pressure acting on it increases while its volume decreases. Consequently, the supporting (buoyancy) force also decreases and the body continues to move downwards. A similar instability takes place when the body is moved upwards from its equilibrium position. In the latter case all the quantities involved undergo reverse variations: the pressure reduces, the volume increases, the buoyancy force increases, and the body continues to move upwards.

The phenomenon we have described can be demonstrated by using as a body whose volume decreases when the pressure grows a floating test tube with a load attached to its open end which makes it float with that end down. The test tube is placed in a high glass vessel filled with water whose opening is covered with an elastic membrane (see Fig. 281). The test tube is partly filled with water and its remaining part contains some air.

When the elastic membrane is pressed the air pressure above the water in the vessel and, together with it, the pressure in the water itself increase. The

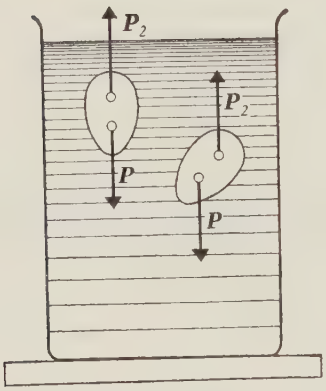

FIG. 280

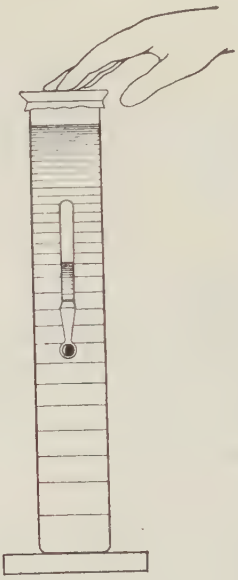

FIG. 281

air in the test tube is compressed, the volume of the water displaced by the float decreases and the test tube moves downwards to the bottom. When the pressure on the membrane is decreased the test tube moves backwards. The pressure on the membrane can be regulated so that the test tube is made to move in any (upward or downward) direction or to occupy any intermediate position between the bottom and the membrane.

The equilibrium conditions for a body floating in a gas are the same as in the case of a fluid. If the volume of a body in a gas does not depend on the pressure the equilibrium is always stable because the specific weight of the gas increases as the altitude decreases. In the case when a body floating in a gas changes its volume under the pressure variations the investigation of its equilibrium is more complex. In such a case both the variation of the volume of the body and the variation of the volume of the gas should be taken into account.

CHAPTER 12

Motion of Fluids and Gases

100. Stationary Fluid Flow

When a fluid or a gas is in motion there arise forces of internal (viscous) friction between its particles. The viscosity coefficient of some substances (for instance, of air or water) is comparatively small. Therefore, under some conditions (these conditions will be discussed in more detail later) the motion of such substances can be approximately regarded as a flow of an *ideal* (*perfect*) *fluid* having no viscosity. In reality there are of course no perfect fluids; however, the motion of a fluid or a gas can often be approximately treated as a flow of a perfect fluid.

After the laws of motion of a perfect fluid have been established we can introduce the necessary corrections into these laws taking into account the influence of the viscosity. In this way it is possible to elucidate the complex laws governing the motion of a viscous fluid or gas.

In order to investigate the motion of a flowing fluid (or gas) it is convenient to describe it with the aid of the *velocity field* for the particles of the medium under consideration. By a velocity field we mean a region in space at whose each point $\boldsymbol{r}$ for any time t a vector $\boldsymbol{v}(\boldsymbol{r}, t)$ is defined which is the velocity of a particle passing at instant t through that point $\boldsymbol{r}$. The vector function $\boldsymbol{v}(\boldsymbol{r}, t)$ depends on the position vector $\boldsymbol{r}$ of the point and on time t.

A fluid (or gas) flow is said to be *stationary* when all the quantities involved, that is the velocity, the pressure, the density, the temperature, and so on remain *constant* all the time at every point in the spatial region occupied by the moving fluid (or gas). If otherwise, we speak of a *non-stationary* motion; in the latter case the laws of motion are more intricate.

Even the kinematical description of a stationary flow of a gas in a pipe or of a stationary motion of water in pipes, rivers, and channels is rather complex. In the general case the velocities of the particles of a moving fluid differ in their magnitude and direction for different points of the region in space occupied by the flow. The pressure acting on the moving particles also differs from point to point, its variation being in a certain connection with the motion of the particles. In a gas flow the density also varies from place to place since the density, the temperature, and other quantities vary.

To simplify the investigation of a stationary flow let us (mentally) divide the flowing fluid* into sufficiently thin *stream tubes*. Imagine a small closed curve A placed in the flow and the trajectories of all the particles which pass

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* In what follows by a fluid flow we shall mean, without special stipulation, a flow of a fluid or of a gas unless the distinction between a fluid and a gas becomes important.

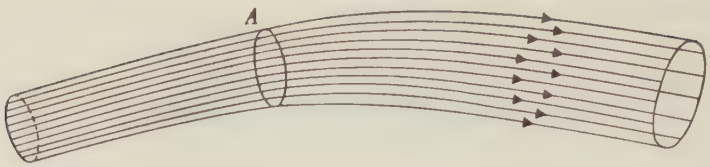

FIG. 282

through the points of that contour (see Fig. 282). All these trajectories form the surface bounding a stream tube.

We consider a fluid continuous and therefore the walls of a stream tube are also thought of as a continuous surface. On the lateral boundary of a stream tube the velocity vectors of the particles are tangent to the bounding surface. The whole region occupied by a fluid flow can be divided into such tubes. The stream tubes can be made visible in a dye experiment. For instance, we can introduce some smoke into an air flow or put some dye into a stream of water as shown in

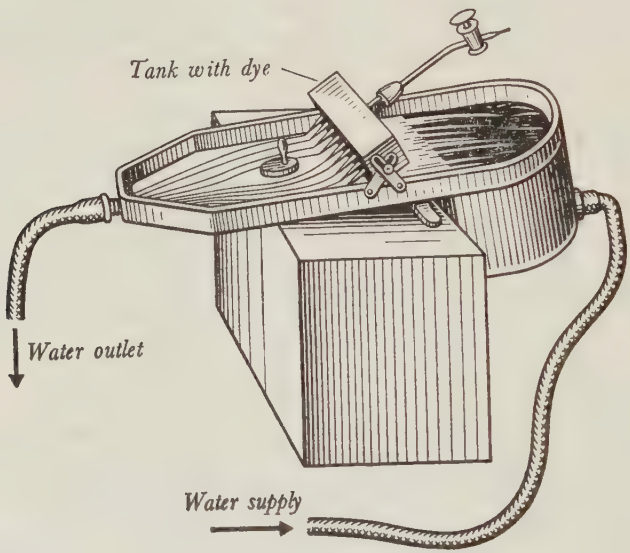

FIG. 283

Fig. 283 where is depicted an apparatus serving for the observation of the flow-around of a body placed into the stream. The smoke or dye traces indicate the trajectories of the corresponding particles and make it possible to see some stream tubes.

The walls of a stream tube are formed of trajectories of particles of the flowing fluid. A particle of the fluid which is within a stream tube at a certain instant remains within this tube all the time. We can always take a stream tube of a sufficiently small cross section so that it is allowable to assume that the velocities of the particles are the same for all the points of a given cross section of the tube and are perpendicular to the normal sections of the tube.

The motion of a fluid within a stream tube resembles a flow without friction in a pipe with rigid walls whose cross section varies sufficiently gradually.

For a non-stationary fluid flow we can also speak of stream tubes but in this case they are no longer formed of trajectories of particles. Indeed, let us consider a non-stationary velocity field $\boldsymbol{v}(\boldsymbol{r}, t)$ at some time t. In this field we can mentally draw the stream lines by which are meant curves at whose every point the direction of the tangent line coincides with that of the velocity field $\boldsymbol{v}$. The stream lines passing through the points of a closed contour form a stream tube. However, generally speaking, it is clear that the shape of such a stream tube formed by the lines passing through the given contour is dependent on time. Besides, the stream lines may essentially differ from the real trajectories of the particles. For, when a particle passes from a point $\boldsymbol{r}$ to another point $\boldsymbol{r} + d\boldsymbol{r}$ the velocity vector at the new point may change during the corresponding time dt, the tangents to a stream line going in the direction of the vectors $\boldsymbol{v}(\boldsymbol{r}, t)$ at any given time t. In other words, in the case of a non-stationary motion a stream line indicates displacements of different particles and does not coincide with a trajectory which is the trace of a single particle.

Let us examine the implications of the condition that the mass flux across different sections of a stream tube of a stationary fluid flow must be constant. It is clear that for a stationary flow the amount of fluid passing through a cross section of a stream tube in unit time must be the same for all the cross sections.

Let us consider a (thin) stream tube with cross-section area S. If the velocity of the particles in that cross section is v then the mass of the fluid passing through the section during one second is equal to

$$Q = \rho v S \tag{100.1}$$

where ρ is the density of the fluid in that cross section. For another cross section of the same tube having an area of S_1 we similarly have

$$Q = \rho_1 v_1 S_1 \tag{100.2}$$

where v_1 and ρ_1 are the velocity and the density of the fluid in the section S_1. Indeed, if otherwise, the amount of fluid placed between these sections would increase or decrease, which is impossible since we have supposed that the flow is stationary and therefore all the quantities involved are time-independent.

Thus, we can write down the *constancy law for mass flux*:

$$Q = \rho v S = \text{const} \tag{100.3}$$

Relation (100.3) holds for any stream tube because every stream tube can be divided into sufficiently thin tubes (the constant on the right-hand side of (100.3) can be different for different stream tubes).

When we deal with an incompressible fluid (for instance, such as water under ordinary conditions) the density ρ of the fluid remains constant and therefore, by the constancy of the mass flux (see (100.3)), the velocity in any cross section of a thin stream tube is inversely proportional to the cross-section area. Hence, the shape of the tube determines the variation of the velocity which increases where the cross-section area is smaller and, conversely, reduces where the tube is wider (Fig. 284).

Let us find the relationship between the variations of the velocity and of the pressure along a (thin) stream tube (see Fig. 284). To this end we (mentally) isolate a particle which occupies a (short) part of the stream tube and whose

cross section coincides with that of the tube (see the shaded part in Fig. 284). As the particle moves along the tube it undergoes a deformation. It is evident that when this particle moves along a part of the stream tube where the cross section is constant the velocity of the particle also remains constant and hence the particle has no acceleration when moving through that part of the tube. When the cross-section area of the tube decreases along the direction of the flow (see part *1-2* in Fig. 284) the particle receives an acceleration and its velocity increases. Conversely, when the cross section increases along the flow (see part *2'-3*

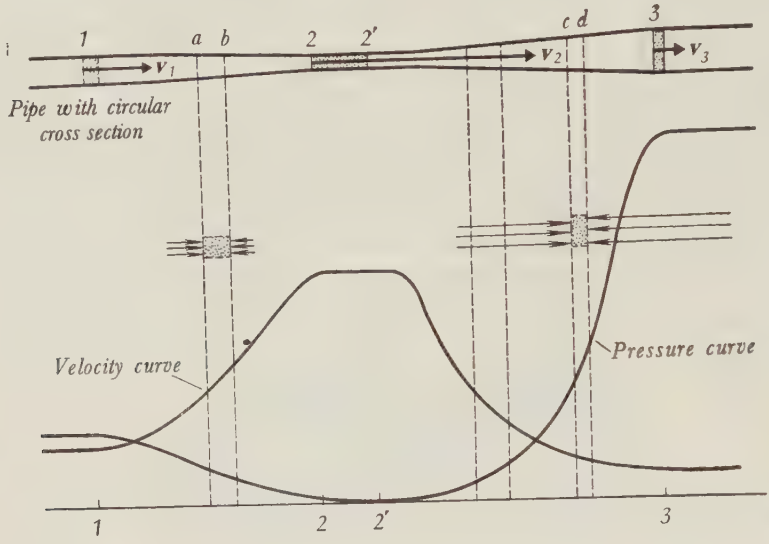

FIG. 284

in the figure) the motion of the particle is decelerated and its velocity decreases.

It is evident that if the tube is horizontal the particle can only receive an acceleration under the action of the adjoining particles. Consequently, for a part of the tube where the cross-section area decreases along the flow (part *1-2*) the pressure must reduce along the flow, that is the pressure behind the particle *ab* (Fig. 284) must be greater than the pressure before it so that the acceleration producing the increase of the velocity appears. In the part of the tube where the cross-section area increases (part *2'-3*) and the velocity decreases along the flow the pressure must grow so that the particle *cd* has a negative acceleration and therefore the pressure before the particle is greater than behind it. We see that knowing the character of the variation of the cross-section area of a stream tube we can describe qualitatively the variation of the pressure along the tube. The diagram of pressure distribution along the tube is shown in Fig. 284.*

* The distribution of pressure obeys Bernoulli's law which will be established in the following sections.

101. Fundamental Law of Dynamics for a Particle of an Ideal Fluid

Every particle in a fluid flow is under the action of the adjoining particles surrounding that particle; the measure of this action is the pressure p. As was shown, the pressure variation determines the acceleration of a moving particle. Proceeding from this fact we can derive the fundamental law of dynamics for a particle of an ideal (perfect) fluid.

Let us (mentally) separate a particle of the shape of a very small cube of volume $d\tau = dx_1\, dx_2\, dx_3$ placed at a point $\boldsymbol{r}$ (x_1, x_2, x_3) (see Fig. 285). Each face of the cube is acted upon by a force of pressure. For example, the face $dx_1\, dx_2$ is under the action of the force $p\, dx_1\, dx_2$ directed upwards while the opposite face is acted upon by the force

$$-\left(p+\frac{\partial p}{\partial x_3}\,dx_3\right) dx_1\, dx_2$$

Therefore, the component of the force acting on the cube along axis *3* is equal to

$$p\, dx_1\, dx_2-\left(p+\frac{\partial p}{\partial x_3}\,dx_3\right) dx_1\, dx_2=$$

$$=-\frac{\partial p}{\partial x_3}\,d\tau$$

Besides, the particle is under the action of the gravity force equal to

$$-\gamma\, d\tau$$

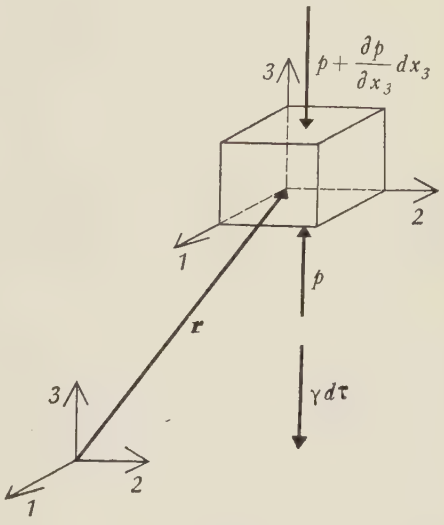

FIG. 285

and directed along axis *3* where γ is the specific weight of the fluid. Therefore, by the second law of dynamics,

$$\rho\, d\tau\, \frac{dv_3}{dt}=-\frac{\partial p}{\partial x_3}\,d\tau-\gamma\, d\tau$$

that is

$$\rho\,\frac{dv_3}{dt}=-\frac{\partial p}{\partial x_3}-\gamma \tag{101.1}$$

where v_3 is the component of the velocity along axis *3*.

The volume $d\tau$ being very small, the density ρ is considered constant over the volume of the fluid we have separated. Similarly, the pressure p is considered constant over each of the faces of the cube and so are the velocities v.

For the other coordinate axes we derive analogously the relations

$$\rho\,\frac{dv_1}{dt}=-\frac{\partial p}{\partial x_1}\quad\text{and}\quad\rho\,\frac{dv_2}{dt}=-\frac{\partial p}{\partial x_2} \tag{101.2}$$

not involving the gravity force since it goes along axis *3*.

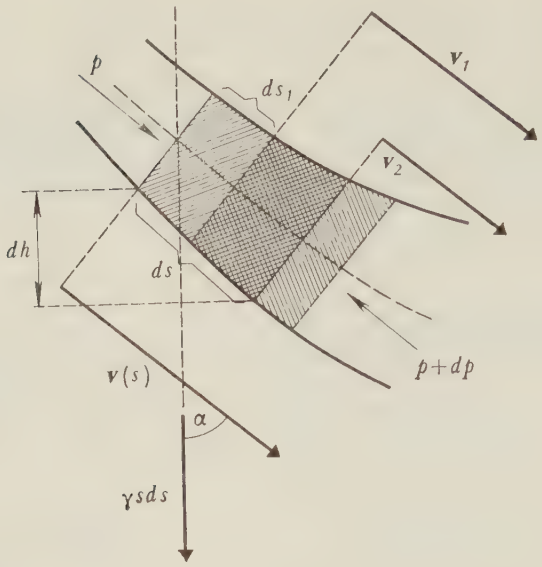

FIG. 286

We can now write the three formulas we have obtained (formulas (101.1) and (101.2)) in the vector form:

$$\rho \frac{d}{dt}(v_1 e_1 + v_2 e_2 + v_3 e_3) = -\left(\frac{\partial p}{\partial x_1} e_1 + \frac{\partial p}{\partial x_2} e_2 + \frac{\partial p}{\partial x_3} e_3\right) - \gamma e_3$$

where e_1, e_2, and e_3 are unit vectors along the coordinate axes. The last relation can be rewritten as

$$\rho \frac{dv}{dt} = -\operatorname{grad} p + \rho g \tag{101.3}$$

where grad p symbolizes the vector $\frac{\partial p}{\partial x_1} e_1 + \frac{\partial p}{\partial x_2} e_2 + \frac{\partial p}{\partial x_3} e_3$ (this vector is called the gradient* of the pressure p) and $\rho g = -\gamma e_3$ (g is a vector of magnitude g directed downwards; it is equal to the acceleration produced by gravity).

Formula (101.3) expresses the fundamental law of hydrodynamics of an ideal fluid or gas (whose particles move without friction). For a non-stationary flow all the quantities involved, that is ρ, v, and p, may depend both on the position vector r and on time t. In a stationary flow they depend on the vector r solely. Therefore, when studying stationary flows, we can use the notion of a stream tube, which simplifies the analysis because in this case the stream tubes are invariable. For a stationary flow we can take a sufficiently thin stream tube and derive the law of dynamics of an ideal fluid in the following way. The magnitude v of the velocity v is in this case a function $v = v(s)$ depending only on

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* The operation of taking the gradient is sometimes symbolized by the vector $e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3} = \nabla$: grad $p = \nabla p$ (the symbolic vector ∇ is called the *Hamiltonian operator* or *nabla* or *del*).

the coordinate s reckoned along the axis of the tube. A particle having a coordinate s at time t gains a displacement ds_1 during time dt (Fig. 286). The velocity v_1 of the particle in its new position $s + ds_1$ may differ from its value v at the point s. It can be represented in the form

$$v_1 = v(s) + \frac{dv}{ds} ds_1$$

Consequently, the difference between the values of the velocity of the particle at time t and at time $t + dt$, that is the increment of the velocity, is expressed as

$$dv = v_1 - v(s) = \frac{dv}{ds} ds_1$$

On replacing the displacement ds_1 of the particle by $v(s)\,dt$ we obtain

$$\frac{dv}{dt} = v \frac{dv}{ds} = \frac{d}{ds}\left(\frac{v^2}{2}\right) \tag{101.4}$$

Thus, the acceleration of a particle in a stationary fluid flow is equal to the derivative of half the square of the velocity of the flow with respect to arc length of the axis of the stream tube. Therefore, the fundamental dynamical equation for a particle of an ideal fluid (formula (101.3)) can be written for a stationary flow in the form

$$-\frac{dp}{ds} + \gamma \cos\alpha = \rho v \frac{dv}{ds} = \rho \frac{d}{ds}\left(\frac{v^2}{2}\right) \tag{101.5}$$

where α is the angle between the vertical and the direction of the tangent line to the axis of the stream tube at the given cross section. This equation is applicable to a stationary flow of both an incompressible non-viscous fluid and a compressible gas without internal friction.

Let us dwell on the determination of the acceleration $\frac{dv}{dt}$ of a particle in the general case of a non-stationary flow for which the vector field $\boldsymbol{v}(\boldsymbol{r}, t)$ is known. As has been shown, for a stationary flow the acceleration of a particle is equal to $v\frac{dv}{ds}$, that is it is determined by the variation of the velocity along the stream tube. It turns out that the same result can be derived without using the notion of a stream tube from the general expression of the acceleration.

The velocity of a particle passing through a point $\boldsymbol{r}$ at time t is equal to $\boldsymbol{v}(\boldsymbol{r}, t)$; at time $t + dt$ this particle is at a point $\boldsymbol{r} + d\boldsymbol{r}$ and its velocity is equal to $\boldsymbol{v}(\boldsymbol{r} + d\boldsymbol{r}, t + dt)$. Therefore, the increment of the velocity of the particle is

$$d\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{r} + d\boldsymbol{r}, t + dt) - \boldsymbol{v}(\boldsymbol{r}, t) \tag{101.6}$$

and the acceleration of the particle is equal to $\frac{d\boldsymbol{v}}{dt}$. Let us represent the increment $d\boldsymbol{v}$ as a sum of two parts which we denote $(d\boldsymbol{v})_t$ and $(d\boldsymbol{v})_r$. Let the first part $(d\boldsymbol{v})_t$ be the increment of $\boldsymbol{v}$ which is connected with the time variation solely and let the other part $(d\boldsymbol{v})_r$ be the increment appearing because of the change of the position of the particle. The vectors $(d\boldsymbol{v})_t$, $(d\boldsymbol{v})_r$, and the other vectors involved in our consideration are shown in Fig. 287 (in the figure we also see the vector $\boldsymbol{v}(\boldsymbol{r} + d\boldsymbol{r}, t)$ representing the velocity of some other particle which passes through the point $\boldsymbol{r} + d\boldsymbol{r}$ at time t). Thus,

$$d\boldsymbol{v} = (d\boldsymbol{v})_t + (d\boldsymbol{v})_r \tag{101.7}$$

where

$$\left.\begin{aligned}(dv)_t &= v(r + dr,\ t + dt) - v(r + dr,\ t)\\ (dv)_r &= v(r + dr,\ t) - v(r,\ t)\end{aligned}\right\} \tag{101.8}$$

In a stationary flow the acceleration is determined by $(dv)_r$ solely since $(dv)_t = 0$ in this case because the velocity is independent of time at each point. In the general case, for a non-stationary flow both terms are non-zero. The term

$$(dv)_t = \frac{\partial v}{\partial t}\,dt \tag{101.8}$$

is specified by the partial derivative of v with respect to t for $r = \text{const}$. The other term is the partial differential $(dv)_r$ of v and it has a more complex structure: it depends on the directional derivative of v along dr for $t = \text{const}$. This derivative is sometimes denoted $\frac{dv}{dr}$. The differential $(dv)_r$ should be found as the increment which the vector v would receive if the flow were stationary. An increment of this type was considered in Sec. 86 where, in connection with the study of small deformations, we dealt with an increment in a vector field corresponding to a displacement dr. Each of the components v_1, v_2, and v_3 of the velocity is a function of the three variables x_1, x_2, and x_3 (and of time t). The vector v can be represented in the form $v = v_1e_1 + v_2e_2 + v_3e_3$ where e_1, e_2, and e_3 are unit vectors in the directions of the coordinate axes. Therefore, the increments of the components of the velocity vector can be written as

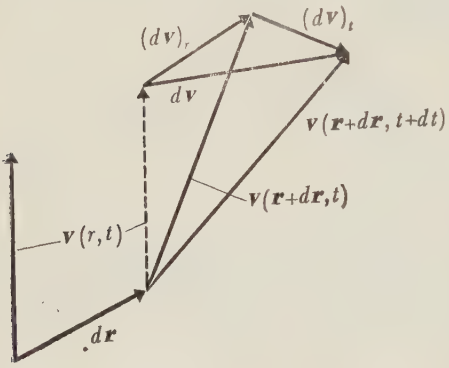

FIG. 287

$$\left.\begin{aligned}dv_1 &= \frac{\partial v_1}{\partial x_1}\,dx_1 + \frac{\partial v_1}{\partial x_2}\,dx_2 + \frac{\partial v_1}{\partial x_3}\,dx_3\\ dv_2 &= \frac{\partial v_2}{\partial x_1}\,dx_1 + \frac{\partial v_2}{\partial x_2}\,dx_2 + \frac{\partial v_2}{\partial x_3}\,dx_3\\ dv_3 &= \frac{\partial v_3}{\partial x_1}\,dx_1 + \frac{\partial v_3}{\partial x_2}\,dx_2 + \frac{\partial v_3}{\partial x_3}\,dx_3\end{aligned}\right\} \tag{101.9}$$

and $(dv)_r = dv_1e_1 + dv_2e_2 + dv_3e_3$. As is seen from formulas (101.9), the increment $(dv)_r$ can be represented as the product of a *tensor* U by the vector $dr = dx_1e_1 + dx_2e_2 + dx_3e_3$:

$$(dv)_r = \mathrm{U}\,dr \tag{101.10}$$

where

$$\mathrm{U} = \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{pmatrix} \tag{101.11}$$

Since we speak of the increment of the velocity of a definite particle which moves from the point r to the point $r + dr$ during time dt there must be $dr = v\,dt$.

On substituting this expression in (101.10) we obtain

$$(d\boldsymbol{v})_r = \mathrm{U}\boldsymbol{v}\,dt \tag{101.12}$$

Therefore, taking into account formulas (101.7) and (101.12) we can write the acceleration $\frac{d\boldsymbol{v}}{dt}$ of the particle in the form

$$\frac{d\boldsymbol{v}}{dt} = \frac{(d\boldsymbol{v})_t}{dt} + \frac{(d\boldsymbol{v})_r}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + \mathrm{U}\boldsymbol{v} \tag{101.13}$$

Formula (101.13) gives the general expression for the acceleration of a particle. The first term on the right-hand side of (101.13) is the partial derivative of $\boldsymbol{v}$ with respect to t and the second term is the product of the tensor U (formula (101.11)) by $\boldsymbol{v}$.

For a stationary flow we have $\frac{\partial \boldsymbol{v}}{\partial t} = 0$ and therefore

$$\frac{d\boldsymbol{v}}{dt} = \mathrm{U}\boldsymbol{v} \tag{101.14}$$

when the velocity, the density, and the pressure depend on one coordinate solely and when the velocity is directed along that coordinate axis, for instance, when $v_1 \neq 0$, $v_2 = 0$, and $v_3 = 0$ and all the derivatives with respect to x_2 and x_3 are equal to zero then $\frac{dv_1}{dt} = v_1 \frac{\partial v_1}{\partial x_1}$. We thus obtain the expression which coincides with the result of our previous investigation expressed by formula (101.4) if we put $x_1 = s$ and $v_1 = v$. In this case the pressure p is also a function of the single argument x_1 and therefore, if the influence of gravitation is neglected, the equation of hydrodynamics (101.3) takes the form

$$\rho v_1 \frac{\partial v_1}{\partial x_1} = -\frac{\partial p}{\partial x_1}$$

which also coincides with our previous results (see (101.5)).

102. Bernoulli's Equation for a Stationary Flow of an Incompressible Fluid

For a stationary flow of an incompressible ideal fluid the basic dynamical equation describing the motion of a particle along a stream tube implies a simpler equation which will be deduced in this section.

In the case under consideration the density and the specific weight of the fluid remain constant and equation (101.5) can therefore be rewritten in the form

$$-\frac{dp}{ds} + \gamma \cos\alpha = \frac{d}{ds}\left(\frac{\rho v^2}{2}\right) \tag{102.1}$$

Let us denote by h the altitude of the point where the particle with coordinate s is located. Then connection between the displacement of the particle by a distance of ds and the increment dh of the altitude (see Fig. 286) is expressed by the relation

$$-dh = ds \cos\alpha \tag{102.2}$$

Now, replacing $\cos\alpha$ by $-\frac{dh}{ds}$ in (102.1) we obtain

$$-\frac{dp}{ds} - \gamma \frac{dh}{ds} = \frac{d}{ds}\left(\frac{\rho v^2}{2}\right) \tag{102.3}$$

where all the terms are derivatives with respect to the coordinate s. Consequently,

$$\frac{d}{ds}\left(p+\gamma h+\frac{\rho v^2}{2}\right)=0 \tag{102.4}$$

The derivative with respect to s being equal to zero, the sum of the three quantities in the brackets remains constant along the stream tube, that is

$$C=p+\gamma h+\frac{\rho v^2}{2}=\text{const} \tag{102.5}$$

This is *Bernoulli's equation* for a stationary flow of an incompressible ideal fluid. It plays a fundamental role in hydrodynamics. In Bernoulli's equation p is the *static pressure* which compresses the fluid particles, γh is the increment of pressure corresponding to the increment h of the altitude, and $\frac{\rho v^2}{2}$ is the *dynamic pressure* (see Sec. 106).

Bernoulli's equation (102.5) provides a simple method for the solution of many important problems. Its application is based on the fact that if a flow of a fluid is divided into stream tubes and the pressure p_0 and the velocity v_0 are found in a certain way for a point whose altitude h_0 is known then, irrespective of the variations of the velocity, of the pressure, and of the altitude along the stream tube passing through that point, the sum on the right-hand side of (102.5) remains invariable. This makes it possible to determine the unknown quantities for other points in the flow. The application of this technique will be demonstrated later by some examples and problems.

Bernoulli's equation is a consequence of the energy conservation law for a particle moving along a stream tube. This equation is implied by the fact that the work of the forces of pressure is equal to the corresponding increment of the sum of the kinetic and the potential energies of the particle since the forces of pressure acting on the particle are external forces.

Let us consider the variation of the energy of a particle and its connection with the work of the forces of pressure during time dt as the particle which occupies a part of length ds of the corresponding stream tube at time t (see Fig. 286) moves along the tube. Let the displacement during time dt of the back cross section of the tube bounding the moving particle be equal to ds_1. Generally, ds_1 is not equal to the length ds of the particle (in the figure ds_1 is smaller than ds). The changes connected with the displacement of the particle can be described as follows. The left-hand shaded part (see Fig. 286) of volume $dQ = S\,ds_1$ displaces the right-hand shaded part having the same volume dQ while the middle part does not change its state during time dt although the material points forming that part at the instant t and at the instant $t + dt$ are different. Therefore, by (102.2), the increment of the potential energy of the particle (in Fig. 286 the particle moves downwards and therefore this increment is negative) can be represented in the form

$$-\gamma\,dQ\cos\alpha\,ds=\gamma\,dQ\,dh \tag{102.6}$$

The corresponding increment of the kinetic energy is equal to

$$\left(\frac{\rho v_2^2}{2}-\frac{\rho v^2}{2}\right)dQ=\frac{d}{ds}\left(\frac{\rho v^2}{2}\right)dQ\,ds \tag{102.7}$$

where v_2 is the velocity of the front cross section bounding the moving particle of length ds. The work performed by the forces of pressure as the back cross section travels the distance ds_1 is equal to $pS\,ds_1 = p\,dQ$; similarly, the work of

the forces of pressure connected with the displacement of the front boundary of the particle is equal to $-(p + dp)\, dQ$. The resultant work of the forces of pressure is

$$[p - (p + dp)]\, dQ = -dp\, dQ \tag{102.8}$$

Now equating the work of the forces of pressure to the variation of the sum of the potential and the kinetic energies we arrive at the equality

$$\gamma\, dQ\, dh + \frac{d}{ds}\left(\frac{\rho v^2}{2}\right) dQ\, ds = -dp\, dQ \tag{102.9}$$

On cancelling (102.9) by $dQ\, ds$ we obtain (102.4). Finally, the integration of (102.4) along the stream line results in Bernoulli's equation (102.5).

103. Outflow of a Heavy Fluid from a Vessel

Using Bernoulli's equation (102.5) we can easily find the velocity of the jet of a heavy fluid flowing out of a vessel. Let the fluid flow out of the vessel shown in Fig. 288. The opening is placed below on the wall of the vessel and is equipped with nozzle directing the outflowing jet. During the outflow the whole volume of the fluid in the vessel is in motion and it can be divided into stream tubes. The exact determination of the shape of the stream tubes is a rather complicated problem even for a vessel of a simple form but for our aims it is quite sufficient to know that the back boundaries of all the stream tubes are on the free surface of the fluid in the vessel and that the stream tubes pass through the opening in the nozzle.

The velocity on the free surface of the fluid has the same value v_0 for all the stream tubes. The same applies to the pressure p_0 on the free surface and to the altitude h_0 (because when the free surface is at an altitude considerably greater than the dimension of the opening it remains horizontal when moving down as the fluid flows out of the vessel). Hence, the constant C in Bernoulli's equation has the same value for all the stream tubes and is equal to

$$\frac{\rho v_0^2}{2} + h_0\gamma + p_0 = \text{const} \tag{103.1}$$

It should be noted that what has been said applies to any motion in which all the moving particles of an ideal fluid are in identical initial conditions. In all such cases the constant entering into Bernoulli's equation has the same value not only along a given stream tube (as was established earlier) but also for the whole region occupied by the moving fluid; this simplifies the analysis of the motion.

It is supposed of course that the diameter of the opening is small in comparison with the altitude of the free surface of the fluid in the vessel and we can therefore assume that the pressure i constant over the cross section of the running jet and is equal to the outer pressure p_0. Similarly, the velocities of the particles in the jet can be considered equal to the same value v_0 for all the stream tubes. Hence, by Bernoulli's equation (102.5), we have

$$C_0 = \frac{\rho v^2}{2} + \gamma h + p_0 = \frac{\rho v_0^2}{2} + \gamma h_0 + p_0 \tag{103.2}$$

whence

$$\frac{\rho}{2}(v^2 - v_0^2) = \gamma\,(h_0 - h) \tag{103.3}$$

where h is the altitude of the opening and h_0 is the altitude of the free surface of the fluid in the vessel.

In case the area of the opening is negligibly small relative to the cross-section area of the vessel the velocity v_0 is very small in comparison with the velocity v and the term in equation (103.3) involving v_0^2 can be dropped. Then the velocity of the fluid flowing out of the vessel is approximately equal to

$$v = \sqrt{\frac{2\gamma}{\rho}(h_0 - h)} = \sqrt{2g(h_0 - h)} \tag{103.4}$$

because $\frac{\gamma}{\rho} = g$ where g is the acceleration of gravity.

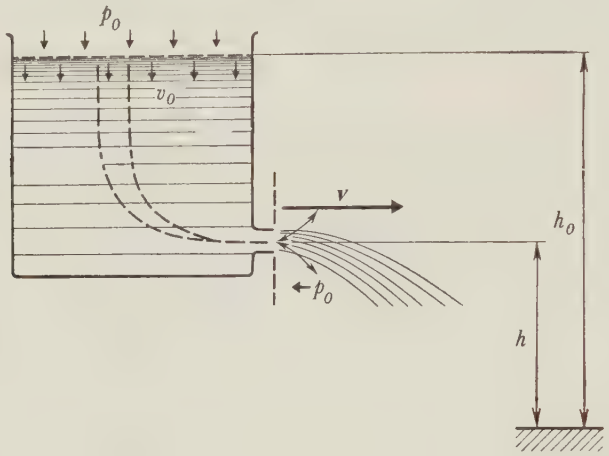

FIG. 288

Relation (103.4) is known as *Torricelli's formula*. The velocity of the outflow of a heavy fluid from an opening in the vessel containing the fluid is equal to the velocity of a body falling from a height of $h_0 - h$ equal to the altitude difference for the opening and the free surface. It should be noted that the velocity v is completely independent of the direction of the jet relative to the horizontal line; it is the same for all the values of the angle of inclination of the jet. Therefore, when the nozzle directs the jet vertically upwards the particles of the fluid must reach the level of the free surface.* But this is a purely theoretical conclusion because the friction in the fluid and the air resistance, however small, prevent the particles in the jet from reaching exactly the level of the free surface (Fig. 289*a*). The particles come closer to that level when the jet is directed at a small angle to the vertical (Fig. 289*b*) because in this case the friction produced by the drops of the fluid falling down reduces.

The validity of Torricelli's formula can be verified in various ways. For example, we can observe two intersecting jets flowing out of the vessel (see Fig. 290) in horizontal direction from two openings placed at different altitudes. The

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* In our consideration we neglect the forces of viscous friction acting on the particles of the fluid as they move along a rubber pipe connecting the nozzle with the vessel (see Fig. 289). In case the diameter of the pipe is large in comparison with the diameter of the jet the influence of these forces is in fact negligibly small.

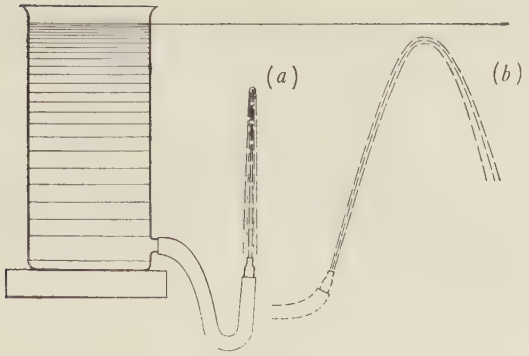

FIG. 289

position of the point of intersection can readily be determined when the forces of viscous friction are so small that they can be neglected. In this case the calculations show that the point of intersection lies below the lower opening at a distance equal to the difference between the altitude of the free surface of the fluid in the vessel and the altitude of the upper opening. The experiment confirms this theoretical inference. The indicated property can be more easily observed when an amount of fluid is additionally poured into the vessel so that the point of intersection changes its position.

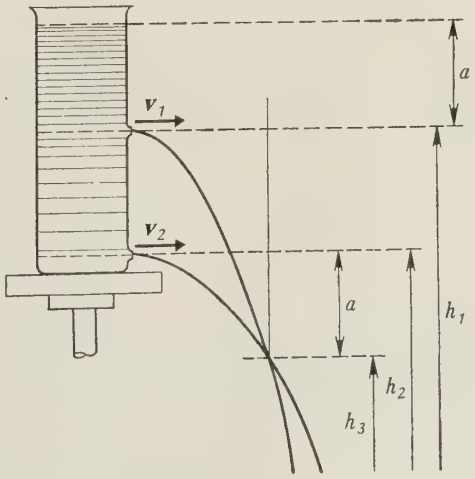

FIG. 290

The calculations leading to the conclusion we have mentioned are rather simple. Assuming that the velocity of the fluid flowing out of the vessel is specified by Torricelli's formula we can write

$$v_1 = \sqrt{2ga} \text{ and } v_2 = \sqrt{2g\,(a + h_1 - h_2)}$$

where the notation corresponds to Fig. 290. The time periods during which the particles of the two jets issued simultaneously from the openings reach the point of intersection are determined by the formulas

$$t_1 = \sqrt{\frac{2(h_1 - h_3)}{g}} \quad \text{and} \quad t_2 = \sqrt{\frac{2(h_2 - h_3)}{g}}$$

At the point of intersection the particles are at equal distances from the vessel. Since the horizontal component of the velocity of a falling particle remains constant we have $v_1 t_1 = v_2 t_2$ whence

$$a(h_1 - h_3) = (a + h_1 - h_2)(h_2 - h_3)$$

that is

$$h_2 - h_3 = a$$

which is what we intended to prove.

104. Pressure in Fluid Flow in a Pipe with Variable Cross Section

In Fig. 291 we see a scheme of a device meant for measuring pressure in a pipe with variable diameter. The pipe made of glass is placed horizontally. It has small openings to which manometric tubes are attached. These manometers measure the pressure in the corresponding cross section of the horizontal pipe (the pressure is proportional to the height of the water columns in the vertical tubes). The fluid in the manometric tubes is at rest and, consequently, the particles of the fluid in the openings are also at rest. The particles of the fluid passing under the openings are compressed under the action of the pressure p. Since the forces of pressure act in the same way in all directions, for the fluid particle in an opening to be at rest it is necessary that the pressure produced by the water column in the corresponding manometric tube should be equal to the pressure p in the fluid moving along the pipe.

It is supposed that the horizontal pipe is sufficiently thin so that it is legitimate to consider the pressure in the moving fluid constant over each cross section of the pipe. If the diameter of the cross section of the pipe varies gradually along the flow the whole horizontal pipe can be regarded as one stream tube.

Varying the velocity of the fluid in the pipe we can watch the levels of the fluid in the manometric tubes and draw some conclusions concerning the distribution of the pressure p along the pipe. The experiment shows that the narrower the cross section of the pipe, the smaller is the pressure in that cross section. It can also be observed that, for a given cross section, the greater the velocity, the smaller the pressure. These conclusions are in a complete agreement with Bernoulli's equation.

If the cross-section areas are known for two places in the horizontal pipe where manometric tubes are attached the measurement of the pressure difference for the two cross sections makes it possible to determine the amount of fluid passing through the pipe every second (the flow rate).

Indeed, let the cross-section areas be S_1 and S_2, the corresponding velocities be v_1 and v_2, and the pressure p_1 and p_2.

Then, according to Bernoulli's equation (102.5), we have

$$p_1 + \frac{\rho v_1^2}{2} = p_2 + \frac{\rho v_2^2}{2} \tag{104.1}$$

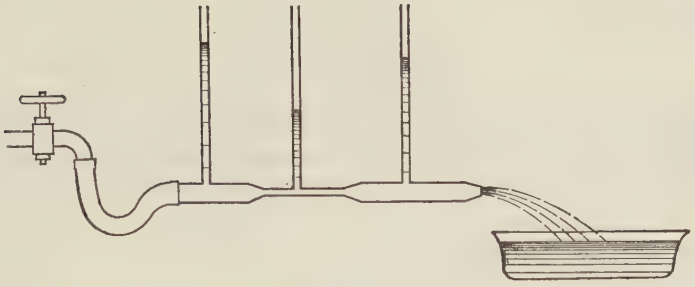

FIG. 291

and, by the constancy of the fluid flux in any cross section,

$$Q = \gamma v_1 S_1 = \gamma v_2 S_2 \tag{104.2}$$

Here, as usual, $\gamma = \rho g$ is the specific weight of the fluid.

On solving the two equations (104.1) and (104.2) with the two unknowns v_1 and v_2 we find the velocities v_1 and v_2 after which the (weight) flow rate is readily determined:

$$Q = g \sqrt{\frac{2(p_2 - p_1)\rho}{\frac{1}{S_1^2} - \frac{1}{S_2^2}}} \tag{104.3}$$

Formula (104.3) specifying the dependence of the flow rate on the pressure difference serves as the theoretical foundation for the construction of a device known as water flowmeter which makes it possible to determine (from the pressure difference) the amount of water passing through the cross section of the pipe in unit time.

105. Outflow of Fluid or Gas from a Vessel Under the Action of Pressure

When a fluid (or gas) is in a vessel under a pressure which is much greater than the pressure produced by the weight of the fluid, the variation of the pressure in the vertical direction can be neglected. In these circumstances it is allowable to assume that the outflow of the fluid from the vessel obeys the laws governing the motion of a fluid flowing out of a closed vessel under a constant pressure p_i inside the vessel. This approximate approach makes it possible to compute in a simple manner the outlet velocity for the water flowing out of a boiler in which it is under the action of a constant pressure of the vapour of the order of ten atmospheres or the velocity of the outflow of a gas from a pressure gas bottle (see Fig. 292) in which a constant pressure is kept with the aid of a compressor. In all such cases the constant entering into Bernoulli's equation can be assumed to be the same throughout the volume of the fluid or gas in the vessel and to be equal to p_i because the velocity of the motion of the fluid within the vessel can be neglected since the cross-section area S of the vessel is much greater than the area s of the opening.

The outlet velocity of a fluid flowing from a vessel can readily be found using equation (102.5):

$$v = \sqrt{\frac{2(p_i - p_0)}{\rho}} \tag{105.1}$$

where p_0 is the pressure outside the vessel and ρ is the density of the fluid.

The outlet velocity of a gas cannot be found using formula (102.5) because the density ρ of the gas varies as the particles of the gas move towards the opening.

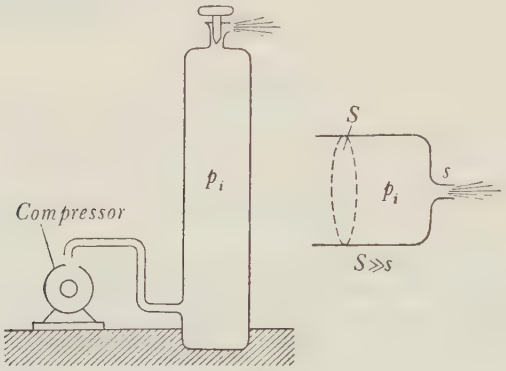

FIG. 292

For a stationary flow the variation of the pressure along a stream tube can be described by (101.5):

$$-\frac{dp}{ds} = \rho v \frac{dv}{ds} \tag{105.2}$$

In (105.2) the density ρ is no longer constant and is dependent on the pressure p. The pressure p must drop as the particles approach the opening because the particles are accelerated in the direction of motion. The velocity of their motion also depends on the character of the variation of the density as the pressure changes.

In the general case the relationship between the pressure and the density is rather complex because it also involves the variation of the temperature along the stream tube. However, as experiments show, when the particles move sufficiently fast it can be assumed that the process is adiabatic; then the connection between the pressure and the density is given by the relation

$$\frac{p}{\rho^{\varkappa}} = \frac{p_i}{\rho_i^{\varkappa}} = \text{const} \tag{105.3}$$

where $\varkappa$ is the adiabatic exponent dependent on the nature of the gas (for the air it is equal to 1.4) and ρ_i is the density of the gas inside the vessel. The adiabatic relation (105.3) is a consequence of the fact that during the expansion of a particle there is no heat exchange with the surrounding particles, that is there is no loss or gain of heat.

On finding from (105.3) the density as function of the pressure and substituting it in (105.2) we obtain the equation

$$-p^{-1/\varkappa}\frac{dp}{ds}=\frac{\rho_i}{p_i^{1/\varkappa}}v\frac{dv}{ds} \tag{105.4}$$

This equation should be integrated along the stream tube. Since the pressure in the vessel is p_1 and the pressure outside the vessel is p_0 we must integrate from p_1 to p_0 with respect to the pressure and from zero to $v = v_0$ ($v = v_0$ is the outlet velocity) with respect to the velocity:

$$-\int_{p_i}^{p_0} p^{-1/\varkappa}\,dp=\frac{\rho_i}{p_i^{1/\varkappa}}\int_0^{v_0} v\,dv$$

On performing the integration we find the outlet velocity:

$$v=\sqrt{\frac{2\varkappa}{\varkappa-1}\frac{p_i}{\rho_i}\left[1-\left(\frac{p_0}{p_i}\right)^{\frac{\varkappa-1}{\varkappa}}\right]} \tag{105.5}$$

If the gas were assumed to be incompressible, formula (105.1) would give us the following expression for the outlet velocity v_{inc} of the incompressible gas:

$$v_{\text{inc}}=\sqrt{\frac{2(p_i-p_0)}{\rho_i}} \tag{105.6}$$

Expression (105.5) can be rewritten in the form

$$v=v_{\text{inc}}\sqrt{\frac{\varkappa}{\varkappa-1}\frac{1-\left(\frac{p_0}{p_i}\right)^{\frac{\varkappa-1}{\varkappa}}}{1-\frac{p_0}{p_i}}} \tag{105.7}$$

Now we can easily estimate the error arising in the calculations when the gas is considered incompressible. To this end we should only estimate the square root in (105.7) for the given pressure difference. When the relative value of the pressure difference between p_1 and p_0 is very small, say of several per cent, the direct computation readily shows that the value of the root is very close to unity. In such cases the velocity of the gas can be found as in the case of an incompressible fluid.

Let us estimate more exactly the error appearing when the air is considered incompressible under a pressure close to the atmospheric pressure. Suppose that the difference between the pressures inside and outside the vessel is 10% of the atmospheric pressure; for definiteness, let the pressure p_1 inside the vessel be 1 atm and the pressure p_0 outside the vessel be 0.9 atm. If the air were an incompressible fluid we would substitute the value

$$\rho_1 = 1.293 \text{ kg/m}^3$$

of the air density and the value

$$p_1 = 1.0133 \times 10^5 \text{ N/m}^3$$

of the atmospheric pressure in (105.6) to find

$$v_{\text{inc}}=\sqrt{\frac{2\times 0.1\times 1.0133\times 10^5}{1.293}}=125 \text{ m/s}$$

Now let us estimate the value of the square root in (105.7). Denoting $\Delta = 1 - \frac{p_0}{p_1}$ and $\frac{\varkappa-1}{\varkappa} = a$ we write the root in the form

$$\sqrt{\frac{1-(1-\Delta)^a}{a\Delta}}$$

and expand $(1 - \Delta)^a$ in Tailor's series in powers of Δ:

$$(1-\Delta)^a = 1 - a\Delta + \frac{a(a-1)}{2}\Delta^2 + \dots$$

On substituting this expression in the radicand we obtain

$$\sqrt{1-\frac{\Delta}{2}(a-1)+\dots} = \sqrt{1+\frac{\Delta}{2\varkappa}+\dots} \approx 1+\frac{\Delta}{4\varkappa}+\dots$$

Now, replacing Δ by 0.1 and $\varkappa$ by 1.4 we find that the error in the determination of the velocity is about 2%. Consequently, when it is sufficient to determine the velocity for the given pressure difference without a high accuracy the compressibility of the air can be neglected for the pressure difference not exceeding 10% of the atmospheric pressure; in these conditions the motion of the air can be regarded as a flow of an incompressible fluid.

It is obvious that for such a small pressure difference along a stream tube the density also varies very slightly; the percentage of the pressure variation is approximately of the order of the percentage of the density variation. Indeed, when a gas undergoes a very small adiabatic expansion the corresponding relative variation of the pressure is $\varkappa$ times the relative variation of the density because (105.3) implies $\frac{dp}{p} = \varkappa\frac{d\rho}{\rho}$. A small variation of the density along a stream tube does not affect the magnitude of the velocity and, consequently, it does not affect the character of the flow either.

106. Pressure at the Stagnation Point in a Flow Around a Body

One of the most widely used devices meant for measuring the velocity of a fluid flow is the Pitot tube. This device is also used for measuring the velocity of motion of a body relative to the air (for instance, the velocity of an airplane). The theoretical foundation of the Pitot tube is the relationship between the pressure at the *stagnation point* in a flow around a body and the velocity of the flow.

When a fluid or a gas flows around a body the stream tubes diverge near the surface of the body (see Fig. 293). Therefore, there is a point (see the point A in Fig. 293) on the part of the surface of the body facing the flow at which the stream tubes diverge in the various directions. It is this point that is called the stagnation point and it is obvious that the velocity of the flow at the stagnation point must be equal to zero. By the continuity of the flow, the velocities of the particles passing near the stagnation point are very small. Let us consider a very thin stream tube (it is dashed in Fig. 293) which, so to say, ends at the stagnation point.

When the body is in a homogeneous flow of an incompressible fluid the pressure p_0, the density ρ_0 and the velocity v_0 have constant values at all the points in the flow which are sufficiently far from the body. Consequently, the constant entering into Bernoulli's equation is the same for all the stream tubes, that is for all the points in the flow. In the case under consideration this constant is

equal to $p_0 + \frac{\rho_0 v_0^2}{2}$ *. Since the velocity of the flow at the stagnation point is equal to zero, the pressure p_s at this point is equal to

$$p_s = p_0 + \frac{\rho_0 v_0^2}{2} \tag{106.1}$$

which is implied by Bernoulli's equation (102.5). From (106.1) we find

$$v_0 = \sqrt{\frac{2(p_s - p_0)}{\rho_0}} \tag{106.2}$$

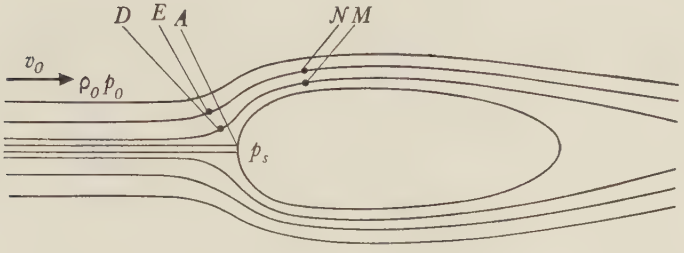

FIG. 293

It should be noted that the connection between the pressure at the stagnation point and the velocity of the flow far from the body described by formula (106.2) is quite similar to the relationship between the velocity and the pressure of a compressed fluid outflowing from a vessel (see (105.1)).

For a body placed in a flow the pressure at the stagnation point can be measured with the aid of a manometer. To this end an opening made in the surface of the body at the stagnation point (or, more precisely, near the stagnation region) is connected with the manometer by means of a pipe. The pressure at the stagnation point determines the constant in Bernoulli's equation for the flow. This is the *total pressure* (*total head*). When the total pressure is known it is possible to determine the velocity of the flow at any point provided that the static pressure p_0 in the flow and the density ρ_0 of the fluid are known.

As a body placed in the flow we can simply take a tube whose opening faces the flow. The other end of the tube is connected with a manometer measuring the pressure in the tube. Sometimes instead of the tube is taken a cylindrical body with a rounded end (Fig. 294) and with a thin hollow along its axis whose opening A is connected with a manometer by means of a pipe. The cylinder is placed so that the opening A faces the flow and the stagnation region lies near the opening.

To determine the velocity v_0 of the flow we must know, besides the total pressure p_s, the static pressure p_0 in the flow. The static pressure in the flow can be measured with the aid of the method similar to that used for the measurement of the pressure in the pipe with a flowing fluid (see Fig. 291). In the pipe the openings were made to which the manometers were attached. For the measurement of the static pressure in a flow the cylindrical body is placed so that its lateral

* Since the variation of the altitude is very small we assume here that the term γh can be neglected. When necessary, this term can also be taken into account.

surface goes along the stream lines of the unperturbed flow.* The pressure is measured in a small opening in the lateral surface of the body. If for the stream tube passing near the opening the cross-section areas close to the opening and far from the body are the same, the pressure near the opening is equal to the pressure far from the body.** The opening is connected by means of a pipe with a manometer which shows the static pressure p_0.

The openings for the measurement of the static pressure in a flow are often made in the surface of the cylindrical body by means of which the total pressure

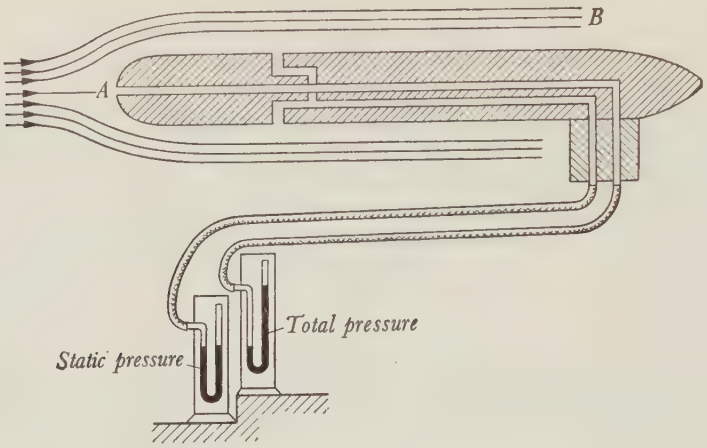

FIG. 294

is measured. In the Prandtl tube (its axial section is shown schematically in Fig. 294) the openings for the measurement of the static pressure are at some distance (of the order of 3-5 diameters of the cylinder) from the front end of the cylinder; near the openings the stream lines are almost rectilinear. The openings are connected by means of a rubber pipe with the manometer measuring the static pressure p_0 in the flow.

Knowing the total pressure p_s and the static pressure p_0 we can find the velocity of the flow incident on the body from equation (106.1).

The quantity $p_d = \frac{\rho_0 v_0^2}{2}$ is called the *dynamic pressure* (or *head*) or the *kinetic (velocity) pressure* (or *head*). It can be measured if we apply the total pressure p_s to one end of a manometer and the static pressure p_0 to the other end; then the manometer will measure the pressure difference, that is p_d. Knowing the dynamic pressure we can find the velocity of the flow.

It should be noted that the velocity of the air can be approximately found by the formula

$$v = 4\sqrt{h} \tag{106.3}$$

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* This condition is always fulfilled when the axis of the cylinder is directed along the flow and the diameter of the cylinder is negligibly small in comparison with the transversal dimensions of the flow.

** Strictly speaking, the pressure is the same when the temperature at these points is the same.

where the velocity v is measured in m/s and the pressure difference, that is the dynamic head $p_d = h$, in millimetres of mercury. Formula (106.3) is used in practical applications; it readily follows from (106.2) if we assume, as is usually done, that for the air the density is

$$\rho \approx \frac{1}{8} \frac{\text{kgf} \cdot \text{s}^2}{\text{m}^4}$$

When the measurement in a compressible gas (for instance, in the air) is done for large values of the velocity* it is necessary to take into account the variation of the density and to consider the relationship between the pressure and the velocity in a stream tube as it was done in Sec. 105. Formulas (105.5) and (105.7) can be used for the calculation of the velocity from a given pressure; to this end p_1 should be replaced by the pressure p_s at the stagnation point. However, when the velocity of the flow is very large, for instance, when it is close to the speed of sound in the gas, these relations become inapplicable because for such values of the velocity of the flow there arises a new phenomenon of the velocity and pressure jump in front of the body, which will be discussed in Sec. 120.

107. Variation of Pressure Across a Stream Tube

In Sec. 100, when determining the pressure in a stream tube, we assumed that the pressure was constant throughout every cross section of the tube. This was allowable because we supposed that the stream tube was very thin and also because we were only interested in the acceleration of the particles along the axis of the tube. However, when the diameter of a stream tube is not small and when the stream tube is not rectilinear the velocity is no longer constant across the stream tube. In this section we shall study the variation of the velocity over the cross section of such a stream tube.

If a stream tube is rectilinear in a given place, that is if the axis of the tube is a straight line, then a particle of the fluid passing through this place can have an acceleration only in the direction of the axis of the tube. For, in these circumstances the pressure must be the same near the opposite walls of the tube in any cross section and it is therefore constant throughout the cross section. On the contrary, in the places where the axis of the tube is curved the pressure cannot be constant over the cross section. Indeed, a particle moving with velocity v along a curved part of the stream tube possesses the centripetal acceleration $\frac{v^2}{R}$ where R is the radius of curvature of the axis of the tube (see Fig. 295). Therefore, the particle must be acted upon by a force of magnitude

$$\rho S \frac{v^2}{R} ds$$

acting in the plane of the axis of the tube and directed perpendicularly to the stream lines where S is the cross-section area of the stream tube and ds is the length of the particle of the fluid (Fig. 295).

This force can only be created by the pressure of the surrounding layers of the flowing fluid. That is why there must exist a pressure difference on the opposite sides of the stream tube in the plane where the curved part of the axis of

* As it follows from Sec. 105, when the velocity is less than 130 m/s the error arising owing to the assumption that the air is incompressible does not exceed 2%.

the tube lies. This pressure difference can easily be computed if we assume that the cross section of the stream tube is a rectangle with sides h and a; then $S = ah$. By the second law of dynamics, we can write

$$(p_1 - p_2)\, a\, ds = \rho ah \frac{v^2}{R} ds \tag{107.1}$$

(where $p_1 - p_2$ is the pressure drop in the direction towards the centre of curvature).

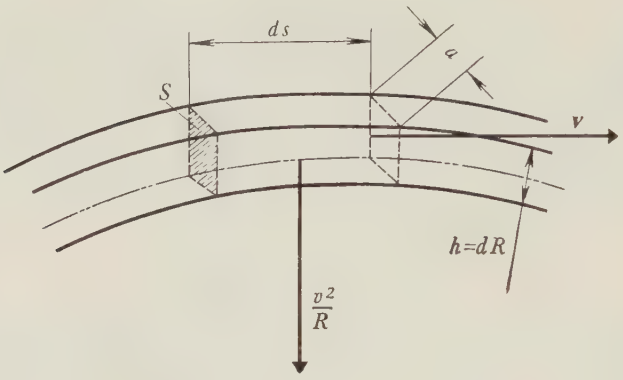


FIG. 295

On cancelling by $ah\, ds$ we obtain

$$\frac{p_1 - p_2}{h} = \frac{\rho v^2}{R} \tag{107.2}$$

If the stream tube is sufficiently thin the ratio $\frac{p_1 - p_2}{h}$ can be replaced by $\frac{\partial p}{\partial R}$ and formula (107.2) can be rewritten as

$$\frac{\partial p}{\partial R} = \frac{\rho v^2}{R} \tag{107.3}$$

This equality means that when a stream tube is curved the pressure varies across its section and drops in the direction towards the centre of curvature of the axis of the stream tube $\left(\text{the pressure drop is characterized by } \frac{\partial p}{\partial R}\right)$.

Let us come back to Fig. 293. It is readily seen that at the point D the pressure is higher than at the point E and, conversely, at M the pressure is lower than at N. Hence, knowing the character of the variation of the curvature of the stream lines we can draw some conclusions concerning the variation of the pressure in the direction perpendicular to the stream lines because the particles can only move along curvilinear trajectories when a definite pressure difference is present.

Since the pressure on the surface of a body placed in a fluid or gas flow determines the forces exerted by the flow on the body, the analysis of the variation of pressure across the stream tubes leads to a number of useful conclusions concerning the forces acting on the body.

108. Distribution of Pressure in a Rotating Fluid

Let us consider a cylindrical vessel containing a heavy fluid which is placed on the rotating disk as shown in Fig. 296.

After a short transient period following the beginning of the rotation of the disk with angular velocity ω all the particles of the fluid move in circular trajectories so that the fluid is at rest relative to the walls of the vessel. In such a motion the particles within a thin stream tube move in a circle of radius r and therefore the pressure in the horizontal plane increases in the direction from the axis of rotation. By formula (107.3), the gradient of the pressure along the radius r is equal to*

$$\frac{dp}{dr} = \frac{\rho v^2}{r} \tag{108.1}$$

On replacing v by ωr in (108.1) we obtain

$$\frac{dp}{dr} = \rho\omega^2 r \tag{108.2}$$

The integration of (108.2) with respect to r results in

$$\int_{p_0}^{p_1} dp = \rho\omega^2 \int_0^{r_1} r\, dr$$

whence

$$p_1 - p_0 = \frac{\rho\omega^2}{2} r_1^2 \tag{108.3}$$

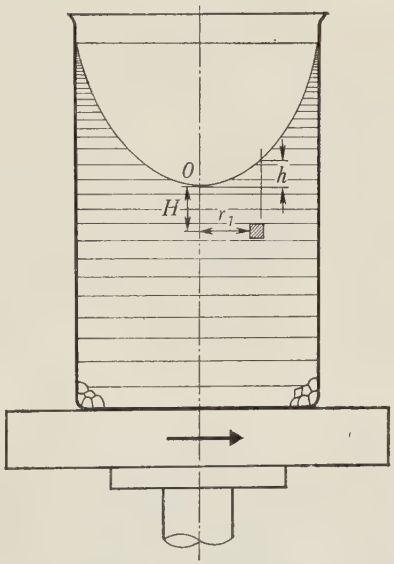


FIG. 296

It follows that the pressure in the horizontal section of the vessel increases proportionally to the square of the distance from the axis of rotation. As is known, the pressure is the same in all the directions at every point within the fluid and therefore the altitude of the fluid column must increase together with the distance from the axis. Indeed, the variation of the pressure in the vertical direction can only be produced by the weight of the fluid. Therefore, for a particle of the fluid to be at rest relative to the vessel it is necessary that the fluid column over the ring-shaped area of radius r_1 should be higher by h than the level of the fluid at the centre of the vessel (r_1 is the distance from the particle to the axis of rotation and h is the difference of the heights). The pressure created by the weight of the fluid at the horizontal level of the lowest point O (see Fig. 296) of the free surface of the fluid is equal to $h\gamma$ and it must be equal to the pressure $\frac{\rho\omega^2}{2} r_1^2$. Hence,

$$h\gamma = \frac{\rho\omega^2}{2} r_1^2$$

whence

$$h = \frac{\omega^2}{2g} r_1^2$$

* Since the motion is stationary the pressure p in the horizontal plane can be regarded as being dependent on r solely.

because $\gamma = \rho g$ where g is the acceleration of gravity. Thus, the altitude of the fluid increases proportionally to the square of the distance from the axis of rotation, which means that the free surface is a paraboloid of revolution; this very shape of the free surface is observed in the experiment.

The shape of the free surface indicates the variation of the pressure along the radius. This variation can be checked experimentally: if we put small pieces of a substance heavier than the fluid in the rotating vessel then in some time they all gather below near the walls of the vessel; if the substance is lighter than the fluid, the floating pieces of the substance gather near the point O.

It is interesting to observe the behaviour of a piece of lead connected by means of a thread with a wax ball; the two connected bodies are put into a rotating vessel containing water (wax is lighter than water). As an exercise, let the reader analyse the results of this experiment. The reader should also answer the following questions:

(1) What is the distribution of pressure in a rotating vessel when it is sealed?

(2) What is the distribution of pressure and the shape of the free surface when the axis of the cylindrical vessel with fluid does not coincide with the axis of rotation of the disk on which it is placed?

In the case under consideration when the particles of the fluid in the rotating vessel move in circular trajectories the constant entering into Bernoulli's equation retains its value only for every very thin stream tube and assumes different values for different stream tubes. For a given thin stream tube we can write the relation

$$p_1 + \frac{\rho v^2}{2} = p_0 + \frac{\rho\omega^2 r^2}{2} + \frac{\rho\omega^2 r^2}{2} = p_0 + \rho\omega^2 r^2 = \text{const}$$

which follows from (102.5) and (108.3) (since the stream tube lies in a horizontal plane it is allowable not to take into account the term involving h). The pressure p_0 at the points lying on the axis of rotation depends on the depth H solely and is equal to γH (see Fig. 296). Thus, the constant in Bernoulli's equation varies and is dependent both on the depth and on the distance from the axis of rotation.

109. Momentum of Fluid or Gas

There are many cases when the analysis of a compound motion of a fluid or a gas admits of the application of the law of variation of the momentum. To determine the forces acting within a fluid flow we separate (mentally) a volume in the spatial region occupied by the fluid and apply the law of variation of the momentum to the amount of fluid passing through that volume. For a stationary fluid flow the law of variation of the momentum can be stated as follows: *the sum of the external forces acting on the particles of the fluid within the given volume is equal to the variation of the momentum of the fluid in the volume during unit time.*

Here by the external forces are meant the forces applied to all the particles of the fluid lying within the given volume (very often such forces simply reduce to the forces of gravity) and also the forces of pressure acting on the surface bounding the separated volume. In the general case of a stationary flow, in order to determine the variation of the momentum during one second we should begin with the determination of the variation of the momentum of the fluid passing through a very small part of the surface.

For a small surface element dS represented by the vector $dS = n \cdot dS$ (see Fig. 297) for which the velocity can be considered constant the amount of fluid passing through the area dS in one second (that is the mass flux) is equal to

$$\rho v \, dS = \rho v \, dS \cos \alpha \qquad (109.1)$$

where α is the angle between the outer normal to the surface and the velocity. As we have mentioned, the mass flux is the amount of fluid passing through the given area during one second; it is a scalar quantity. The flux is considered positive when the fluid flows out of the given volume through the given area and negative when the fluid flows into the volume.

The product of the mass flux (formula (109.1)) by the velocity vector v is equal to

$$\rho v \, dS \cos \alpha \cdot v \qquad (109.2)$$

This product is a vector quantity; it is equal to the momentum of the amount of the fluid leaving the given volume through the given surface area during one second.

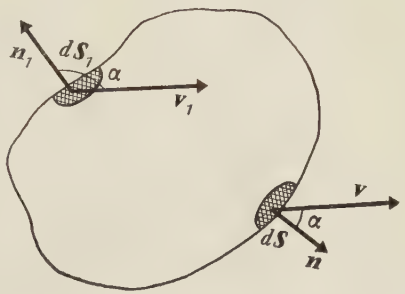


FIG. 297

For the part of the surface bounding the given volume through which the fluid flows into the volume the same expression (109.2) determines the momentum of the amount of the fluid flowing into the volume during one second (see Fig. 297). On adding together (or integrating) the variations of the momenta over all the small areas constituting the whole surface bounding the separated volume of the fluid we find the total increment of the momentum of the fluid in the given volume during one second. The total momentum thus found is a vector quantity; it is equal to the sum of the external forces (which are the gravity forces and the forces of pressure applied to the surface of the given volume) acting on the volume.

It should be noted that the general law of variation of the momentum is applicable to both fluids and gases with the only distinction that when a motion of a gas is considered then in the computation of the momentum it is necessary to take into account the dependence of the density of the gas on the pressure.

110. Reaction Force of Flowing Fluid

When a fluid flows along a pipe it acts on it with a reaction force. One can observe the action of such forces when the water starts running through a flexible hose (Fig. 298). If the hose is bent before the water starts running, the reaction forces sharply straighten the hose. Since the hose is flexible it cannot produce the force of a sufficiently large magnitude needed for changing the direction of the momentum of the water flowing through it and therefore the hose straightens so that the direction of the momentum of the water flowing out of the nozzle of the hose coincides with that of the momentum of the water entering the hose. In this section we shall discuss some examples in which the law of the variation of momentum is applied to the determination of the reaction force of a flowing fluid.

Let us determine the reaction force of the water running through the tap shown schematically in Fig. 299. Let us consider the volume of the fluid marked

by dotted line in Fig. 299. The surface bounding this volume consists of the inner side of the tap and of the two cross sections A and B. Since the cross-section area of the tap is constant the absolute value v of the velocity v is the same for every section. Therefore the momentum of the amount of fluid flowing into the given volume every second is equal to

$$K_A = \rho S v_A v \tag{110.1}$$

where ρ is the density of the water, v_A is the velocity in the section A and S is the cross-section area of the tap. The vector K_A is directed perpendicularly to

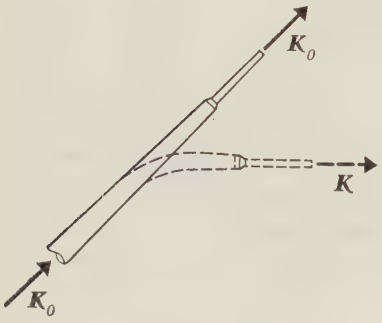


FIG. 298

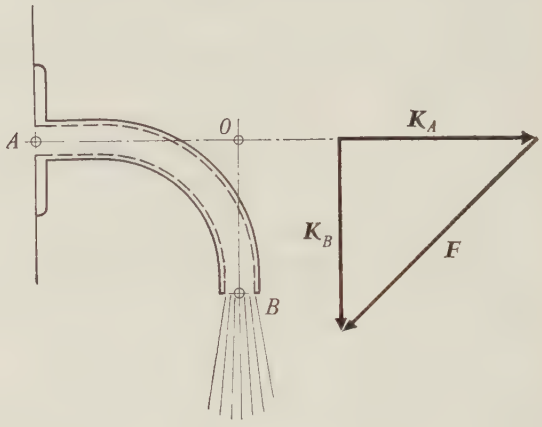


FIG. 299

the section A. The momentum K_B of the amount of fluid leaving every second the given volume through the cross section B has the same magnitude as the vector K_A and forms an angle of 90° with the latter. Therefore, the vector representing the increment of the momentum is directed as shown in Fig. 299 and is equal to the corresponding force:

$$K_B - K_A = F \tag{110.2}$$

The magnitude of the force F is

$$F = \sqrt{2}\, K = \sqrt{2}\rho S v^2$$

Now let us consider the forces acting on the volume of water under consideration. The action of the gravity force can be neglected* and therefore it only remains to consider the forces of pressure applied to the surface bounding the volume. The forces of pressure in the inlet and outlet sections A and B are equal (on condition that we neglect the water viscosity which is in fact very small). Indeed, Bernoulli's equation implies that when the magnitude of the velocity is constant along a stream tube the pressure is also constant. The pressure in the outlet section A is equal to the atmospheric pressure. The forces of atmos-

* The force of gravity acting on the fluid within the volume is practically very small in comparison with the sought-for reaction force. It is evident that, when necessary, this force can also be readily taken into account.

pheric pressure acting on the inlet and outlet sections of the tap are balanced by the pressure exerted on the tap from the outside and therefore the resultant of these forces is zero (the situation is similar to the case of an empty tap when the resultant of the forces of atmospheric pressure acting on the tap is equal to zero).

Now we should only take into account the action of the forces of pressure with which the tap acts on the remaining part of the surface bounding the given volume. The sum of these forces equal to the resultant of all the forces of pressure exerted by the tap on the given volume of water coincides with $\boldsymbol{F}$ (that is

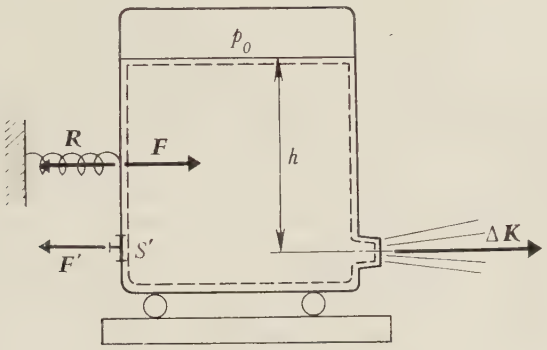


FIG. 300

with the per-second variation of the momentum whose magnitude and direction are specified by (110.2)). It follows that the reaction force of the outflowing fluid applied to the tap is equal to the vector $\boldsymbol{F}$ in its magnitude, opposite in the direction, and passes through the point O (see Fig. 299) at which the lines of the momenta of the water in the inlet and outlet sections meet.

Now let us determine the reaction force of the fluid flowing out of the vessel shown in Fig. 300. By Bernoulli's equation (102.5), the outlet velocity of the fluid is

$$v = \sqrt{\frac{2}{\rho}(p_0 + \gamma h)} \tag{110.3}$$

where p_0 is the pressure above the free surface of the fluid in the vessel* and h is the height of the fluid column over the opening. Let us consider the volume of the fluid shown in dotted line in Fig. 300. To simplify the calculations we shall assume that the vessel has a rectangular horizontal cross section. Then it is sufficient to consider the forces of pressure and the variations of the momentum of the fluid only on the surfaces normal to the jet; the variation of the momentum of the fluid in all the other directions is equal to zero. Therefore, we shall only consider the forces and the variations of the momentum in the direction of the velocity of the jet.

Denoting by S_0 the cross-section area of the jet we can write the per-second increment of the momentum in the direction of the jet in the form

$$\Delta K = \rho S_0 v^2 = Qv \tag{110.4}$$

* More precisely, p_0 is the difference between the pressure under the free surface of the fluid in the vessel and the atmospheric pressure.

where Q is the mass flux during one second. The vessel acts on the fluid with the force $\boldsymbol{F} = \Delta \boldsymbol{K}$ and the reaction force $\boldsymbol{R}$ of the fluid exerted on the vessel is equal to $\boldsymbol{F}$ in its magnitude and opposite in the direction. The vector $\boldsymbol{R}$ represents the reactive force; it turns to be exactly equal to the reactive force specified by Meshchersky's equation (see Sec. 27). The reactive force of the discharging jet can be determined experimentally by measuring the force acting on the vessel with the aid of a dynamometer as is shown schematically in Fig. 300. In principle, the thrust force of a reaction or rocket engine can be measured in the same way.

Let us discuss the way in which the reactive force is transmitted to the walls of the vessel. The reactive force appears owing to the difference in the forces of pressure acting on the walls of the vessel. The pressure on the back wall (see Fig. 300) can be considered equal to $p_0 + \gamma h$ (where γh is the hydrostatic pressure, γ is the specific weight of the fluid, and h is the depth) because the velocity of the flow near that wall is small (more precisely, it is very small in comparison with the velocity of the particles in the discharging jet). The pressure on the front wall (that is on the wall in which the opening is made through which the fluid outflows; see Fig. 300) is not equal to the pressure on the back wall. This can be proved as follows. Suppose that the pressures on both walls are equal. Owing to the forces of pressure acting on the area $S' = S_0$ (lying on the back wall) which is opposite to the area S_0 of the opening in the front wall, the resultant force of pressure acting on the front wall must be less than the one acting on the back wall. Thus, the force exerted by the fluid on the back wall exceeds the force exerted on the front wall by a magnitude F'. Let us determine the force F' (that is the force acting on the area S') under the assumption that the dimensions of the area S' are very small in comparison with h. It is evident that

$$F' = S_0 (p_0 + \gamma h) \tag{110.5}$$

Taking into account formula (110.3) we can write

$$F' = \frac{1}{2} \rho v^2 S_0$$

Comparing the last expressions with (110.4) we see that the force F' is twice as small as the reactive force R. This discrepancy shows that the assumption that the pressure is the same on the opposite walls of the vessel is incorrect. Consequently, it is not allowable to neglect the increase of the velocity of the particles in the fluid flow adjoining the front wall near the opening. Because of this increase the pressure in a region near the opening reduces. For the complete description of the distribution of pressure on the walls of the vessel it is necessary to analyse thoroughly the whole motion of the fluid within the vessel, which is a very complicated problem. However, it turns out that the application of the law of variation of momentum makes it possible to determine the reactive force in a rather simple way without an exhaustive analysis of the motion.

Our argument shows that the pressure drop near the opening in the front wall results in a force equal to half the reactive force R. It should be noted that this inference only holds when the minimum cross-section area of the discharging jet is equal to the area S_0 of the opening, which is by far not always the case.

When the passage from the vertical wall to the nozzle is smooth all the stream tubes in the discharging jet are parallel to the axis of the jet (see Fig. 301*a*) and the minimum cross-section area of the jet is equal to S_0. When the passage from the wall to the nozzle is not smooth (Fig. 301*b*) the jet becomes narrower and its minimum cross-section area is less than S_0. This phenomenon can easily

be explained. When the passage is not smooth (see Fig. 301*b*) the particles of the fluid approaching the opening along the vertical wall tend to the centre of the jet by inertia, and it is the pressure produced by the particles moving nearer to the centre of the jet that straightens the stream lines adjoining the walls at a certain distance from the opening. This results in a jet whose minimum cross section is less than the area of the opening. The ratio of the minimum cross-section area of the jet to the area of the opening depends on the shape of the edge of the opening and is determined experimentally.

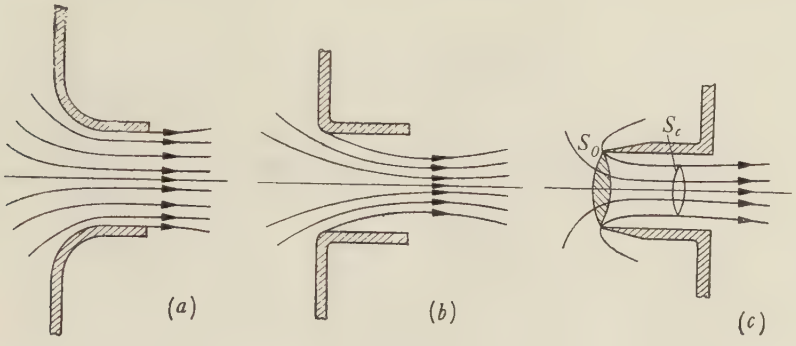


FIG. 301

When the edge of the opening is sharp the minimum cross-section area of the jet can be considerably smaller than the area of the opening but it always exceeds half the area of the opening. In case the fluid outflows from the vessel through a pipe built in the wall of the vessel and having a sharp edge directed inside the vessel (see Fig. 301*c*) the minimum cross-section area of the jet is exactly equal to half the area of the opening. For, in this case, the velocity of the flow along the vertical wall of the vessel into which the pipe is fitted is very small and can be neglected because the vertical wall of the vessel is far enough from the opening. Then the pressure is the same on the opposite parts of the walls and, by (110.5), the reactive force must be equal to

$$F' = S_0 (p_0 + \gamma h) \tag{110.6}$$

where S_0 is the area of the opening in the pipe. The law of variation of momentum implies that the reactive force is equal to

$$F' = S_c \rho v^2 = S_c \cdot 2 (p_0 + \gamma h) \tag{110.7}$$

where S_c is the minimum cross-section area of the jet (see Fig. 301*c*). The comparison of (110.6) and (110.7) shows that

$$2S_c = S_0 \tag{110.8}$$

whence it follows that the ratio of the minimum cross-section area of the jet to the area of the opening is equal to 1/2. This relationship can readily be confirmed experimentally.

The application of the law of variation of momentum is particularly useful in those cases when it is easy to determine the direction and the velocity of the jet. For instance, when the jet is incident on the blade of a turbine of the form

shown in Fig. 302*b* (the blade has two troughs) the force of pressure acting on the blade is twice as great as the force exerted by the jet incident on the plane blade shown in Fig. 302*a* (provided that the velocities and the cross-section areas of the two jets are equal). Of course, what has been said applies only when it is allowable to assume that the velocity of the jet retains its magnitude after the impact against the blade with the troughs and when the velocities of both blades are equal and are sufficiently small in comparison with the velocity of motion of the particles of the fluid in the jet.

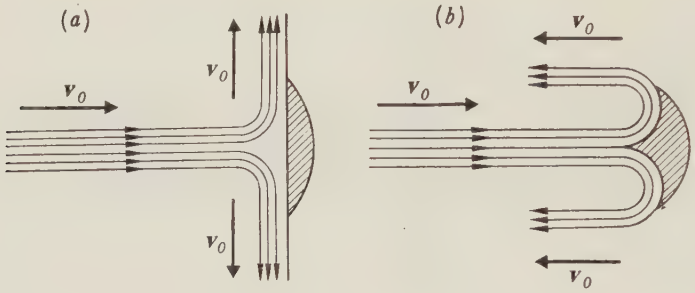


FIG. 302

111. Flow of a Viscous Fluid in a Pipe

When a motion of a fluid is investigated there are many cases in which it is allowable to neglect viscous forces and to consider the motion as if there were no viscosity. This approach proves useful, firstly, because the general methods for the thorough analysis of a flow of a viscous fluid have not yet been elaborated and, secondly, because there are a large number of important cases in which the results of the experiments with a viscous fluid coincide, with an admissible accuracy, with the results of the theoretical analysis of the flow of a perfect (non-viscous) fluid. On the other hand, there are of course many situations in which the neglect of viscosity leads to essential errors.

As is known, the viscous forces are proportional to the variation of the velocity of the flow in the direction perpendicular to the velocity and therefore the action of these forces is particularly noticeable in the places where the variation of the velocity is large. When a viscous fluid flows around a solid body the particles of the fluid adjoining the body stick to it and have zero velocity relative to the body. Therefore the increase of the velocity of the flow from zero to non-zero value takes place in the vicinity of the surface of the body. In this region the viscosity of the fluid plays an essential role. On the other hand, in the parts of the flow lying far from the body the variations of the velocity are comparatively small and the influence of the viscosity is negligible.

The layer of the fluid surrounding the body in which the magnitude of the velocity grows and for which the influence of the viscosity is essential is called a *boundary layer*. In some cases this layer is very thin and its influence can be neglected; then the flow of the viscous fluid is quite similar to the flow of a perfect (non-viscous) fluid around the same body. In some other cases the boundary layer is not thin and then it is not allowable to neglect the viscosity. For instance, when a viscous fluid flows in a narrow pipe the boundary layer can fill

the whole volume of the fluid; in this case the analysis of the flow must involve the viscous forces.

Let us consider the experiment whose scheme is shown in Fig. 303. In this experiment a viscous fluid flows in a horizontal pipe with a constant cross section and the distribution of pressure along the pipe is measured with the aid of manometric tubes. When the fluid is sufficiently viscous (for instance, such is glycerine) or when the pipe is sufficiently thin (or both), the pressure drop is uniform along the pipe. This conclusion follows from the experimental fact that

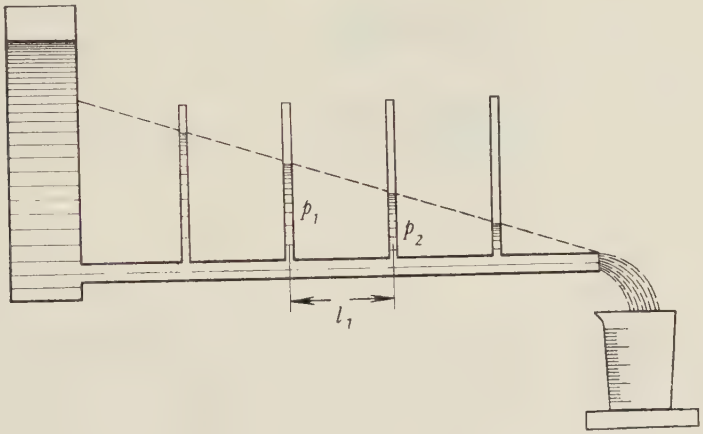


FIG. 303

the levels of the fluid in the equidistant manometric tubes lie on an inclined straight line (see Fig. 303).

If the fluid were non-viscous the levels would be the same in all the tubes and the pressure would be constant along the pipe. Indeed, the fluid can be considered almost completely incompressible and therefore the velocity of the flow is the same in all the cross sections of the pipe because the cross-section area is constant. Consequently, when the fluid is non-viscous Bernoulli's equation implies that in the case under consideration the pressure must also be constant along the pipe.

For a viscous fluid, besides the forces of pressure, every particle of the fluid in the stationary flow along the pipe with a constant velocity is acted upon by viscous forces, which produces a pressure drop along the pipe. Since the flow along the pipe is rectilinear and the velocities of all the particles are directed along the axis of the pipe, the viscous forces act only in the direction of that axis. The pressure drop in the pipe is balanced by the viscous forces and therefore the velocity of the fluid remains constant along the pipe.

Let us dwell in more detail on a stationary flow of a viscous fluid along a rectilinear horizontal pipe of constant circular cross section. In this case the pressure is the same in every cross section. For, if otherwise, the stream tubes would be curved and there would appear fluid streams across the pipe. All the particles of the fluid adjoining the walls of the pipe stick to the walls and their velocity is equal to zero. By the symmetry, all the particles in a thin ring-shaped layer of the fluid (with the axis coinciding with that of the pipe) must have the

same velocity. The whole fluid can be divided into sufficiently thin concentric ring-shaped layers of this kind and the velocity is the same within every layer. Therefore, the magnitude of the velocity of the flow must depend solely on the distance r between the particle of the fluid and the axis of the pipe.

Let us mentally separate from the flowing fluid a cylinder of radius r and length dl (see Fig. 304). Since the fluid is in a uniform motion the sum of all the forces

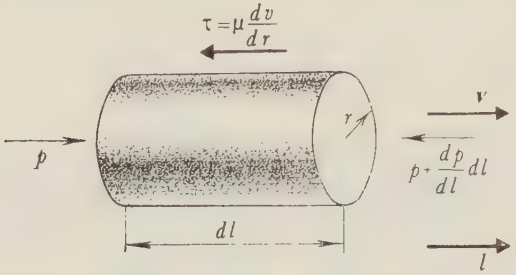


FIG. 304

applied to it is equal to zero. The difference between the forces of pressure acting on the opposite bases of the cylinder is equal to

$$\left[p - \left(p + \frac{dp}{dl} dl\right)\right] \pi r^2 = -\frac{dp}{dl} dl \, \pi r^2$$

It follows that this difference must be balanced by the viscous forces applied to the lateral surface of the cylinder. The resultant of the viscous forces equals the area $2\pi r \, dl$ of the lateral surface times the tangential stress τ produced by the viscous forces. Consequently, the condition that the sum of all the external forces acting on the cylinder is equal to zero can be written in the form

$$-\frac{dp}{dl} dl \, \pi r^2 - \tau \cdot 2\pi r \, dl = 0$$

whence

$$\frac{dp}{dl} r = -2\tau \qquad (111.1)$$

By Newton's law (see formula (39.1)), the tangential stress produced by the viscous forces is proportional to the derivative of the velocity in the direction perpendicular to it, that is in the direction of the radius r:

$$\tau = -\mu \frac{dv}{dr} \qquad (111.2)$$

where μ is the viscosity coefficient of the fluid (the minus sign on the right-hand side of (111.2) is placed because the velocity reduces as the radius increases). On substituting (111.2) in equality (111.1) we obtain

$$\frac{dp}{dl} r = 2\mu \frac{dv}{dr} \qquad (111.3)$$

The gradient of the pressure $\frac{dp}{dl}$ along the axis of the pipe is independent of the radius because the pressure p is constant throughout any cross section. There-

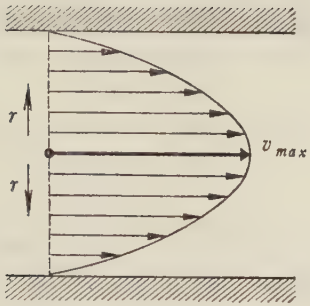


FIG. 305

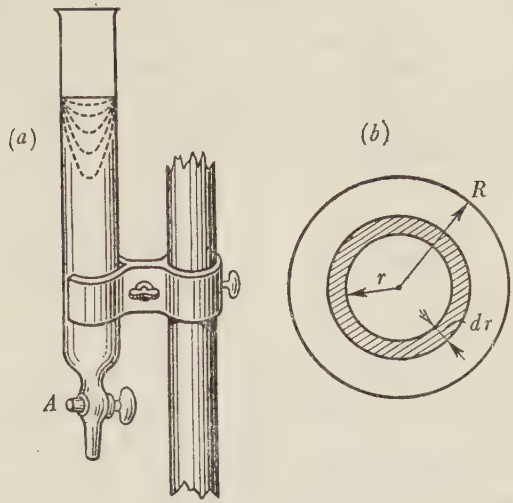


FIG. 306

fore, equation (111.3) enables us to find the dependence of the velocity of the flow on the radius by integrating the equation with respect to the radius and taking into account that for the velocity of the particles of the fluid adjoining the walls of the pipe there must be $v(R) = 0$. The integration results in

$$\frac{dp}{dl}\int_R^r r\,dr = 2\mu\int_0^v dv \tag{111.4}$$

where R is the radius of the pipe. On performing the calculations we obtain

$$\frac{1}{2}\frac{dp}{dl}(r^2 - R^2) = 2\mu v$$

whence

$$\frac{1}{4\mu}\left(-\frac{dp}{dl}\right)(R^2 - r^2) = v \tag{111.5}$$

Since the pressure drop in the direction of the velocity is uniform, the expression $-\frac{dp}{dl}$ is positive and constant. The velocity achieves its maximum value on the axis of the pipe and the distribution of the magnitude of the velocity over the diameter of the pipe obeys a parabolic law (Fig. 305). The maximum velocity is given by the formula

$$v_{\max} = \frac{R^2}{4\mu}\left(-\frac{dp}{dl}\right)$$

The distribution of the velocity of a flow of a viscous fluid in a pipe can be demonstrated by a dye experiment in which the interface between two layers of the fluid is observed. The scheme of the experiment is shown in Fig. 306*a*. The vertical glass pipe is filled with a dyed viscous fluid over which an amount of the same fluid without the dye is carefully poured. In the state of rest the interface is horizontal. After the tap below the pipe is opened the viscous fluid

starts to move slowly and to outflow from the pipe which causes the interface to change its shape and elongate along the axis of the pipe (see Fig. 306*a*).

Knowing the distribution of the velocity we can readily compute the volume flow rate Q (that is the volume of the fluid passing through the cross section of the pipe during one second). The volume of fluid passing every second through the ring of radius r and width dr (the area of the ring is $2\pi r\,dr$; see Fig. 306*b*) is equal to $dQ = v2\pi r\,dr$. Consequently, for the whole cross section we obtain

$$Q = \int dQ = 2\pi \int_0^R vr\,dr = \frac{2\pi}{4\mu}\left(-\frac{dp}{dl}\right)\int_0^R (R^2 - r^2)\,r\,dr = \frac{\pi R^4}{8\mu}\left(-\frac{dp}{dl}\right) \qquad (111.6)$$

(in the derivation of (111.6) we have taken into account formula (111.5)).

Relation (111.6) can be used for the construction of a device for the measurement of the viscosity coefficient μ of a fluid. The scheme of the device is shown in Fig. 303. The value of the gradient $\frac{dp}{dl}$ can be found from the results of the measurement of the pressure at different points of the pipe. Since the pressure drop is proportional to the length reckoned along the axis of the pipe we have

$$Q = \frac{\pi R^4}{8\mu}\,\frac{p_1 - p_t}{l} \qquad (111.7)$$

where p_1 is the initial pressure (at the beginning of the pipe) and p_t is the terminal pressure (at the end of the pipe; the length of the pipe is denoted l). The volume flow rate Q can be determined directly by measuring the amount of fluid passing through the pipe during a definite time interval. Knowing the radius R of the pipe we can calculate the viscosity coefficient μ of the fluid after the quantities p_1, p_t, and Q have been measured.

A flow of a fluid in a cylindrical pipe for which the velocities of the particles are everywhere directed along the axis of the pipe is an example of a *laminar* flow. This type of a flow of a viscous fluid is observed when the velocity of the flow is small. As the velocity of the flow increases and the pressure drop between the ends of the pipe grows there arises a new phenomenon in which the flow is of an essentially different character: instead of a quiet laminar motion there arises a *turbulent* (*eddying*) flow.

The appearance of a turbulent flow can easily be observed in an experiment with a horizontal glass pipe through which the water from the vessel flows out (see Fig. 307). A thin jet of dyed water is added to the water stream as shown in Fig. 307*a*. When the velocity of the water in the pipe is not high the flow is laminar and the dyed jet is seen as a thin almost straight line going along the axis of the pipe (Fig. 307*b*). As the velocity of the flow is increased gradually, for a certain value of the velocity there suddenly arises a turbulent motion of the fluid and the dyed jet takes the form of a wide band with uneven edges (see Fig. 307*c*).

For a stationary turbulent motion the velocity of the fluid particles at a given place does not remain constant either in its magnitude or in direction. Both the magnitude and the direction of the velocity fluctuate irregularly. At the same time the mean (average) value of the velocity is a definite vector quantity directed along the axis of the pipe.

The phenomenon of the appearance of turbulence will be discussed in more detail in Sec. 113. Here we confine ourselves to the remark that the distribution

of the mean velocity of a turbulent flow over the diameter of the pipe (see Fig. 308) is completely different from the one observed in a laminar motion (Fig. 305). In a turbulent motion the mean velocity remains constant over almost the whole cross section of the pipe and it drops up to the zero value only near the walls of the pipe. The boundary layer near the walls of the pipe occupies a comparatively small part of the flow and in a wide region surrounding the cen-

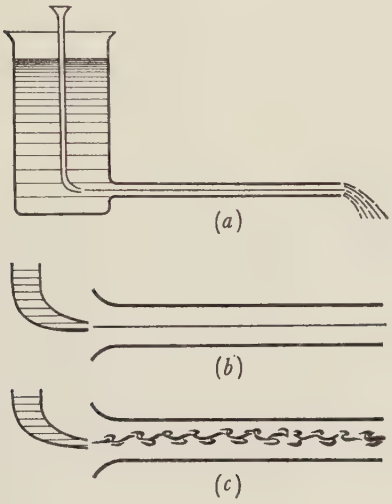


FIG. 307

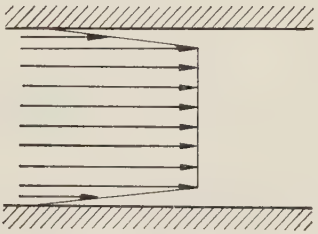


FIG. 308

tre of the flow the velocity field is almost homogeneous and is very similar to the one observed in a flow of a non-viscous fluid in a pipe. For a laminar flow (see Fig. 305) we can say that there is no boundary layer at all (since owing to the viscous forces the velocity field varies in all the parts of the pipe including the central layer and the layers adjoining the walls) or, alternatively, we can say that the boundary layer occupies the whole flow of the fluid.

CHAPTER 13

Action of a Fluid or Gas Flow on a Body

112. Drag Force

The forces with which a flow acts on a solid body placed into it can always be reduced to one resultant force and one resultant moment. The resultant force can always be resolved into two mutually perpendicular components one of which is directed along the flow and the other perpendicularly to it. The component directed along the flow is the *drag force* (or, simply, the *drag*) caused by the resistance of the fluid or gas. For a symmetric body placed so that its axis of symmetry goes along the flow the force exerted on the body by the flow is obviously directed along the latter, that is the whole force reduces to the drag.

Experiments show that the drag force depends on the shape and on the dimensions of the body, on the velocity of the flow and on the physical properties of the fluid. It turns out that for geometrically similar bodies the drag force is proportional to the area of the section of the body perpendicular to the direction of the flow (i.e. perpendicular to the velocity v of the flow), to the dynamic head $\rho v^2/2$ and to a coefficient C_x called the *coefficient of drag* for the bodies of the given shape. In the general case the coefficient of drag C_x does not remain constant and depends on the *Reynolds number* $\mathrm{Re} = vl\rho/\mu$ where l is the characteristic length scale for the given body, v is the velocity of the flow, ρ is the density of the fluid and μ is the viscosity coefficient of the fluid. The physical nature of this dependence will be discussed in the next section.

In Fig. 309 we see the diagram representing the relationship between the coefficient of drag and the Reynolds number Re for a ball. Since the Reynolds number is proportional to the velocity of the flow, the drag force is also proportional to the velocity for small values of the number Re (approximately, for the values not exceeding 100). For the values of Re ranging, approximately, from 100 to 10^5 the coefficient of drag remains almost constant, which means that in this transitional region the drag force is proportional to the square of the velocity. In the vicinity of $\mathrm{Re} \approx 1.5 \times 10^5$ the coefficient C_x sharply changes its value and then remains almost invariable as the number Re increases further. For a ball the dependence of the magnitude F of the drag force on the velocity is represented by the diagram in Fig. 310. The region a in the figure corresponds to the linear dependence; the region b is the first region of quadratic dependence to which there corresponds the coefficient of drag $C_x \approx 0.4$; the region c is the second region of quadratic dependence which is characterized by the values of C_x ranging from 0.1 to 0.2.

Before proceeding to the analysis of the characteristic features of a flow of a viscous fluid around a body and to the investigation of the phenomena causing the changes of the drag depending on the velocity we make some remarks con-

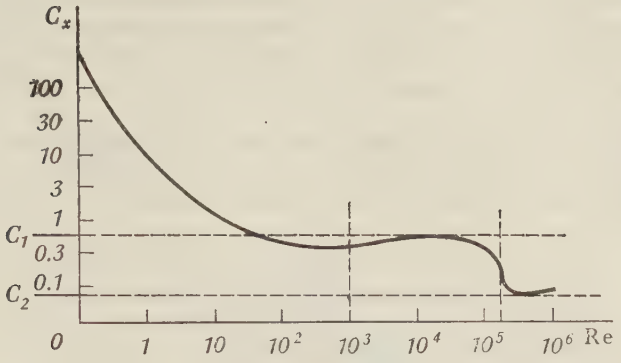


FIG. 309

cerning the drag force exerted by a perfect (ideal) fluid (that is by a non-viscous fluid).

When the velocity of the flow of a perfect fluid around a body with a smooth surface is small the flow-around is smooth; for instance, the stream tubes of a flow of an ideal fluid around a ball are entirely symmetric relative to the latter in the case of a smooth flow-around (see Fig. 311). Since we assume that there are no viscous forces the surface of the ball is only acted upon by forces of static pressure. Because of the symmetry of the flow, the cross section of a stream tube

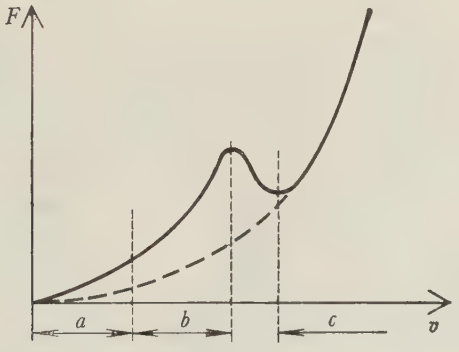


FIG. 310

at any point A in front of the ball coincides with the cross section at the point B which lies behind the ball and is symmetric to A. Consequently, the velocity of the flow is the same at these points and so is the pressure. Therefore, the resultant force acting on a ball placed in a homogeneous flow of a perfect fluid is equal to zero. A thorough theoretical analysis of a smooth flow-around for any body placed in the flow shows that in the case when there is no flow separation the drag force is equal to zero (this is known as *D'Alembert's paradox*).

However, for a body placed in a fluid flow a smooth flow-around without flow separation is only observed when the velocity of the flow is very small. Such a

flow-around can easily be observed with the aid of the device shown in Fig. 283; on placing a small smooth cylinder into the flow we can see the stream lines diverging in front of the cylinder and converging behind it. This case is shown in Fig. 312*a* where the stream lines of a smooth flow-around are depicted for the cylinder.

As the velocity of the flow is increased and achieves a certain value the character of the flow-around essentially changes (see Fig. 312*b*). The stream lines no longer converge behind the cylinder; there arises a stalled flow-around (with

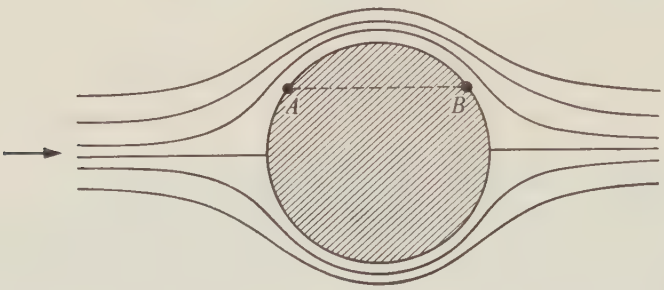


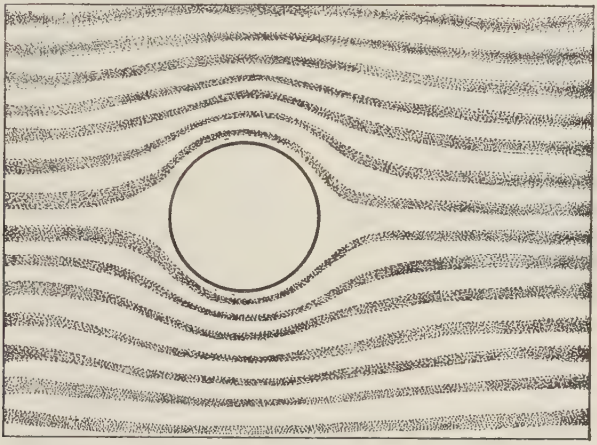
FIG. 311

separation of the flow), and behind the body there appears a region filled with eddies (vortices). Fig. 312*b* shows the stream lines of the stalled flow around the cylinder. In the region with vortices behind the cylinder it is difficult to observe distinct stream lines. The flow in this region is highly irregular; this part of the flow is clearly separated from the region of a regular flow where the stream lines are easily observable. In this case the distribution of pressure in front of the body and behind it is no longer symmetric. In front of the body the distribution of pressure is approximately similar to the one in the smooth flow-around, that is the pressure in the vicinity of the stagnation point and at that point itself exceeds the static pressure by, approximately, $\rho v^2/2$ (that is by the magnitude of the dynamic head). On separating from the body the stream lines become more rectilinear behind the body. In the region with vortices behind the body the pressure is less than in front of the body and is approximately equal to the static pressure in the unperturbed flow far from the body. Consequently, even in the case of a perfect fluid the appearance of the phenomenon of flow separation results in a force directed backwards. It is the change in the distribution of pressure behind the body that accounts for the drag force exerted on the body by a stalled flow.

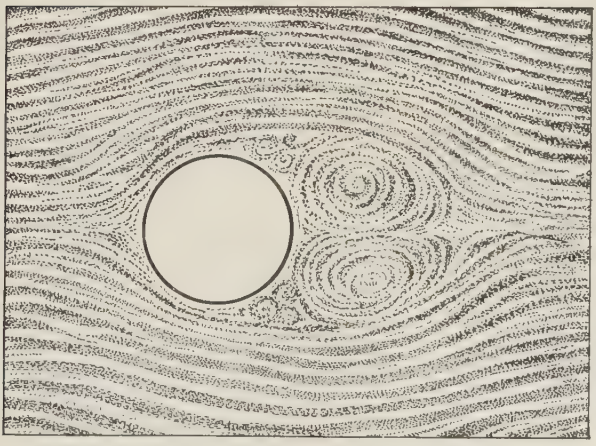
In the general case when a perfect fluid flows around an arbitrary body the flow separation causes a redistribution of pressure over the surface of the body such that the resultant force is different from zero and the flow of the nonviscous fluid exerts a definite drag force on the body.

When a body is placed in a flow of a viscous fluid the latter exerts on the body tangential viscous forces whose resultant is directed along the flow. Experiments show (see Fig. 312*a*) that the viscous fluid exerts a drag force on the body even when there is no flow separation and even when the flow is symmetric with respect to the body. This drag force is due to the tangential viscous forces whose distribution in the case of a cylinder is shown schematically in Fig. 313.

The character of the motion essentially changes when the flow separation arises in a viscous fluid. In this case, besides the tangential forces acting owing to the viscosity, an important role is played by the redistribution of the forces of pressure caused by the flow separation, which results in an additional resistance force acting on the body in the direction of the flow. When the velocity of the flow is large (more precisely, when the Reynolds number Re is large) the forces generated by the redistribution of pressure near the surface of the body play a predominant role.



(*a*)



(*b*)

FIG. 312

Another phenomenon accompanying the flow separation is the generation of discrete vortices behind the body (in the region filled with eddies) near the part of the surface of the body where the flow separates from it. These vortices appear regularly or irregularly and fill the region with eddies. It is clear that these

vortices generated in the vicinity of the body and carried away by the flow produce fluctuations of the flow and, consequently, fluctuations of the pressure near the surface of the body. These fluctuations can be regular or irregular depending on whether the generation of the vortices is periodic. It is rather difficult to measure these fluctuations of the pressure or to calculate them theoretically. However, it is clear that the time average of the fluctuations of the pressure gives an additional component of the drag force (this part of the drag force is called the *eddy-making resistance*).

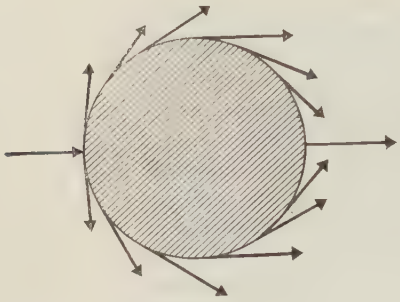


FIG. 313

The appearance of this additional drag force can be confirmed by the following argument. Suppose that a body moves uniformly in a fluid so that there appear vortices behind it (the *vortex wake*). After the body has passed, the fluid in the wake is in a rotary motion and hence possesses some kinetic energy. The only possible source of this energy is the work of a force applied to the body as it moves uniformly. By the law of conservation of energy, the kinetic energy of the eddying motion in the vortex wake must be equal to the work of the drag force.

Thus, we can indicate the following three factors causing the drag force acting on a body placed in a flow of a viscous fluid: (*a*) the viscosity of the fluid producing tangential viscous forces, (*b*) the redistribution of pressure because of the

Shape of the body and direction of flow		c_x	Re
→ ▯	Disk	1.11	0-5×10⁶
→ ◗	Hemispherical cup	1.35-1.40	
→ ◗		0.30-0.40	
→ ○	Ball	0.4	2×10³-2.5×10⁵
→ ○		0.1-0.2	3×10⁵-7×10⁶
→ ◖	Drop-shaped solid of revolution	0.045	1.5×10⁵-6×10⁶
→ ◖		0.1	

flow separation behind the body, and (*c*) the fluctuations of the pressure appearing because of the generation of vortices behind the body which are carried away by the flow.

In the table above are given the mean values of the coefficients of drag for the bodies of various shape.

As is seen from the table, the shape of the rear part of the body plays a very important role: among the bodies having the same cross-section area the one of a drop-shaped form with its blunt nose facing the flow and with a smooth sharpening of its rear part undergoes the minimum resistance. The smooth sharpening makes the stream lines of the flow around the body converge behind it; this reduces the region of the separation of the flow or even prevents the flow from the separation. The flow around the body is smooth over most part of its surface and resembles the one observed in the case of a perfect fluid. The pressure in the region behind the body where the stream lines converge becomes greater and the drag force reduces.

Conversely, when the sharpening of a drop-shaped body faces the flow the drag force increases because in this case almost the whole rear part of the body is in the region where the flow separates from the body, the stream lines do not converge smoothly behind the body and the resistance grows. That is why the various parts of an airplane are designed so that they have a stream-lined form, which reduces the air resistance exerted on the airplane.

113. Dynamical Similarity of Bodies

As it follows from the foregoing sections, the phenomena in a flow of a viscous fluid around a body are rather complex and in most cases it is not yet possible to determine the drag force theoretically. That is why the forces exerted on a body by a flow are usually determined experimentally. However, there are many cases of practical importance when a direct measurement of the forces acting on a body (for instance, on an airplane, on a ship, and the like) is very difficult or very expensive or even practically impossible. It is therefore expedient to try to measure such forces for a smaller body which is geometrically similar to the body in question and then, proceeding from the data obtained for the small model, to draw the necessary conclusions concerning the action of the flow on the real body. It turns out that for this approach to be applicable it is necessary that, besides the geometrical similarity of the bodies, the *dynamical similarity* of the flows should take place.

A simple geometrical similarity of the bodies is obviously insufficient because, depending on the velocity of the flow and on the properties of the fluid, the character of the flow around the bodies can be quite different. Therefore, when speaking of a dynamical similarity, we mean not only the similarity of the shapes of the bodies and of the stream lines of the flows but also the similarity of the relationships between the forces of different physical nature for every part of the flow both for the model and for the body under consideration.

Every particle in a flow of a viscous fluid is acted upon by two forces, namely by the force of the pressure and by the viscous force (provided that the force of gravity of the particle is negligibly small). The sum of these forces equals the mass of the body times its acceleration. For the sake of brevity, we shall temporarily refer to the product of the mass by the acceleration taken with opposite sign as the "inertia force".* We can say that every particle of the fluid is always

* Here the term "inertia force" is used in the sense different from the one we mean throughout the book.

under the action of the three mutually balancing forces: the "inertia force", the force of pressure, and the viscous force. The sum of the three forces being equal to zero, only two of them are independent. Therefore, as a measure of the dynamical similarity we can take the ratio of any two of these forces. Usually the ratio of the "inertia force" to the viscous force is taken as this measure. This ratio is a dimensionless quantity; it is proportional to the Reynolds number, which is also dimensionless. When there are two geometrically similar bodies placed in flows of viscous fluids with similar configuration of the stream lines for which these ratios are equal we have what we call the dynamical similarity.

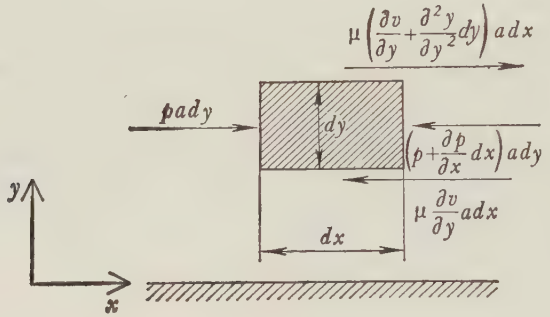


FIG. 314

It follows that when the Reynolds numbers are equal for the flows around a given body and around its model, the ratio of the "inertia force" to the viscous force is the same for both flows and, consequently, the flows around the model and around the body are similar not only geometrically but also dynamically.

Let us take a very simple example and determine the ratio of the "inertia force" to the viscous force for a particle of a viscous fluid in the presence of forces of viscous friction. To this end we shall write the equation of motion for a flow of a viscous fluid in a constant direction near a plane wall (Fig. 314).

Let the velocity v depend on time t and on the coordinate y reckoned in the direction perpendicular to the wall, the x-axis going along the wall in the direction of the velocity. An element $a\,dx\,dy$ of the volume of the fluid (here a is the dimension of the element in the direction perpendicular to the plane of the figure) is acted upon by the force of pressure $pa\,dy$ exerted on the left-hand face of the element, by the viscous force $\mu\frac{\partial v}{\partial y}a\,dx$ exerted on the lower face, by the viscous force $\mu\left(\frac{\partial v}{\partial y}+\frac{\partial^2 v}{\partial y^2}dy\right)a\,dx$ exerted on the upper face and by the force of pressure $\left(p+\frac{\partial p}{\partial x}dx\right)a\,dy$ exerted on the right-hand face. The sum of these forces must be equal to the product

$$\rho\frac{dv}{dt}a\,dx\,dy$$

of the mass by the acceleration. Hence, the dynamical equation has the form

$$\rho\frac{dv}{dt}a\,dx\,dy=-\frac{\partial p}{\partial x}a\,dx\,dy+\mu\frac{\partial^2 v}{\partial y^2}a\,dx\,dy \qquad (113.1)$$

By (101.3) for a stationary flow along the x-axis we have

$$\frac{dv}{dt} = v\frac{\partial v}{\partial x}$$

and therefore (113.1) can be rewritten as

$$\left(\rho v\frac{\partial v}{\partial x} + \frac{\partial p}{\partial x} - \mu\frac{\partial^2 v}{\partial y^2}\right) a\,dx\,dy = 0 \tag{113.2}$$

This is the basic relation expressing the equilibrium of the forces acting on a particle in a stationary flow of a viscous fluid along the x-axis.

Let us agree to supply the quantities concerning a real body (for instance, an airplane, a ship, etc.) with the subscript r and the quantities concerning a model of the body (which is geometrically similar to the given body) by the subscript m (the same notation will be used for the fluid flows around the body and around its model). On the basis of (113.2) we can write the similarity condition

$$\frac{\rho_r v_r \dfrac{\partial v_r}{\partial x_r}}{\mu_r \dfrac{\partial^2 v_r}{\partial y_r^2}} = \frac{\rho_m v_m \dfrac{\partial v_m}{\partial x_m}}{\mu_m \dfrac{\partial^2 v_m}{\partial y_m^2}} \tag{113.3}$$

Relation (113.3) can be transformed to the equality

$$\frac{\rho_r v_r}{\mu_r}\frac{\partial y_r}{\partial v_r} = \frac{\rho_m v_m}{\mu_m}\frac{\partial y_m}{\partial v_m}\frac{\partial^2 v_r}{\partial^2 v_m}\frac{\partial v_m}{\partial v_r}\frac{\partial x_r}{\partial x_m}\frac{\partial y_m}{\partial y_r} \tag{113.4}$$

Let l_m be a characteristic length scale for the model and l_r be the analogous length scale for the real body. Then for the dynamically similar flows we can write

$$\frac{\partial x_r}{\partial x_m} \sim \frac{l_r}{l_m},\quad \frac{\partial y_r}{\partial y_m} \sim \frac{l_r}{l_m},\quad \frac{\partial v_r}{\partial v_m} \sim \frac{v_r}{v_m} \text{ and } \frac{\partial^2 v_r}{\partial^2 v_m} \sim \frac{v_r}{v_m} \tag{113.5}$$

Now, taking into account conditions (113.5) we can rewrite the basic condition of the dynamical similarity (113.4) in the form

$$\frac{\rho_r v_r l_r}{\mu_r} = \frac{\rho_m v_m l_m}{\mu_m} \tag{113.6}$$

Since $\mathrm{Re} = \dfrac{\rho v l}{\mu}$, we can say that, the greater the Reynolds number Re, the smaller is the relative magnitude of the viscous forces, that is the more similar is the flow of the viscous fluid to a flow of a perfect fluid (without viscosity). When this number is very small the viscous forces dominate in the given flow.

The Reynolds numbers can be compared in order to draw a definite conclusion only when the given bodies and the fluid flows around them possess geometrical similarity. In this case an increase of the Reynolds number means that the ratio of the "inertia forces" to the viscous forces has increased the same number of times.

When the bodies are of different shape but resemble each other in a certain sense (for instance, when we deal with pipes of different cross sections, with wings of different airplanes, and the like) the comparison of the corresponding numbers Re only approximately indicates a change in the relationship between the "inertia forces" and viscous forces. The character of a flow around a body (for instance, the appearance of a flow separation) is determined not only by the shape of the body but also by the relationship between the "inertia forces" and the viscous forces. To investigate in more detail the influence of this relationship

on the character of a flow-around we shall consider in the next section the boundary layer, its connection with the generation of vortices and the separation of the flow.

The appearance of a turbulent motion in a pipe instead of a laminar flow (see the foregoing section) is also specified by the relationship between the "inertia forces" and the viscous forces. In a pipe with a circular cross section a laminar flow is observed for the velocities corresponding to the numbers Re not exceeding, approximately, 1000 while for greater velocities (when $\mathrm{Re} > 1000$) the flow usually becomes turbulent.

114. Boundary Layer

As was indicated earlier, when a fluid or a gas with small viscosity flows around a body the role of the viscous forces is essential only in the vicinity of the body or, as we say, in a thin boundary layer within which the velocity grows from a zero value on the surface of the body to a non-zero value at a certain distance from the body. The formation of the bounadry layer is specified not only by the properties of the fluid but also by the shape of the body. For instance, when a plane plate is placed in a flow so that it is parallel to the direction of the flow (see Fig. 315) the boundary layer widens along the flow. Under the action of the viscous forces in the fluid this layer becomes wider towards the back edge of the plate; Fig. 315 shows schematically the velocity field near the plate. For the sake of visuality the width of the boundary layer in the figure is enlarged and the arrows representing the velocities of the particles are depicted more densely within the layer.

As can be shown by means of theoretical calculations, the width δ of the boundary layer can be estimated approximately by using the formula

$$\frac{\delta}{l} \approx \frac{1}{\sqrt{\mathrm{Re}}} \tag{114.1}$$

The ratio $\frac{\delta}{l}$ of the width δ of the boundary layer to the characteristic length scale l of the body (this ratio characterizes the relative width of the boundary layer) is inversely proportional to the square root of the Reynolds number. We do not present the derivation of this formula and confine ourselves to the remark that this simple relation proves useful for approximate calculations. For instance, for a ball of diameter 10 cm placed in an air flow with velocity 30 m/s the Reynolds number is $\mathrm{Re} = \frac{10 \times 30 \times 10^2}{0.15} = 2 \times 10^5$ (for the air we have $\mu/\rho \approx 0.15$ cm^2/s for 20 °C). Consequently, in this case the width of the boundary layer for the ball is approximately equal to

$$\delta \approx \frac{10}{\sqrt{20} \cdot \sqrt{10^5}} \approx 0.022 \text{ mm}$$

As the velocity of the flow grows the width of the boundary layer reduces.

For very small values of the Reynolds number (of the order of unity) the width of the boundary layer is of the order of the dimensions of the body (see (114.1)). Although in this case formula (114.1) is not applicable in the strict sense the conclusion we have drawn does not contradict reality because for such small values of the Reynolds number it is no longer possible to speak of a definite boundary layer since the region of the variation of the velocity field involves the whole flow or its considerable part surrounding the body. We dealt with motions of this kind when the example of a laminar flow of a viscous fluid in a pipe was

considered (Sec. 111) and when the fall of a small ball in glycerine was studied (Sec. 40). For instance, the velocity of the fall of a steel ball with diameter 2 mm in glycerine is approximately equal to 2 cm/s for, by (40.3), we have

$$v = \frac{2}{9}\frac{\rho_b - \rho_a}{\mu} g r_0^2 = \frac{2}{9}\frac{8-1}{8.5} 10^3 \times 0.1^2 \approx 2 \text{ cm/s}$$

Therefore, the value of the Reynolds number is $\mathrm{Re} = \frac{2 \times 0.2}{6.8} \approx 0.06$ because

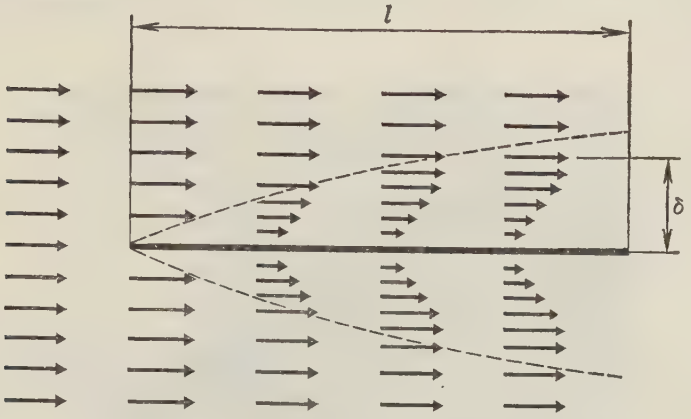


FIG. 315

$\frac{\mu}{\rho} \approx 6.8$ cm²/s for glycerine. We see that in this case the width of the "boundary" layer exceeds considerably the dimensions of the ball.

When the values of the Reynolds number exceed 10^4 the width of the boundary layer is less than 0.01 the dimension of the body. Consequently, for large values of the Reynolds number (for $\mathrm{Re} \geqslant 10^4$) it makes sense to speak of a thin boundary layer surrounding the body. In what follows we shall always mean a boundary layer of this kind.

A flow of a fluid with small viscosity around a cylinder can be approximately characterized as follows. In the vicinity of the surface of the cylinder there appears a layer of fluid with zero velocity or with a velocity very close to zero, that is the boundary layer. Within this layer the velocity of the flow changes sharply and increases from the zero value on the surface of the body to a non-zero value at a short distance from the body while outside the layer the flow is almost completely similar to a flow of a perfect (non-viscous) fluid. One may think that the presence of the boundary layer only results in the appearance of small tangential forces and in a small change of the effective dimensions of the cylinder. But it turns out that for the large values of the Reynolds number the character of the flow-around essentially changes because of the flow separation and the generation of turbulence.

Let us consider qualitatively the influence of the boundary layer on the character of the flow-around. On the front part of the cylinder where the velocity of the flow outside the boundary layer increases in the direction of the flow while the pressure drops, the presence of the boundary layer does not affect noticeably

the character of the flow and the distribution of pressure in this part of the boundary layer is almost the same as in the flow of the fluid near the layer. The pressure on the front surface of the cylinder drops in the direction of the flow, which reduces the boundary layer and the deceleration of the particles of the fluid produced by the viscous forces. We can say that the pressure drop along the flow "reduces the number of the particles" in the boundary layer.

The character of the flow is quite different behind the cylinder (or near the rear part of a ball or some other body) where the stream tubes widen along the flow, the velocity of the particles decreases in the direction of the flow while the pressure grows. In this zone the decelerative action of the pressure on the parti-

FIG. 316

cles grows within the boundary layer so that a further increase of the velocity of the flow resulting in a further increase of the pressure drop can stop the particles near the body or even make them move in the direction opposite to the flow (see Fig. 316). The particles moving in the opposite direction along the surface of the body and the particles of the incident flow meet on a certain "separation" boundary and the former particles start to rotate under the action of the latter. Therefore, there arises a rotational motion of the particles which involves more and still more amounts of fluid. These rotating amounts of fluid are carried away from the body by the flow and proceed to rotate due to the inertia. As a result, there appears a zone of eddies, that is a region filled with vortices. When the Reynolds number exceeds a certain value (for a ball this value is approximately equal to 3×10^5) the boundary layer itself becomes turbulent, which reduces the region of the flow separation and decreases the coefficient of drag (see Fig. 309).

115. Measuring Forces Acting on a Body in a Flow of Fluid or Gas

The action of a flow on a body depends on the motion of the body relative to the particles of the fluid. It is evident that the acting force is the same when the body moves relative to a motionless medium or when the medium moves with the same velocity relative to the body.

When an airplane, or a ship, or a submarine is designed it is necessary to know the magnitude of the corresponding force exerted by the air or by the water on the body. To determine such forces special models are tested. To this end the model of the ship is towed with a definite velocity over the water surface in a model testing basin and the corresponding forces are measured, a model of an airplane is tested in the air flow in a wind tunnel (see the scheme in Fig. 317), and the like.

A model is always tested in a bounded flow (within a testing basin or in a wind tunnel) and there is therefore a difference between experiments in which

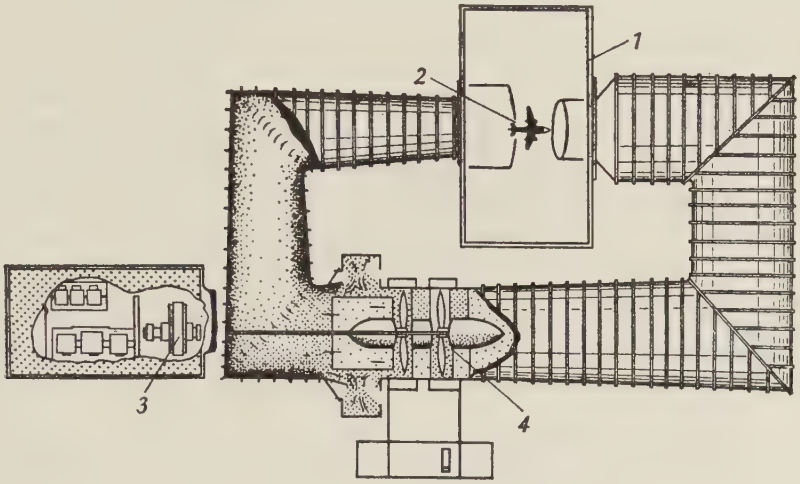


FIG. 317

a body moves in an unbounded motionless medium and experiments in which a model is placed in a flow. The matter is that the boundaries of the basin or the walls of the tunnel (that is the boundaries of the flow) always introduce some distortion in the flow-around. Therefore, to eliminate the distortion (practically, to make it negligible) the dimensions of the model must be sufficiently small in comparison with the dimensions of the flow (that is in comparison with the width and the depth of the basin or with the diameter of the tunnel).

In Fig. 317 is shown a scheme of a wind tunnel where *4* is the ventilator rotated by motor *3*. The air flow in the tunnel is almost closed; it has gap *1* in which an almost homogeneous flow is created. Model *2* is placed on the wind tunnel balance by means of which the forces acting on the model are measured. The dimensions of the flow, of the tunnel and of the model are chosen so that the influence of all the other objects surrounding the flow is negligibly small.

The measurement of the forces by means of a balance is sometimes inconvenient; in these cases an alternative method can be used which is based on the law of variation of momentum (see Sec. 109). This method makes it possible to determine the forces exerted by the flow on the given body by measuring the velocity and pressure fields around the body.

Suppose that a small axially symmetric body (Fig. 318) is placed in an air flow. On measuring the velocity and pressure fields in front of the body and

behind it we can determine the corresponding drag force. Indeed, let K_0 be the momentum of the amount of air passing during one second through the plane area AB and K_1 be the momentum of the amount of air passing during one second through the plane area $A'B'$ parallel to AB, the corresponding forces of pressure being p_0 and p'. Then the force R' with which the body acts on the flow can be found from the equality

$$K_1 - K_0 = p_0 + p' + R'$$

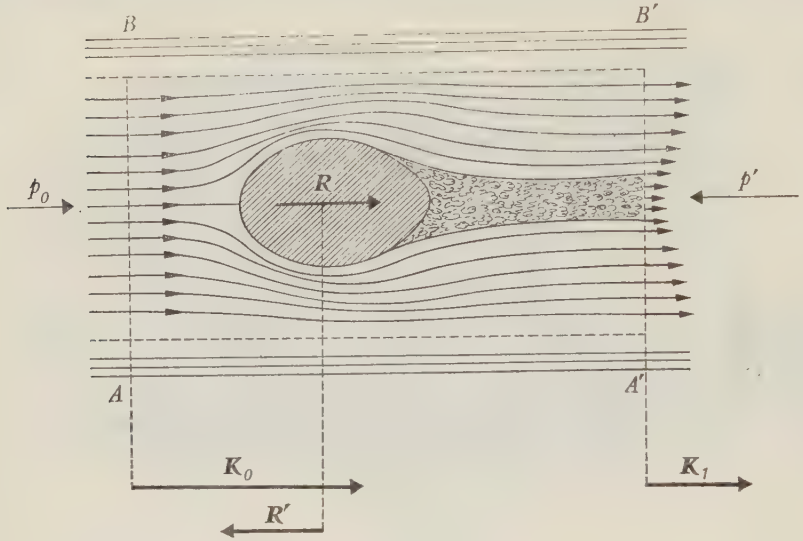


FIG. 318

which results in

$$R' = K_1 - K_0 - (p_0 + p')$$

Of course we suppose that the areas AB and $A'B'$ are sufficiently large and that the momenta of the particles passing through the lateral cylindrical surface (this surface is shown schematically in dotted line in Fig. 318) can be neglected.

It often happens that the magnitudes of the pressures in front of the body and behind it are almost the same, that is $p' + p_0 \approx 0$. In this case the pressure can be neglected and the drag force $R = -R'$ acting on the body is equal to the difference between the per-second fluxes of the momenta of the fluid in front of the body and behind it:

$$R = K_0 - K_1$$

In the general case the body is acted upon not only by the drag force (directed along the flow) but also by a force normal to the flow. In this case the resultant force with which the flow acts on the body can also be found from the per-second variation of the flux of the momentum of the fluid passing through a closed surface surrounding the body provided that the velocity and the pressure at the points of this surface have been measured.

116. Lift Force of a Wing of an Airplane

An important example of a force with which a flow acts on a streamlined body is the *lift force* acting on a wing of an airplane. A wing can be considered as a plate of a definite profile; as a crude model of a wing we can take a plane plate placed at an angle α to the flow (the *angle of incidence* or *angle of attack*; see Figs. 319 and 320). A real wing has a rounded front part (the head edge) and a sharp rear part. They are called the leading edge and the trailing edge of the wing, respectively.

The reaction force of the air (or fluid) acting on a wing can always be resolved into two components: the normal component F_l (the lift force) and the drag force R (see Fig. 319). When the angle of attack α is small the force F_l is much greater than R. In a flight of an airplane the angle of attack is usually such that the lift force F_l is much greater than the force R.

One of the first elementary explanations of the appearance of the lift force was suggested by Newton. In this explanation the particles of the air which do not adjoin the wing are considered completely motionless. When the wing approaches such particles it makes them move downwards and forwards due to the increase of the pressure under the wing. On calculating the per-second variation of the momentum of the particles and the difference between the pressures on the upper and the lower surfaces of the wing we can determine the lift force.

Thus, the wing imparts a momentum directed downwards to the particles it meets and the particles, in their turn, exert on the wing a force directed upwards. The pressure on the lower surface of the wing exceeds that on the upper surface, which results in a lift force acting on the wing and balancing the weight of the airplane.

As is seen from Fig. 319, the velocity of the particles of the air flowing near the upper surface of the wing exceeds the velocity near the lower surface because the stream tubes above the wing are narrower than below the wing. Bernoulli's equation implies that to a greater velocity there corresponds a lower pressure; consequently, the force of pressure on the upper surface is less than the force acting on the lower surface, which causes the lift force. The difference of the pressures on the upper and on the lower surfaces can readily be checked experimentaly.

Newton himself made an attempt to calculate theoretically the lift force. As a model of a wing he considered a plane plate of area S inclined at an angle α to the flow. The total mass of the particles which impact against the plate during one second is equal to

$$\rho S v_0 \sin \alpha$$

where ρ is the density of the air and v_0 is the velocity. The velocity of these particles changes: it receives a component of magnitude $v_0 \sin \alpha$ directed downwards. Consequently, every second the air receives a momentum of magnitude

$$\rho S v_0^2 \sin^2 \alpha$$

directed downwards. The lift force must be equal to this momentum. Since this is an approximate computation method it is necessary to introduce a correction coefficient k different from unity and to use the formula

$$F_l = k \rho S v_0^2 \sin^2 \alpha \qquad (116.1)$$

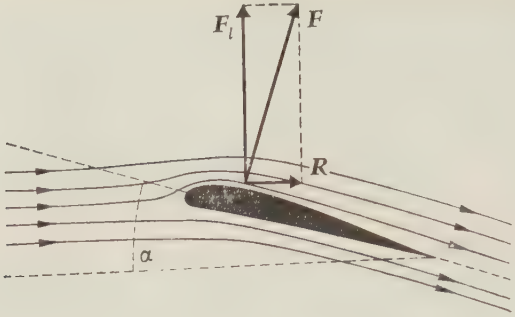


FIG. 319

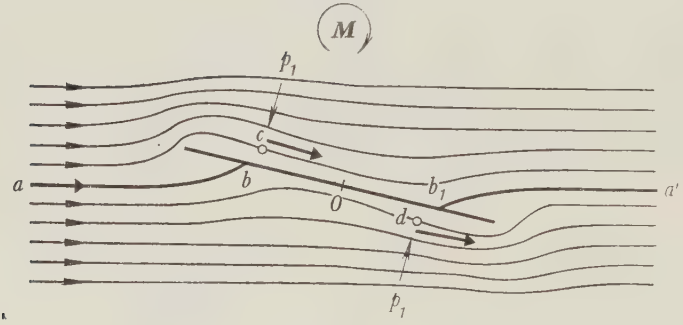


FIG. 320

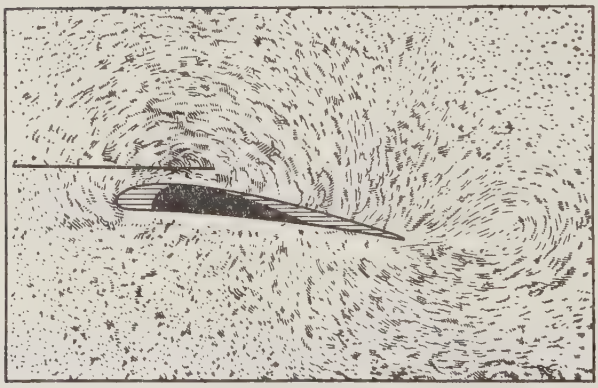


FIG. 321

for the lift force where the value of the coefficient k should be found experimentally.

However, the simplest experiments in which the lift force is measured for a wing (for a plate) show that for small values of the angle α formula (116.1) does not hold. It turns out that the lift force is in fact proportional to v_0^2 and to $\sin \alpha$ (or, for small angles of incidence, to α) instead of $\sin^2 \alpha$. Consequently, Newton's suggestion is wrong.

In Newton's argument it is supposed that the particles of the air do not interact with each other and that they only interact with the wing the moment they impact against it. Before the interaction with the wing every particle is considered to be at rest. But in reality the course of the phenomenon is quite different. The increase of the pressure near the wing arising because of the motion of the particles in front of the wing is transmitted to the other parts of the air and, generally speaking, all the particles start moving, particularly, in the vicinity of the wing (see Fig. 321). The interaction between the particles makes the phenomenon of the flow-around much more complex. The particles which are near the wing at a given instant move more intensively and the greatest variations of the pressure take place in this region. The particles which are far from the wing move very slightly but, since the number of these particles is very large, their momentum cannot be neglected.

However, when the velocity of the flight is much greater than the speed of sound in the given fluid or gas (see Sec. 121) the character of the flow-around changes sharply: for such *supersonic* velocities only the particles near the wing lying in a narrow layer surrounding it interact with the wing. That is why Newton's formula (116.1) proves practically applicable to the calculation of the lift force in a supersonic flight and is confirmed by the results of experiments although the phenomenon of the flow-around becomes extremely complicated.

117. Fluid Flow Around a Wing. Circulation. Lift Force

Before proceeding to the phenomena accompanying a flow of a viscous fluid or gas around a wing let us discuss some theoretical aspects of a flow of an ideal (perfect) fluid around an infinite plate (or wing).

The plate is supposed to be infinite in the direction perpendicular to the plane of the figure (Fig. 320). Under this assumption all the stream tubes lie in the planes parallel to each other and, in particular, to the plane of the figure, the shape of the stream lines being the same in any of these planes. In other words, in this model we deal with a plane (two-dimensional) flow; when the plate touches the walls of the channel in which the fluid or the gas flows this model of a plane motion also applies with a certain accuracy.

In these calculations we neglect the viscosity but take into account the motion of the particles of the fluid (or gas) in the whole spatial region surrounding the wing.

Here there can be two different cases, namely a smooth (continuous) flow-around and a discontinuous one. In the former case the pressure and the velocity are continuous at all the points of the flow and in the latter case the pressure in the fluid also varies continuously while the velocity may change discontinuously from place to place.

As was mentioned in Sec. 112, when a perfect fluid flows smoothly around a body it exerts no force on it. In a smooth flow around a plate the stream tubes are of a shape shown in Fig. 320. On the lower and on the upper surfaces of the

plate there are two stagnation points b and b_1 at which the velocity of the flow is equal to zero. The configuration of the stream lines is symmetric with respect to the midpoint O of the plate: on any straight line in the plane of the figure passing through the centre O there are two points (lying at equal distances from the centre) at which the velocities of the particles in the flow are equal. For instance, the velocities are equal at the points c and d and therefore the pressure at these points is also the same. Consequently, the resultant force of pressure acting on the plate is equal to zero. Hence, the system of forces exerted by the flow on the wing reduces to a moment M making the plate tend to turn clockwise. When a real wing is sufficiently thin the flow around it is approximately the same as in the case of a plate.

When a real (viscous) fluid flows around a wing the phenomenon is quite different and cannot be described by the theory of a smooth flow-around. A real configuration of the stream lines is approximately as the one shown in Fig. 319. The flow near the trailing edge does not turn near the edge and goes along the wing both above and below it. The stream lines converge behind the trailing edge so that the velocities of the particles are directed along the wing. Near the leading edge the stream lines are similar to those in a flow of a perfect fluid: there is a stagnation point on this edge (the leading edge is usually rounded and not sharp) and the stream lines adjoin the upper surface of the wing, go along it, and reach the trailing edge.

In typical conditions, after the wing starts moving there appears an eddy (vortex) near the trailing edge of the wing (see Fig. 321). The appearance of the eddy can be explained as follows. At the beginning of the motion when the velocity of the flow is not large the flow-around is approximately the same as in the case of a perfect fluid. Under the action of the pressure the particles of the fluid tend to turn in the counter-clockwise direction about the trailing edge but the action of the viscous forces stops them and they lose their kinetic energy. The incident flow is therefore decelerated by the pressure exerted on it by the particles near the trailing edge, the direction of the motion of the particles of the incident flow changes and they start to rotate near the surface of the wing, which generates an eddy. Then the eddy separates from the trailing edge and is carried away by the flow. After the separation of the eddy the flow becomes stationary and is of a form shown in Fig. 319. The eddy carried away from the wing possesses a certain angular momentum and hence the fluid surrounding the wing loses an angular momentum. This loss of the angular momentum must be compensated for by the appearance of an angular momentum of opposite sign, that is the particles of the fluid near the wing must rotate in the clockwise direction since before the motion has started the total angular momentum of the fluid is equal to zero and, by the law of conservation of momentum, it must retain the zero value.

Thus, there arises a circulatory motion of the particles around the wing. This inference can readily be confirmed by the solution of the equations of hydrodynamics.

We see that a flow of a real fluid around a wing (or around a plate inclined to the flow at an angle α) can be represented as a superposition of a smooth flow-around of a perfect fluid (see Fig. 320) and a circulatory flow around the plate (see Fig. 322*a*).

In a circulatory motion the particles of the fluid move along closed trajectories; every particle undergoes a deformation during its motion and is in a translatory motion. A thorough theoretical investigation of this motion shows that it possesses an important property whose formulation involves the notion of

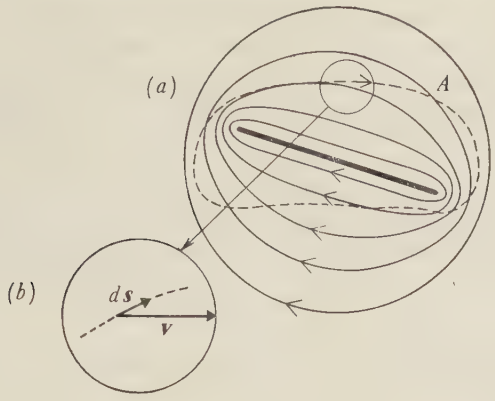


FIG. 322

circulation: by the circulation of the velocity field $\boldsymbol{v}$ over a closed contour is meant the scalar quantity

$$\Gamma = \oint \boldsymbol{v}\, d\boldsymbol{s} \tag{117.1}$$

where $\boldsymbol{v}$ is the velocity and $d\boldsymbol{s}$ is the (vector) element of the contour. The property we have mentioned is that *the circulation of the velocity over any closed contour surrounding the body is a constant quantity.*

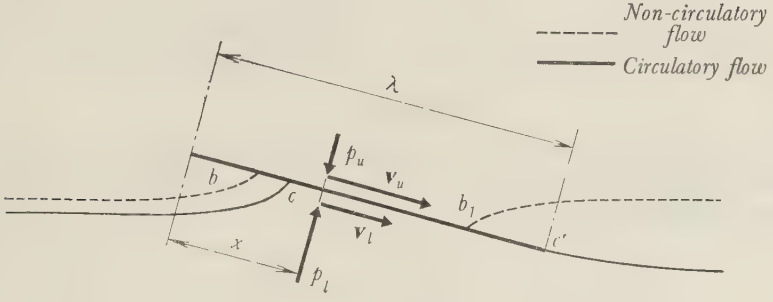


FIG. 323

Formula (117.1) means that to calculate the circulation we must find the scalar product of the velocity v by the element $d\boldsymbol{s}$ (the vector $d\boldsymbol{s}$ is shown in Fig. 322*b*) and then add together (that is integrate) all these products for all the elements $d\boldsymbol{s}$ constituting the given contour.

The circulation Γ is independent of the shape of the contour A for which it is calculated. It retains a constant value for all the closed contours surrounding the wing although the velocities at the points of the contours may be quite different. That is why the magnitude Γ of the circulation specifies uniquely the character of the circulatory flow around the given body. It should be noted that for a closed contour which does not surround the wing the circulation is equal to zero.

The famous Russian scientist N. E. Zhukovsky showed that the real flow around the wing is representable as a flow of a perfect fluid resulting from the superposition of two simultaneous flows: (a) a smooth flow whose stream lines are shown in Fig. 320 and (b) a circulatory motion around the wing whose stream lines are shown in Fig. 322*a*. In this flow-around the value of the circulation is such that the second stagnation point b_1 (see Fig. 320) is on the trailing edge (Fig. 323): the point b_1 goes into the point c' and the point b into c. Under this condition the flow around the trailing edge is smooth and is almost similar to the one observed in the experiment. This choice of the value of the circulation Γ makes it possible to take into account the most significant feature of the influence of the viscosity leading to the formation of the circulation and to a smooth flow-around near the trailing edge.

This condition imposed on the circulation (it is known as Zhukovsky's condition) makes it possible to determine the corresponding value Γ_0 of the circulation; it turns out that

$$\Gamma_0 = \frac{1}{2}\pi\lambda v\alpha \tag{117.2}$$

The derivation of formula (117.2) is rather intricate and we do not present te here. The quantity λ in f rmula (117.2) is the chord of the wing, that is thi distance along the flow from the leading edge of the wing to the trailing edge (Fig. 323) and α is the angle of attack.

Knowing the circulation around the wing we can find the magnitude of the lift force acting on the wing. Indeed, suppose that the value Γ_0 of the circulation of a flow around a wing is known, the angle of attack being α. Let v_0 be the velocity of the flow at a sufficiently large distance from the wing and p_0 be the pressure far from the wing. On denoting by $v_u(x)$ and $p_u(x)$ the velocity and the pressure on the upper surface of the plate (of the wing) where x is the distance reckoned from the leading edge and by $v_l(x)$ and $p_l(x)$ the analogous quantities for the lower surface (see Fig. 323) we can write the expression

$$(p_l - p_u)\, l\, dx$$

for the force exerted by the flow on an element of the wing of width dx and length l. Therefore the lift force exerted by the flow on the whole plate of length l can be written in the form

$$P_{\text{lift}} = \int_0^\lambda (p_l - p_u)\, l\, dx \tag{117.3}$$

By Bernoulli's equation, we have

$$p_l = p_0 + \frac{\rho v_0^2}{2} - \frac{\rho v_l^2}{2} \quad \text{and} \quad p_u = p_0 + \frac{\rho v_0^2}{2} - \frac{\rho v_u^2}{2}$$

It follows that

$$p_l - p_u = \frac{1}{2}\rho\,(v_u^2 - v_l^2) = \frac{1}{2}\rho\,(v_u + v_l)\,(v_u - v_l) \tag{117.4}$$

For small values of the angle α the velocities only slightly differ from v_0 and therefore we can approximately put*

$$v_u + v_l \approx 2v_0 \tag{117.5}$$

* It can be proved rigorously that the symmetry condition for the flow-around (both for circulatory and non-circulatory flows) implies that equality (117.5) holds exactly.

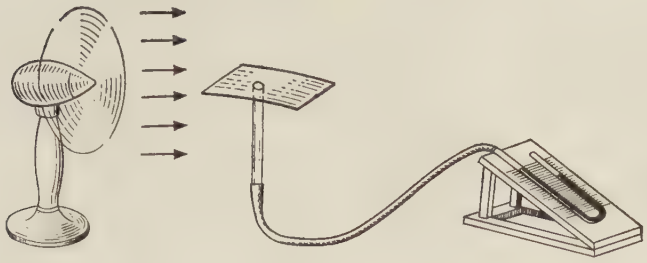


FIG. 324

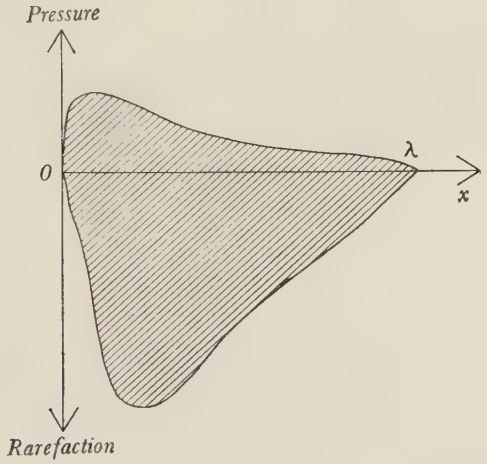


FIG. 325

Taking into account (117.5) and substituting (117.4) in (117.3) we obtain

$$P_{\text{lift}} = \int_0^{\lambda} \frac{\rho}{2} 2v_0 (v_u - v_l)\, l\, dx = \rho v_0 l \int_0^{\lambda} (v_u - v_l)\, dx \qquad (117.6)$$

By definition, the integral

$$\int_0^{\lambda} (v_u - v_l)\, dx = \Gamma_0$$

is equal to the value of the circulation around the wing. Consequently, formula (117.6) can be written as

$$P_{\text{lift}} = \rho l \Gamma_0 v_0 \qquad (117.7)$$

This is the well-known *Zhukovsky formula* determining the lift force of the wing as function of the circulation. According to (117.2), the circulation Γ_0 is proportional to the angle of incidence α and to the velocity; consequently, the lift force of the wing is proportional to the square of the velocity, to the density of

the air, and to the angle of attack. All these theoretical conclusions concerning the lift force acting on a wing having a small value of the angle of attack are in a sufficiently accurate coherence with the results of experiments.

We can easily explain the fact that the appearance of the lift force is connected with the circulation in the clockwise direction. The presence of the circulation makes the velocity v on the upper surface of the wing become greater than v_0, the effect being particularly strong near the leading edge. The velocity of the circulation above the wing is added to the velocity of the circulation-free flow

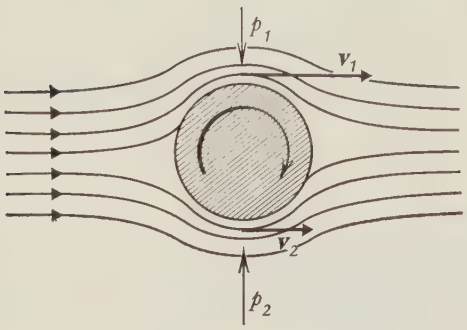


FIG. 326

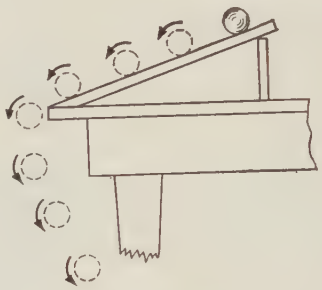


FIG. 327

shown in Fig. 320; below the wing it is subtracted from the latter velocity. That is why the velocity v_u exceeds the velocity v_l. Therefore, the pressure p_u is less than the pressure p_l: there appears a rarefaction above the wing and a compression below the wing. The pressure on the surface of a wing or a plate can be measured by connecting small openings in the surface of the wing with manometers. A simple experiment of this kind showing qualitatively the variation of the pressure can be performed with the aid of the model shown in Fig. 324 where the air flow is created by a ventilator.

For the section of an ordinary wing the diagram showing the variation of the pressure is approximately as the one shown in Fig. 325. The force acting on the wing is proportional to the shaded area in the figure. If a through hole is made in the wing the air starts to flow in the upward direction and the component of the lift force near the hole sharply reduces.

It is known that when a rotating cylinder is placed in an air flow it is subject to the action of a force transverse relative to the direction of the flow. The appearance of this force can be explained in the same way as the lift force acting on a wing. Indeed on the part of the surface of the cylinder where the circumferential velocity coincides with the direction of the flow (see Fig. 326) the velocity of the flow relative to the cylinder increases under the action of the viscous forces and on the other part of the surface of the cylinder the velocity of the flow decreases. Therefore the pressure on the latter part exceeds that on the former, which results in the transverse force. It can be shown theoretically that, generally, the transverse force increases together with the increase of the velocity of the flow and with the increase of the angular speed of the rotation of the cylinder. The phenomenon of the appearance of a transverse force acting on a rotating cylinder is known as the *Magnus effect.*

The presence of the transverse force can readily be demonstrated in an experiment with a light cylinder (for instance, made of paper) rolling down an inclined plane and then falling freely (see Fig. 327). When the cylinder starts falling it does not move vertically downwards and its trajectory is curved as shown in the figure. The matter is that the falling cylinder proceeds to rotate with the angular speed it gained during its rolling down the plane, which results in the appearance of the transverse force directed so that the cylinder moves as shown in the figure.

The attempts to replace the wings of an airplane by rotating cylinders failed although a ship with rotating cylinders (Flettner's rotors) instead of the sails was constructed but proved ineffective.

118. Dependence of the Lift Force on the Angle of Incidence. Wing Drag

Up till now we have considered a wing infinite in one direction. Such a model gives results which are sufficiently accurate only for the forces acting on the middle part of a very long wing or for a wing whose ends touch the fixed walls of a channel in which the fluid flows. In this approximation we assumed that all the forces were parallel to a plane perpendicular to the wing and that the stream lines were in planes parallel to that plane and were the same for all the sections of the wing.

However, a real wing (or a plate serving as its model) has a definite length l and therefore the stream lines are not in the planes perpendicular to the wing, the directions of the stream lines on the upper and on the lower surfaces of the wing being different. Let us consider the top view of such a wing (see Fig. 328). In the cross section of the wing passing through its midpoint the stream lines go along the flow both above and below the wing. But near the ends of the wing the stream lines turn towards the ends under the wing and towards the middle of the wing above it. For, there appears a rarefaction zone above the wing and a compression zone below the wing, which results in an additional force of pressure making the air move from the latter zone to the former. That is why the streams of the air on the upper surface of the wing turn towards the middle of the wing and those on the lower surface turn towards the ends. The air streams along the upper and the lower surfaces of the wing merge behind the trailing edge and start to rotate so that there appear some vortical streams behind the wing. This effect is particularly strong near the ends of the wing. As a result, near the ends of the wing the direction of the flow is inclined relative to the leading edge, which reduces the angle of attack and thus reduces the components of the lift force near the ends. The reduction of the lift force can also be explained on the basis of the simple fact that the motion of the air streams from the lower surface of the wing to its upper surface near the ends decreases the pressure below the wing and increases the pressure above the wing. That is why the lift force is not simply proportional to the area: the lift force per unit area is greater for longer wings and smaller for shorter ones.

In technical calculations the lift force of a wing is found by the formula

$$P_{\text{lift}} = CS\frac{\rho v^2}{2} \tag{118.1}$$

where S is the area of the wing and C is a dimensionless coefficient dependent on the Reynolds number, on the angle of attack, and on the ratio of the length of the wing to its chord (the span-to-chord ratio). The coefficient C increases together with the span-to-chord ratio.

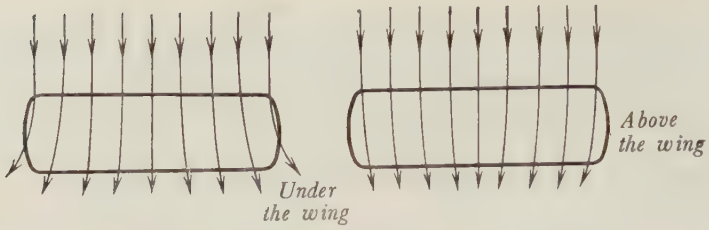


FIG. 328

For small values of the angle of attack the ift force is proportional to α and hence the coefficient C increases proportionally to α (Fig. 329). This type of dependence takes place when the angle of attack α does not exceed a certain value α_1. For $\alpha > \alpha_1$ the lift force and the coefficient C increase slower; the coefficient C achieves a maximum value for some critical value $\alpha = \alpha_{cr}$ of the angle of attack and then starts to decrease as α continues to grow. The critical angle of attack α_{cr} (also called the *incidence of stall*) is of the order of 10°-15° and depends predominatingly on the profile of the wing and on the Reynolds number. For α_{cr} the coeffi-

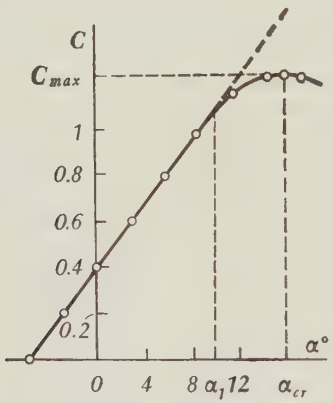


FIG. 329

cient C attains its maximum value C_{max} which is a very important characteristic of the wing since it specifies the minimum possible velocity of the flight. To secure the safety of landing this minimum velocity (the touch-down speed) should be as small as possible. To this end the value C_{max} should be increased

The reduction of the lift force for α exceeding α_{cr} is due to the stall of the flow on the upper surface of the wing, the zone of stall increasing together with α for $\alpha > \alpha_{cr}$. For the values of the angle of attack exceeding the incidence of stall the air flow no longer adjoins the upper surface of the wing and separates from it as shown in Fig. 330. In the zone of stall the pressure is almost equal to the atmospheric pressure; the absence of the rarefaction in this zone must obviously lead to a considerable reduction of the lift force.

FIG. 330

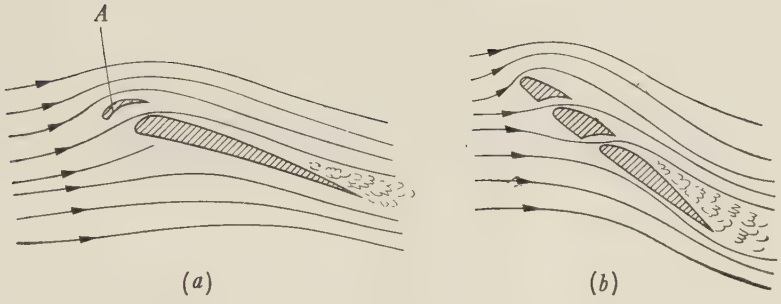


FIG. 331

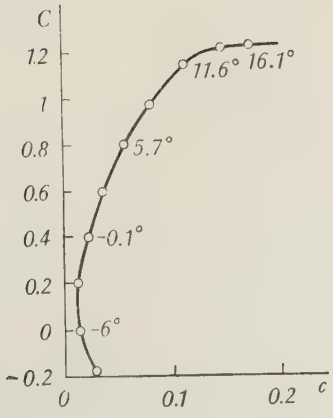


FIG. 332

There are a number of methods preventing the flow from the staff on the upper surface of the wing which lead to the increase of C_{max}; for instance, to this end the leading-edge slats (Fig. 331*a*) or the divided wings (Fig. 331*b*) are used. The flow passes through the slit between the leading-edge slat A and the wing, which makes the flow adjoin the upper surface of the wing; as a result, the value of C_{max} increases and so does α_{cr}. The same phenomena take place in the case of a divided wing.

The total reaction force exerted by the flow on the wing always has two components: the lift force P_{lift} and the drag force R. The expression of the drag force can be written in the form

$$R = cS\frac{\rho v^2}{2} \tag{118.2}$$

where c is a dimensionless coefficient and S is the area of the wing. For the wings of high aerodynamic efficiency the magnitude c is much less than C. The drag force also varies depending on the angle of incidence of the wing. The *aerodynamic efficiency* of the wing (characterized by the *lift-to-drag ratio*) depends on the character of the simultaneous variation of c and C as the angle of incidence changes. The diagram in Fig. 332 represents the connection between C and c (for the sake of convenience the length scales along the coordinate axes taken in Fig. 332 are different).

The drag increases together with the angle of attack. For practical purposes it is important to know the value of the angle of attack for which the lift-to-drag ratio (that is the ratio of the lift force to the drag force) is the greatest. Generally speaking, the wings of high aerodynamic efficiency are preferable since the ratio of the "useful" lift force to the drag force is greater for such wings.

119. Forces in a Flight of an Airplane

When an airplane is in a uniform horizontal flight the sum of all the external forces acting on it is equal to zero. Let us consider the forces applied to the airplane. There are two forces acting in the vertical direction, namely the gravity force $\boldsymbol{P}_g$ and the lift force $\boldsymbol{P}_{lift}$ of the wing; the horizontal forces acting on the airplane are the thrust force $\boldsymbol{F}$ and the drag force $\boldsymbol{R}$. These forces are shown in Fig. 333.

When $\boldsymbol{P}_{lift} > \boldsymbol{P}_g$ the airplane has an upward acceleration; conversely, for $\boldsymbol{P}_{lift} < \boldsymbol{P}_g$ the acceleration is directed downwards. The control of the flight in the vertical plane is performed with the aid of the *elevators* which are usually placed in the horizontal tail unit. To guarantee the stability of the flight in the vertical plane the airplane is supplied with the *horizontal tail*. When a random deviation of the airplane from the equilibrium state occurs the angle of incidence of the horizontal tail is changed so that the force acting on it makes the airplane return to the equilibrium. The elevators (they are marked by the letters *e* in Fig. 334) are placed behind the horizontal tail. When the elevators deviate downwards the force acting on the horizontal tail in the upward direction increases, which makes the nose of the airplane turn downwards. When the elevators deviate upwards the nose of the airplane also turns upwards.

The lateral control of the flight (that is the control of the inclination of the wings of the airplane relative to the horizontal or the turn of the airplane about its horizontal longitudinal axis) is performed with the aid of the *ailerons* (they are marked with the letters *a* in Fig. 334). The mechanism of the action of the ailerons is the same as that of the elevators.

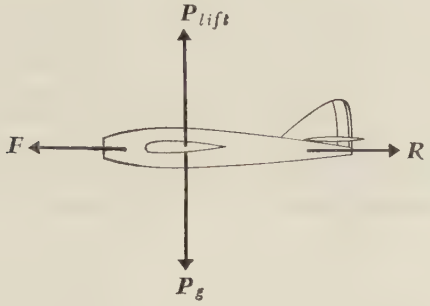


FIG. 333

The turn of the airplane about the vertical axis is performed with the aid of the *rudder* which is usually placed on the vertical tail of the airplane (in Fig. 334 the rudder is marked by the letter *r*).

In the flight of the airplane the drag force $\boldsymbol{R}$ is overcome by the thrust force $\boldsymbol{F}$ produced by a rotating propeller or by a reaction engine (see Sec. 27).

120. Propagation of Pressure Disturbances in a Compressible Fluid. Supersonic Motion of a Body

When the velocity of motion is small the air can be considered an *incompressible fluid*. As was shown in Sec. 105, the error arising in this approximation remains rather small when the velocity does not exceed 100 m/s. In the ordinary

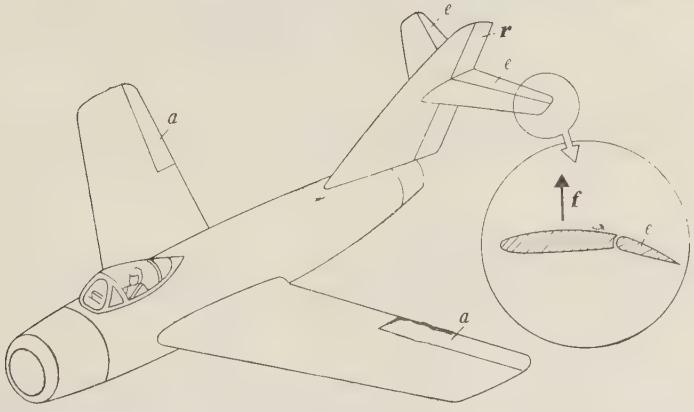


FIG. 334

conditions, say for $t = 15$ °C and $p = 760$ mm Hg, the speed of sound in the air is equal to 340 m/s. When the velocity of motion of a body in the air increases it is necessary to take into account the compressibility of the air. In this section we shall consider the propagation of a pressure pulse in the air (or in some other compressible fluid) when the situation is such that the compressibility should be taken into account.

A pressure pulse in the air (when it is not very intensive) propagates with the speed of sound. When a part of the air medium is compressed and then allowed

to expand on its own the particles of the air in this part start to move and impart their motion to the neighbouring particles which, in their turn, make other particles move and so on. Therefore, there appears a *disturbance* (a *wave*) propagating in the air with the velocity of the order of 340 m/s.

When the part of the medium subject to the initial compression is very small and has a spherical form there arises a *spherical sound wave*: all the particles lying on a spherical surface with centre at that initial point oscillate in the same manner or, as we say, the *wave front* is a sphere. The spherical symmetry implies

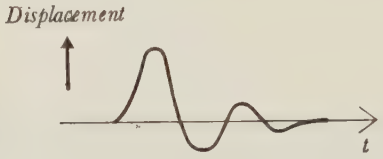


FIG. 335

that the displacements of the oscillating particles are directed along the radius; the character of the oscillation of a particle varies with time approximately as is shown in the diagram in Fig. 335. The intensity of the oscillation decreases together with the distance from the centre of the wave.

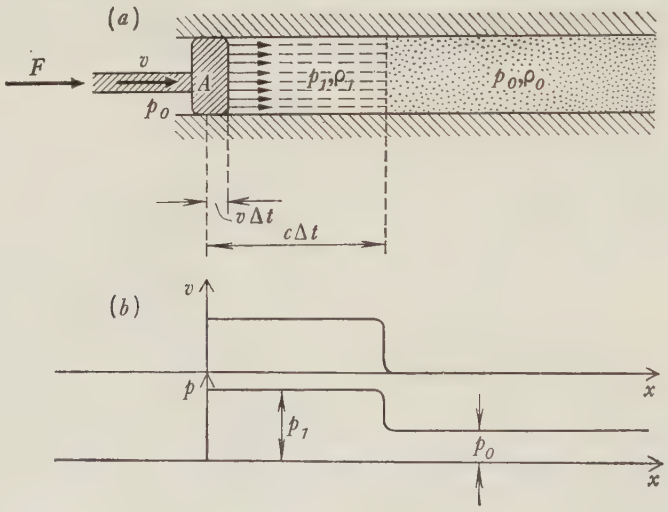


FIG. 336

Experiments show that for small variations of the density the velocity c of the propagation of a sound wave is independent of the character of the oscillation and is a constant quantity specified solely by the physical properties of the medium. Therefore, to determine the speed of sound it is sufficient to consider a very simple example.

We shall consider the propagation of a pressure pulse in a cylindrical pipe filled with the air or with some other compressible gas or fluid. Let a piston A (Fig. 336*a*) be placed in the pipe with gas, the initial pressure being p_0. Suppose

that at the initial instant $t = 0$ the piston instantaneously starts to move with a velocity v which is not very high. This initial pulse generates a *compression wave* propagating to the right of the piston.* During time Δt in front of the piston is formed a layer of the gas of length $c\Delta t$ moving with velocity v; the pressure in this layer is no longer equal to p_0; it has a value of $p_1 = p_0 + \Delta p$. In Fig. 336*b* is shown the graph of the velocity of the particles at time instant Δt; the particles are compressed and move with the velocity of the piston and the number of the moving particles increases with time because the compression wave propagates to the right. The graph of the pressure is similar to the graph of the velocity shown in Fig. 336*b*.

To investigate the propagation of the wave we shall apply the law of variation of momentum. The impulse of the force F with which the piston compresses the gas can be written as

$$F\ \Delta t = \Delta p\ S\ \Delta t$$

This impulse is equal to the momentum

$$(\rho_0 + \Delta\rho)\ cSv\ \Delta t$$

gained by the gas where $\rho_0 + \Delta\rho = \rho_1$ is the density of the gas in the wave, that is in the compressed layer. The indicated equality implies

$$\Delta p = (\rho_0 + \Delta\rho)\ cv \tag{120.1}$$

The condition of the constancy of the mass of the gas in the volume occupied by the propagating wave can be written in the form

$$\rho_0 cS\ \Delta t = (\rho_0 + \Delta\rho)\ (c - v)\ S\ \Delta t$$

whence $$\Delta\rho + \rho_0 = \rho_0 \frac{c}{c - v} \tag{120.2}$$

It follows that

$$\rho_0 + \Delta\rho = \Delta\rho \frac{c}{v} \tag{120.3}$$

On substituting (120.3) in (120.1) we obtain the general expression for the velocity of propagation of the compression wave in a pipe:

$$c^2 = \frac{\Delta p}{\Delta\rho} \tag{120.4}$$

When $\Delta p \ll p_0$ and $\Delta\rho \ll \rho_0$ the velocity of propagation of the wave can be written as

$$c = \sqrt{\frac{dp}{d\rho}} \tag{120.5}$$

Formula (120.5) specifies the sound velocity because the variations dp and $d\rho$ of the pressure and of the density in a sound wave are always very small.

The process of the variation of the density and the pressure in a sound wave is adiabatic** (see Sec. 105):

$$p = \frac{p_0}{\rho_0^{\varkappa}} \rho^{\varkappa}$$

* Another wave propagates to the left of the piston but we do not consider it here.

** In an adiabatic process the pressure and the density are connected by a law of the type $p = \text{const} \cdot \rho^{\varkappa}$. A process of this kind is accompanied by no loss or gain of heat. In the case under consideration the increase of the pressure acting on the particles of the gas arises very fast and therefore, as was indicated by P. S. Laplace, the process can be considered adiabatic.

where p_0 and ρ_0 are the pressure and the density in the absence of the wave. It follows that

$$dp = \frac{p_0}{\rho_0} \varkappa \, d\rho \tag{120.6}$$

On substituting (120.6) in (120.5) we find that the speed of sound in a gas (or fluid) is given by the formula

$$c = \sqrt{\frac{\varkappa p_0}{\rho_0}} \tag{120.7}$$

The velocity of wave propagation in a gas or in a fluid is proportional to the square root of $\frac{\Delta p}{\Delta \rho}$ where $\Delta\rho$ is the variation of the density of a particle produced by the variation Δp of the pressure. If the given variation Δp of the pressure produces only a small variation of the density, the gas or the fluid undergoes a small compression and the velocity of sound is large. The case of an incompressible fluid is characterized by the limiting relation $\Delta\rho \to 0$ for a finite value of Δp, which means that the speed of sound in an incompressible fluid tends to infinity.

Suppose that a small solid body which can be considered a material point is in a rectilinear motion in a gas with a supersonic velocity v. During its motion the body collides with the particles of the medium. These collisions result in the appearance of pulses propagating in all directions in the surrounding medium, the velocity of the propagation being equal to the constant speed of sound c. It is obvious that the wave front is a sphere for every particle because the particles of the medium do not move before the collision. This is a continuous process; if the body is at a point with coordinate s_1 (Fig. 337) at time t_1 it collides with a particle of the gas of length $v\,dt$ during time dt. During the time interval from t_1 to t this particle generates a spherical sound wave of radius $c\,(t - t_1)$. The same occurs during all the following time intervals. For instance, a point with coordinate $s_2 = vt_2$ generates at time instant $t_2 > t_1$ an analogous spherical wave which has the radius $c\,(t - t_2)$ at time $t > t_2$, and so on. The imaginable surface enveloping all these spherical waves is obviously a cone whose vertex is at the point with coordinate $s = vt$ at a given instant t, the moving body being at the vertex of the cone point at that instant. The apex angle of the cone is equal to 2α where α is determined by the equation

$$\sin \alpha = \frac{c\,(t - t_2)}{v\,(t - t_2)} = \frac{c}{v} \tag{120.8}$$

All the parts of the waves adjoining the surface of the cone are of a similar shape; they are superimposed and amplify each other. It should be stressed that only the part of the medium lying within the cone (it is referred to as the *Mach cone*) is perturbed by the moving particle, all the other particles of the medium being at rest at time t. Consequently, the motion of the body (considered a material point, that is a particle) with a supersonic velocity can be thought of as a continuous process of the propagation of the sequence of sound waves inside the Mach cone which moves together with the body. The greater the velocity v, the smaller α. This means that an increase of the velocity results in a decrease of the apex angle of the cone, that is in the reduction of the part of the medium disturbed by the moving particle.

It is evident that to generate the waves some energy should be expended. A part of the kinetic energy of the moving body is transformed into the energy of sound waves and therefore the body must be acted upon by a resistance force (which is called the *wave drag*). The oscillation of the particles in the sound waves is damped as $t \to \infty$, the part of the space occupied by the waves extends and owing to the internal friction in the gas the cone is eventually dispersed in the medium.

If the velocity of the motion of the particle were not supersonic it would also generate sound waves but the character of the waves would be quite different.

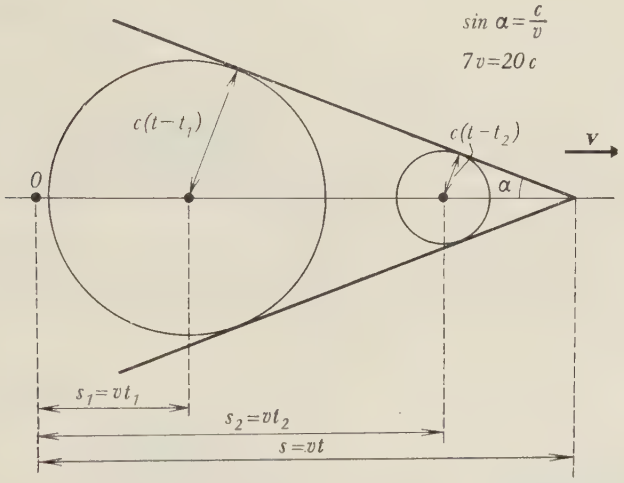


FIG. 337

The disturbance would occupy the whole part of the space filled with the gas. The phenomenon is represented schematically in Fig. 338 where the positions of the moving body and of the wave fronts generated by it are marked by similar numbers. When a particle moves in this way during a very long time period preceding the given instant the waves occupy the whole space.

When the velocity of the moving particle is equal to the speed of sound the region of the disturbance occupies half-space because in this case the apex angle of the Mach cone is $2\alpha = \pi$ (see Fig. 339).

The imaginable example of the motion of a very small body regarded as a material point makes it possible to describe the processes taking place in supersonic motions of large bodies. For instance, when the wing of an airplane is sufficiently thin it can be considered, in the first approximation, as a collection of very small bodies (particles) each of which generates a spherical sound wave in the surrounding medium. The resultant wave can be regarded as the superposition of the elementary waves generated by every point of the surface of the wing and the disturbance region is formed of the collection of the Mach cones moving together with the corresponding points of the surface of the wing. The boundary surface of the disturbance region is no longer a cone but it can readily be constructed because it is the envelope of all the Mach cones. For instance, in

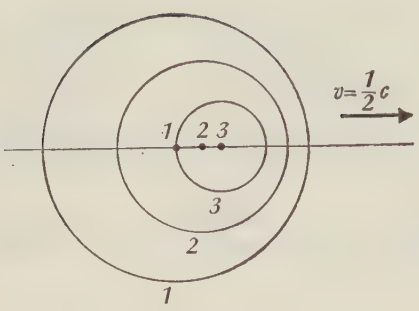


FIG. 338

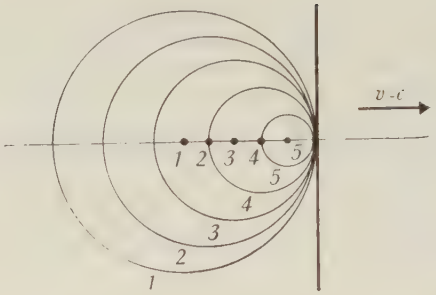


FIG. 339

Fig. 340 we see the enveloping surface of the Mach cones near a thin plate *adbc* (in the figure are shown the sections of this surface by the horizontal plane *AcadB* and the vertical plane *DaC*). In the first approximation the wave drag of the wing and its lift force can be found using this simple model of the propagation of the elementary waves around the wing.

In reality the phenomenon is more complex because the body moving with a supersonic velocity collides very strongly with the particles of the medium, which results in sufficiently large pressure variations and therefore the character

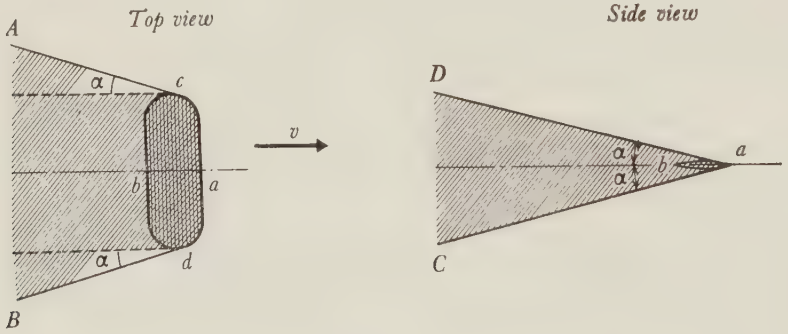


FIG. 340

of the disturbances propagating in the surrounding medium is of a different type; the resultant disturbance can no longer be described as a simple superposition of elementary acoustic waves with small pressure and density variations. The distinction becomes particularly significant when the moving body is not thin and has a blunt nose. This question will be discussed in the next section.

121. Wave with Large Variation of Pressure. Motion of a Body with Large Velocity

In the foregoing section we assumed that the velocity of the piston was very small ($v \ll c$) and that Δp was very small in comparison with p_0. We also supposed that the piston started to move instantaneously and received instantaneously a finite velocity v. If the time $\Delta\tau$ during which the velocity of the piston increases from zero to v is much less than Δt all the conclusions remain valid, the only

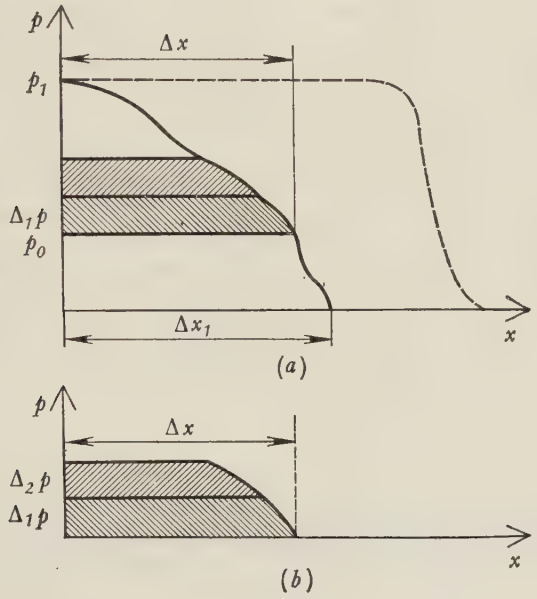


FIG. 341

difference being that the graph of the front of the compression wave is not so steep as in Fig. 336*b*. The velocity of the particle on the wave front increases from zero to v during time $\Delta\tau$.

In this section we shall consider a wave with large variation of pressure. Let us suppose that during time $\Delta\tau$ the pressure of the piston increases up to a large value p_1 and that the distribution of the pressure in front of the piston at instant $t = \Delta\tau$ is as shown in the diagram in Fig. 341*a*. The propagation of a complex wave of this type can be represented as the propagation of a sequence of elementary waves of small intensity which were considered in the foregoing section. Each of these elementary waves propagates one after another and the corresponding variations Δp of the pressure in the waves are small. Then $\Delta x_1 = c\Delta\tau$. Let us suppose that an elementary compression wave with pressure variation $\Delta_1 p$ propagates in the gas which has already been compressed up to the pressure p_0 (see Fig. 341*b*). The next compression wave with a pressure variation $\Delta_2 p$ propagates in the gas which is under the pressure $p_0 + \Delta_1 p$, and so on. The sequence of the elementary waves results in the pressure increase up to the value p_1.

Each of the subsequent elementary waves propagates in the medium with a greater density. The density and pressure variations are connected by the adiabatic law $p = \text{const} \cdot \rho^{\varkappa}$ (for the air we have $\varkappa = 1.4$). The slope of the tangent line to the adiabatic curve is equal to the square of the speed of sound, which follows from (120.5). Therefore, the speed of sound increases together with the density. Consequently, every elementary wave has a velocity exceeding the velocities of the waves generated earlier and the wave front becomes steeper and still steeper. The resultant wave front is therefore of a form shown in dotted line in Fig. 341*a*. This means that there arises a discontinuous compression *shock wave* in which the pressure and the velocity increase almost instantaneously in a jump-like manner, after which the magnitudes of the pressure and of the velocity remain constant.* The velocity of propagation of a shock wave is determined in a more complicated way; it differs from the speed of sound and depends on the magnitude of the pressure jump. Compression shock waves propagate in the air when there appear large and sharp pressure variations, for instance, when a shell blasts. In these waves the pressure jump also decreases with the distance, particularly when the zone of the blast is small and when the generated wave is of an almost spherical form.

Almost the same phenomena take place in front of and on the sides of a blunt body moving with a velocity exceeding the speed of sound in the medium. In this case there appears a leading edge shock wave which is very close to the front part of the body when the nose is sharp and the velocity of motion is sufficiently large. The particles of the medium lying in front of the body are at rest until the shock wave accompanying the moving body has reached them.

The moment the shock wave reaches a particle of the medium the particle undergoes an instantaneous compression and receives a non-zero velocity. The air between the body and the shock wave front moves and the particles go apart as the body displaces them. This phenomenon resembles the wave process in the cylindrical pipe; the distinction is that the layer of the compressed gas between the shock wave and the piston widens all the time while the particles of the compressed gas in front of the moving body go apart all the time (because in this case there are no walls) and the layer of the gas between the shock wave and the body is of constant width. The particles of the air which permanently flow into this layer move apart in all the directions and disturb the surrounding medium. The disturbances propagate with a definite velocity dependent on the magnitude of the pressure jump. As the distance from the moving body increases the pressure jump drops, the wave propagation velocity no longer differs considerably from the speed of sound and the form of the shock wave becomes similar to the corresponding Mach cone.

When the front part of a moving body has a sharpening the layer between the nose of the body and the shock wave can be very thin for $v > c$. When the velocity is very large ($v \gg c$) the shock wave is located very close to the surface of the body and is of the form of a very sharp Mach cone.

When an airplane is in a supersonic flight it is always accompanied by a shock wave and if the height of the flight is not very large the pressure jump near the ground can be of a considerable magnitude. When this wave reaches a person on the ground he hears a sharp sound resembling the blast of a shell. The pilot of the airplane naturally does not hear such blasts because he moves together with

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* Analogous considerations show that when a *rarefaction* (*depression-type*) *wave* propagates in the medium the wave front becomes less steep and disperses as $t \to \infty$.

the wave. In Fig. 342 are shown the bullets moving with the velocities of $1.04c$ and $2.48c$. The dotted areas in the figure show the shock-wave shadow.

In a motion of a body (such as a shell, a bullet, etc.) with supersonic velocity a major part of the energy is expended on the generation of the shock wave accompanying the motion. The resistance force acting on a body moving with a supersonic velocity is mainly due to the wave drag. The particles with which the moving body collides start moving, go apart, and impart the motion to the particles of the surrounding medium which start moving after the leading edge

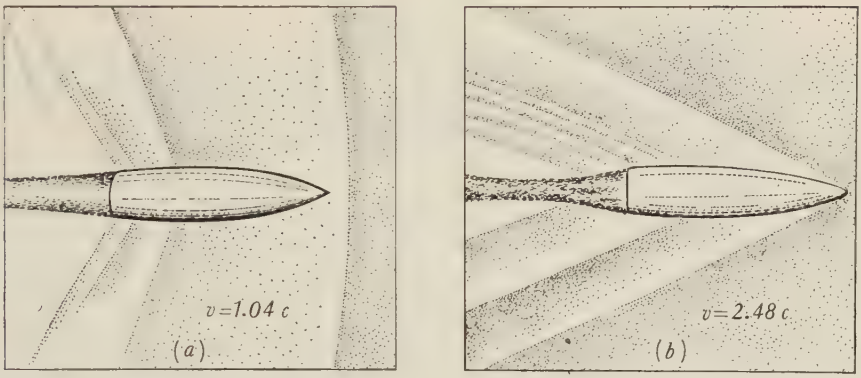

FIG. 342

shock wave has passed. The energy expended on the motion of the particles and on the generation of heat when the body collides with the particles is due to the decrease of the kinetic energy of the moving body or due to the energy source producing the motion of the body. The shape of the front part of the body is essential for the magnitude of the drag force in a supersonic motion: the bodies with sharpened noses and small cross-section areas undergo smaller drag force. In contrast to the case of a flow-around with a small velocity, the shape of the rear part of the body no longer plays an important role in the formation of the drag.

122. Supersonic Flow in a Pipe

As was shown in Sec. 105, the outlet velocity of a gas flowing from an opening in the vessel containing the gas which is under an almost constant pressure p_1 inside the vessel can achieve a very large value. For instance, by formula (105.5), the outlet velocity of the air flowing into the vacuum (with zero pressure) under the pressure of one atmosphere is equal to

$$v_0 = \sqrt{\frac{2\varkappa}{\varkappa - 1}\frac{p_i}{\rho_i}} = \sqrt{\frac{2 \times 1.405 \times 1.0133 \times 10^5}{0.405 \times 1.293}} \approx 750 \text{ m/s}$$

where the adiabatic exponent $\varkappa$ is equal to 1.405, the atmospheric pressure p_i inside the vessel is equal to 1.0133×10^5 kg/m·s^2 and the initial density of the air ρ_i inside the vessel is equal to 1.293 kg/m^3.

This outlet velocity is more than twice as great as the velocity of sound in the air. However, this is to a certain extent only a theoretical result. For instance, for an opening of the form shown in Fig. 301*a* this value of the velocity of the flow cannot be practically achieved. Indeed, suppose that at the beginning the pressure is the same inside and outside the vessel and that the outside pressure is then gradually decreased while the inside pressure is kept constant (equal to one atmosphere). The maximum velocity in the narrow section of the opening will gradually increase. For a certain value of the outside pressure

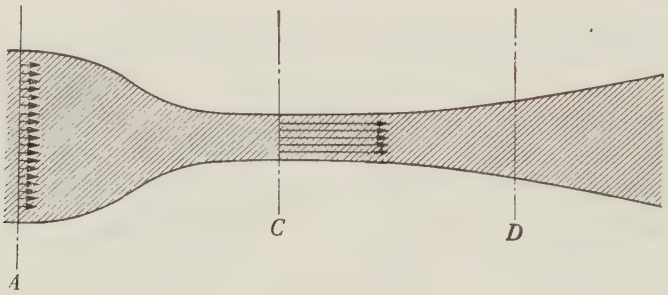

FIG. 343

(which can be found by formula (105.5)) the outlet velocity will achieve the sound velocity and the whole cross section of the opening will be occupied by an amount of the gas moving with the sound velocity.

A variation of the outside pressure can only affect the flow inside the vessel if this variation penetrates into the vessel in a certain way. As we know, pressure variations propagate in a gas with the sound velocity and, consequently, they cannot penetrate into the vessel through the region occupied by particles moving with the velocity of sound. It follows that a further reduction of the outside pressure cannot produce any increase of the velocity of the flow in the opening. The air jet in the opening will have a definite pressure exceeding the outside pressure. We can say that the flow with sound velocity "locks" the opening. This conclusion can be confirmed by the analysis of the conditions under which there can exist a stationary supersonic flow of a compressible gas in a pipe with variable cross-section area.

Let a stream tube of a stationary flow of a compressible gas be described by a function $S = S(x)$ where the coordinate x is reckoned along the axis of the tube in the direction of the flow. Then the condition of the constancy of the mass flux along the tube in the stationary flow can be written in the form

$$\rho v S = \text{const} \tag{122.1}$$

where all the quantities involved depend on x solely. If we pass from the section of the tube with coordinate x to another section with coordinate $x + dx$ all the quantities change so that their differentials are connected by the relation

$$\frac{dS}{S} + \frac{dv}{v} + \frac{d\rho}{\rho} = 0 \tag{122.2}$$

which is obtained from expression (122.1) by differentiation. By the second law of dynamics, we have

$$-dp = \rho v\, dv \tag{122.3}$$

where dp is the increment of the pressure corresponding to the passage from x to $x + dx$. For the speed of sound c we have the relation

$$dp = c^2 \, d\rho \tag{122.4}$$

(see Sec. 120). On substituting (122.4) in (122.3) we obtain

$$\rho v \, dv = - c^2 \, d\rho$$

whence

$$\frac{d\rho}{\rho} = - \frac{v \, dv}{c^2} \tag{122.5}$$

The substitution of (122.5) into (122.2) results in the following relation connecting the velocity of the flow and the cross-section area of the stream tube:

$$\frac{dS}{S} = \frac{dv}{v} \left(\frac{v^2}{c^2} - 1 \right) \tag{122.6}$$

Formula (122.6) describes the relationship between the variation dS of the cross-section area S and the variation dv of the velocity v; it indicates the essential role of the ratio of the velocity of the flow v to the speed of sound c.

If the cross-section area of a stream tube decreases along the flow, that is if dS is negative, formula (122.6) shows that dv can only be positive when $\frac{v}{c} < 1$. This means that in a narrowing stream tube the velocity v can only increase up to a definite value not exceeding the sonic speed c. As is seen from condition (122.5), for $dv > 0$ (when the velocity v increases) the density ρ always decreases. Thus, in a narrowing stream tube the velocity of the flow grows in the direction of the flow but cannot exceed c while the density and the pressure decrease along the flow. The same inference was drawn earlier on the basis of elementary considerations.

If the stream tube widens ($dS > 0$) and the velocity of the flow at the beginning of the tube is less than c, then, according to (122.6), there must be $dv < 0$. Consequently, in this case the velocity of the flow reduces along the flow while the pressure and the density increase, which follows from (122.5). In the case when the velocity v of the flow in a widening stream tube exceeds c we have $dv > 0$, that is the velocity of the flow increases in the direction of the flow while the pressure and the density reduce.

In the case when $v > c$ a flow of a compressible gas essentially differs in its character from a flow of an incompressible fluid and from a flow of a gas with velocity less than c. Indeed, for a flow in a widening tube with velocity not exceeding c the pressure and the density increase along the flow while the velocity decreases. For an incompressible fluid we can put $c \to \infty$ and therefore in this case the qualitative character of the variation of the density and of the velocity in a widening or in a narrowing stream tube is the same as in the case of a gas when $v < c$.

We see that in order to obtain a flow with supersonic velocity in a pipe the latter must first narrow and then widen (see Fig. 343). For an appropriate pressure difference the velocity of the flow will increase from section A to section C up to the speed of sound c and then will continue to grow and remain greater than c from section C to section D. To minimize the energy expenditure needed for the creation of a flow with a given velocity the variation of the cross-section area along the pipe should be chosen in a special manner. When the pipe widens

in an unsuitable manner there may appear shock waves in the supersonic zone which are at rest relative to the walls of the pipe, the velocity of the flow reducing after the pressure jump. Here we shall not discuss the physical causes generating these jumps.

Wind tunnels meant for testing models in a supersonic flow are constructed so that their cross section varies in a way similar to the one in Fig. 343, the model being placed in a zone near section D. Near the model there appear shock waves as in a flight of a body with supersonic velocity in the motionless air.

PART THREE

Oscillations and Waves. Elements of Acoustics. Fundamentals of Special Theory of Relativity

CHAPTER 14

Oscillations

123. Periodic Processes

Among the various phenomena there are very many periodic processes such as alternation of day and night, the rotation of the Moon round the Earth, the motion of the planets round the Sun, oscillation of a pendulum, motions of components of machines. In this section we shall study some important types of periodic processes.

In a periodic process the course of the phenomenon is repeated again and again at regular intervals. The mathematical definition of a periodic quantity reads: a function $f(t)$ is said to be periodic in t with period T if $f(t+T) = f(t)$ for any t. If the range of the independent variable t is separated in an

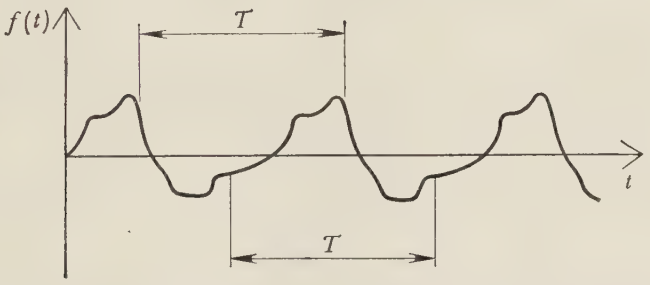

FIG. 344

arbitrary manner into subintervals of length T the graph of the periodic function $f(t)$ is the same in each subinterval (see Fig. 344). We also frequently deal with non-periodic phenomena which resemble periodic processes, for instance, the vibration of a load suspended from a thread which is gradually damped. A general process of this kind is referred to as a *vibration* (*oscillation*), *periodic oscillations* being a special case.

When oscillation is transmitted from one particle of a medium to another, for instance, when the oscillation of the water level appearing after a stone has been thrown into the water is transmitted to the surrounding particles of the water, we speak of a *wave process*.

124. Harmonic Vibrations

A *harmonic vibration* is a periodic process in which the variation of the given quantity goes in a sinusoidal manner. The harmonic vibrations form an important and wide class of periodic processes. For instance, when a point moves uniformly in a circle (see Fig. 345) its projection on a straight line lying in the plane of the motion of the point varies with time according to a sinusoidal law.

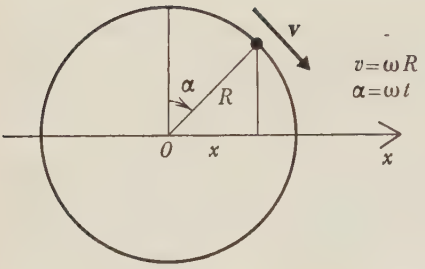

FIG. 345

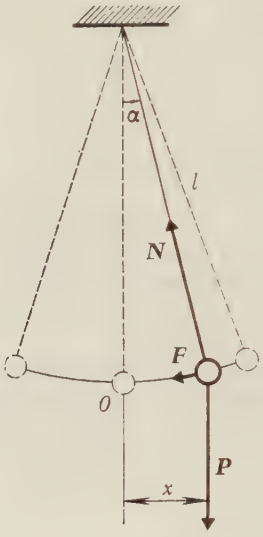

FIG. 346

If the radius of the circle is R and the angular speed of the rotation of the point is ω its projection x is expressed as

$$x = R \sin \alpha = R \sin \omega t \tag{124.1}$$

It is evident that the period of the variation of x is

$$T = \frac{2\pi}{\omega} \tag{124.2}$$

At time intervals T (T is the duration of one revolution of the point) the course of the process is repeated. The quantity T is called the *period* of the harmonic oscillation and ω is referred to as the *circular* (or *cyclic* or *angular*) *frequency*. The number ν of vibrations per unit time is called the *frequency*:

$$\nu = \frac{1}{T}$$

(Sometimes ω is also simply called frequency.)

The unit for the measurement of the frequency ν is called the *hertz* (denoted Hz), the dimension formula being [Hz] = 1/s. To N vibrations per second there corresponds a frequency ν of N Hz.

Here we have given only a kinematic description of a harmonic vibration; physical conditions under which harmonic vibrations occur will be discussed later.

Let us consider a load suspended from a thread (see Fig. 346). If the load is given a displacement from the equilibrium position and then allowed to move on its own it approaches the equilibrium position with an acceleration due to the action of the resultant of the force of tension of the thread N and the force of gravity $\boldsymbol{P} = m\boldsymbol{g}$. On achieving the equilibrium position O where the accelerating force turns into zero, the load passes through the equilibrium position by inertia after which it undergoes the deceleration produced by the same force which accelerated it before. Then the load stops and starts to move backwards; as a result, there appears a *natural vibration* of the pendulum. The term "natural" is used in this case because during the course of the oscillation the load is under the action of the forces specified by its own physical nature and structure solely.

Because of the friction the oscillation of the pendulum is gradually *damped* and the initial energy given to the pendulum when it is displaced from the equilibrium position is transformed into heat. The oscillation of the pendulum is neither harmonic nor periodic but if the initial departure from the equilibrium position and the initial velocity are sufficiently small and if the friction forces are negligible the vibration is very close to a harmonic vibration.

To investigate the oscillation of the pendulum we shall begin with the simplest limiting case when there are no friction forces. Then the vibration of the pendulum is obviously periodic. Let us consider natural vibrations of such a pendulum when the departure of the load from the equilibrium is small.*

Let us denote by α the angular deviation of the pendulum (see Fig. 346). The resultant force acting on the load of mass m has two components: the gravity force $\boldsymbol{P} = m\boldsymbol{g}$ directed vertically downwards and the force of tension of the thread $\boldsymbol{N}$ directed along the thread towards the point of suspension. For a small angle of deviation α the arc of the trajectory of the load can be approximately replaced by a straight line; on denoting by x the linear deviation of the pendulum (see Fig. 346) we can write the approximate relation

$$x \approx l\alpha \tag{124.3}$$

where l is the length of the thread. The force $\boldsymbol{F}$ acting along the arc has a magnitude $\boldsymbol{F}$ equal to $P \sin \alpha$; for small α we can write

$$F \approx mg\alpha \tag{124.4}$$

Therefore, the equation of motion of the load is

$$m\ddot{x} = -F \tag{124.5}$$

where the minus sign is put in front of F because the force $\boldsymbol{F}$ is directed opposite to the positive x-axis. On replacing α by x/l in (124.4) we rewrite equation (124.5) in the form

$$m\ddot{x} = -mg\frac{x}{l}\,. \tag{124.6}$$

On cancelling (124.6) by m we finally obtain the equation

$$\ddot{x} + \frac{g}{l}x = 0 \tag{124.7}$$

The load moves under the action of the force $\boldsymbol{F}$ whose magnitude $F = mg\frac{x}{l}$ (see (124.6)) is proportional to the deviation x of the load from the equilibrium

* If the dimensions of the load are so small that it can be considered a material point and if the thread is weightless the pendulum is called a *simple* (or *mathematical*) *pendulum*.

position $x = 0$ and which is always directed towards the equilibrium position. That is why the force F is referred to as a *restoring* force.

The solution of the equation (124.7) can readily be found. It has the form

$$x = A \sin \left(\sqrt{\frac{g}{l}}\, t + \varphi \right) \tag{124.8}$$

where A and φ are arbitrary constants. To prove that (124.8) satisfies equation (124.7) we differentiate x twice with respect to t to obtain

$$\ddot{x} = -\frac{g}{l} A \sin \left(\sqrt{\frac{g}{l}}\, t + \varphi \right) = -\frac{g}{l} x \tag{124.9}$$

which coincides with equation (124.7). Denoting

$$\frac{g}{l} = \omega^2 \tag{124.10}$$

we can write the formula expressing the deviation of the pendulum in the form

$$x = A \sin (\omega t + \varphi) \tag{124.11}$$

We have thus established that x varies according to a sinusoidal law. The quantity A which is equal to the maximum deviation from the equilibrium position is called the *amplitude* of the harmonic vibration. The amplitude depends on the initial deviation of the load and on the initial velocity given to it. The quantity $\omega t + \varphi$ is called the *phase* of the vibration. The phase depends on time linearly. The term φ is the *initial phase*, that is the value of the phase at the initial instant $t = 0$; the initial phase also depends on the initial deviation and on the initial velocity of the pendulum.

The vibration process is periodic. To find the *natural period* T of the vibration we use the obvious fact that the displacement x of the load and its velocity $\frac{dx}{dt}$ assume the same values when the phase $\omega t + \varphi$ receives an increment of magnitude 2π. To the increase of the phase by 2π there corresponds an increment of time by the period T, whence

$$2\pi = \sqrt{\frac{g}{l}}\, T$$

It follows that the period T is given by the formula

$$T = 2\pi \sqrt{\frac{l}{g}} \tag{124.12}$$

It should be stressed that when time t receives an increment equal to the period T the pendulum returns to its previous position and has the same velocity. The period of vibrations is proportional to the square root of the length l of the pendulum, which can readily be checked experimentally: if the length of the pendulum is increased four times its period becomes twice as large. The period of the pendulum is independent of the mass of the load* and of the amplitude. The independence of the period of the amplitude takes place only for very small values of the angle of deviation. For instance, the periods of the pendulums with the (angular) amplitudes of 2° and 4° practically coincide and for the values of the angle of deviation not exceeding 10° the periods coincide to within 0.2%.

* This assertion only applies when the dimensions of the load are very small in comparison with l and the amplitude is not large.

For large values of the angle of deviation the approximate equation (124.6) no longer holds. In this case the equation describing the oscillations must be written in the form

$$ml^2\ddot{\alpha} = -mgl \sin \alpha \qquad (124.13)$$

It should be noted that equation (124.13) coincides with the one describing the rotation about the horizontal axis passing through the point of suspension. For large values of the angle α the motion is periodic but non-harmonic and its period depends on the amplitude.

For small angles of deviation the *natural frequency* is given by the formula

$$\nu = \frac{1}{2\pi}\sqrt{\frac{g}{l}} \qquad (124.14)$$

which is implied by (124.12). The quantity $\omega = \sqrt{\frac{g}{l}}$ is the *natural (circular) frequency* of the pendulum. Indeed, the comparison of (124.10) and (124.14) shows that

$$\omega = 2\pi\nu = \sqrt{\frac{g}{l}} \qquad (124.15)$$

For small values of the angle of deviation the period and the frequency of a simple pendulum depend solely on its length l and on the acceleration of gravity g at the given place on the Earth's surface.

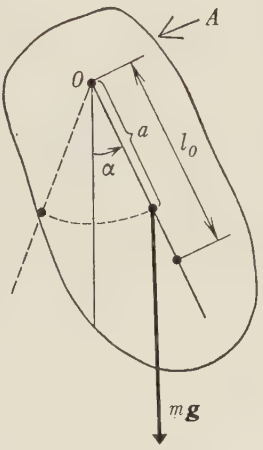

FIG. 347

Let us consider the natural oscillation of a *compound* (or *physical*) *pendulum* by which is meant a heavy solid rotating freely about a fixed axis. As will be shown, the oscillation of the body A (see Fig. 347) which can freely rotate about the horizontal axis O perpendicular to the plane of the figure resembles the oscillation of a simple pendulum. Denoting by a the distance from the centre of mass of the body to the axis of rotation we can write the expression

$$mga \sin \alpha$$

for the restoring moment of the gravity force where α is the angle the turn of the body reckoned from the equilibrium position and m is the mass of the body.

In the oscillation process the body is only acted upon by that moment and, consequently, by the second law of dynamics for a rotating body, we have the equation

$$I\ddot{\alpha} = -mga \sin \alpha$$

where I is the moment of inertia of the body relative to the horizontal axis passing through the point O. For small values of α we have $\sin \alpha \approx \alpha$ and then

$$I\ddot{\alpha} + mga\alpha = 0$$

that is

$$\ddot{\alpha} + \frac{mga}{I}\alpha = 0 \qquad (124.16)$$

Equation (124.16) belongs to the type (124.7). Therefore, α is a harmonic function of time t with frequency

$$\omega = \sqrt{\frac{mga}{I}} \tag{124.17}$$

Formula (124.17) readily follows from the comparison of the formulas (124.16), (124.7), and (124.15). Consequently, a simple pendulum with length

$$l_0 = \frac{I}{ma} \tag{124.18}$$

(l_0 is called the *equivalent length*) has the same frequency as the given physical pendulum. The point lying at a distance of l_0 from the axis of rotation on the straight line passing through the centre of mass is called the *centre of oscillation* of the physical pendulum. If the axis of rotation is placed at the centre of oscillation the frequency of the vibration of the pendulum remains the same. (Such a pendulum with two parallel axes of rotation for which the frequencies of the oscillation coincide is known as Karter's pendulum.)

This can readily be shown by using the Huygens-Steiner parallel axes theorem (see (59.16)). Indeed, the moment of inertia I about the axis O is connected with the moment of inertia I_0 about the parallel axis passing through the centre of mass by the relation $I = I_0 + ma^2$ and equality (124.18) can therefore be rewritten as

$$l_0 = \frac{I_0}{ma} + a \tag{124.19}$$

From (124.19) it follows that

$$a = \frac{I_0}{m(l_0 - a)} \tag{124.20}$$

We see that the centre of oscillation of Karter's pendulum when it rotates about the new axis is at a distance of

$$l_0' = \frac{I_0}{m(l_0 - a)} + l_0 - a \tag{124.21}$$

from the new point of suspension; this follows from (124.19). Now, taking into account (124.20), we find that $l_0 = l_0'$.

Formula (124.19) also shows that when $\frac{I_0}{ma}$ is negligibly small in comparison with a the physical pendulum resembles a simple pendulum for which $I_0 = 0$.

Knowing the length l_0 and measuring the period of oscillation of the physical pendulum we can measure the magnitude g of the acceleration of gravity at the given place. This method is widely used for precise measurements of the gravity force and for the investigation of the variation of the gravity force from place to place at different points on the Earth's surface. This method of measuring g makes it possible to investigate the local variations of the density of the Earth's crust and is used in the gravitational prospecting of natural deposits.

Now let us consider the natural vibrations of a load suspended from an elastic spring (Fig. 348*a*). After the load has been displaced from the equilibrium position and allowed to move on its own it is in a vertical harmonic vibration provided that the force appearing owing to the deformation of the elastic spring is proportional to the elongation of the spring. The dependence of the elastic force $\boldsymbol{F}$ of the spring on the elongation Δl is represented by the graph in Fig. 348*b*, the equation of the graph (which is a straight line) being $F = k\,\Delta l$. Under the action of the force of gravity mg the spring receives an elongation of magnitude

Δl_0; this means that when the load is in the equilibrium position the deformation of the spring is Δl_0. Let us denote the displacement of the load from the equilibrium position by x (the coordinate x is reckoned in the downward direction). To the displacement of the load from the equilibrium position equal to x there corresponds a restoring force of magnitude $f = kx$ equal to the difference between the elastic force and the gravity force.

Denoting by m the mass of the load we can write its equation of motion

$$m\ddot{x} = -f = -kx \tag{124.22}$$

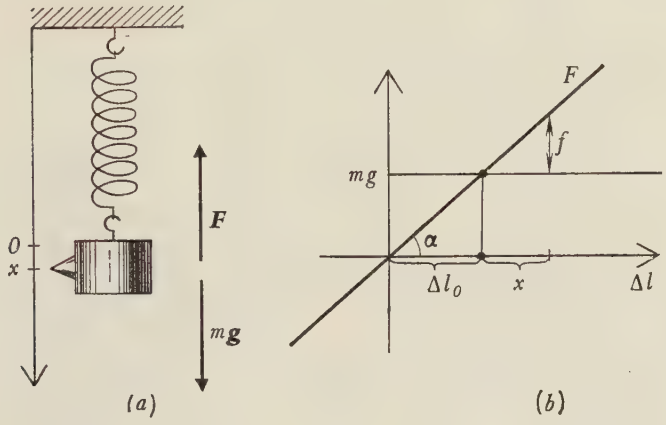

FIG. 348

which is quite analogous to the equation describing small oscillations of a simple pendulum. The coefficient k entering into the expression $f = kx$ and into equation (124.22) is the *stiffness factor* of the spring. The solution of equation (124.22) describes a harmonic vibration:

$$x = A \sin\left(\sqrt{\frac{k}{m}}\, t + \varphi\right) \tag{124.23}$$

The period of the vibration is given by the formula

$$T = 2\pi \sqrt{\frac{m}{k}} \tag{124.24}$$

and the circular frequency is

$$\omega = \sqrt{\frac{k}{m}} \tag{124.25}$$

The circular frequency is equal to the square root of the ratio of the stiffness factor k of the spring to the mass m of the load. If the stiffness of the spring is increased the frequency also increases while the increase of the mass reduces the natural frequency. It should be noted that the magnitude of the gravity force does not affect the character of the vibration of the load suspended from the spring; the given load would oscillate with the same period under the action of the same spring if the system were placed horizontally and the friction force acting on the load were negligible.

It follows that the vibrations of a load suspended from a spring are the same for the various places on the Earth's surface; even if the load with the spring

were placed on some other planet the oscillations would be the same. The character of the natural vibrations depends solely on the variable restoring force of the spring and is independent of the constant gravity force.

125. Natural Oscillation and Variation of Energy in Oscillation Process

Considering various examples (for instance, the vibrations of a mathematical or a physical pendulum with small deviations from the equilibrium, the oscil-

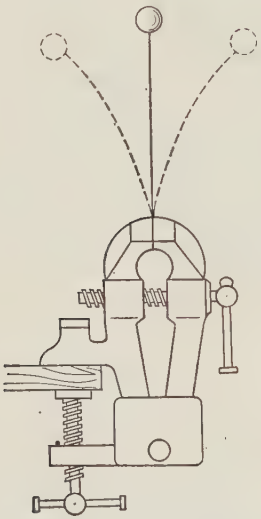

FIG. 349

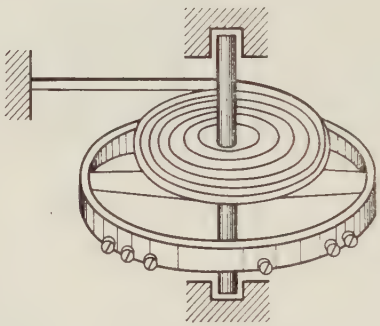

FIG. 350

lation of a load suspended from a spring, etc.) we can draw the following conclusion: *in a natural harmonic vibration the body always oscillates about a stable equilibrium position, the restoring force being proportional to the departure of the vibrating body from the equilibrium.*

Natural vibrations are observed very frequently; to the above examples we can add a number of other cases, such as the vibration of an areometer immersed in a fluid, the vibration of a load attached to an elastic plate clamped in a bench vice (Fig. 349), the oscillation of the torsion pendulum of a watch (Fig. 350), and the like.

A natural harmonic oscillation arises after the body has been given an initial displacement from the equilibrium position or after it has received an initial velocity or both. If there is no friction in the system in question the vibrations arising after an initial perturbation of the system last indefinitely. In other words, after such a system has been given an amount of energy at an initial instant this energy is preserved in the form of the oscillation energy.

Consequently, according to the law of conservation of energy, *the total energy remains invariable in a harmonic vibration* process while the kinetic energy and the potential energy, considered separately, vary and their values oscillate. The moment the vibrating body achieves its extreme position and its velocity

turns into zero *the whole energy turns into the potential energy and the kinetic energy is equal to zero.* When the body passes through the equilibrium position *the whole energy turns into the kinetic energy and the potential energy is equal to zero* (the latter assertion implies that the potential energy is assumed to be zero at the equilibrium position; we remind the reader that the potential energy is always determined to within an arbitrary constant). Since during the period of vibrations the body passes twice through the equilibrium position the period of the oscillation of the kinetic energy is twice as small than the period of the oscillation of the load. The potential energy oscillates with the same period as

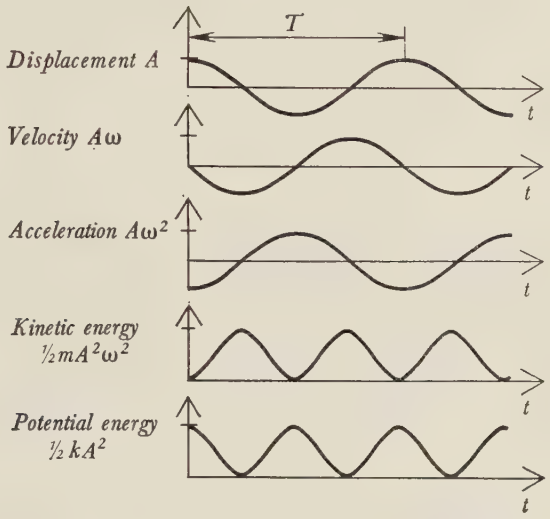

FIG. 351

the kinetic energy. The variation of all the quantities characterizing the oscillation process is represented in Fig. 351.

The energy conservation law allows us to find in a simple way the natural frequency (or period) of the vibrations. Let a body of mass m be in an oscillatory motion described by the equality

$$x = A \sin \omega t$$

Then the velocity of the body is

$$\dot{x} = -\omega A \cos \omega t$$

The velocity achieves its maximum value equal to ωA (this is the amplitude of the velocity) at the time instants when the body passes through the equilibrium position. The value of the total energy (equal to the kinetic energy at these instants) is

$$E_{\text{total}} = \frac{mv^2}{2} = \frac{m\omega^2 A^2}{2} \tag{125.1}$$

It takes the body the quarter of the period to pass from the equilibrium position to an extreme position (characterized by the values $x = \pm A$ and $\dot{x} = 0$

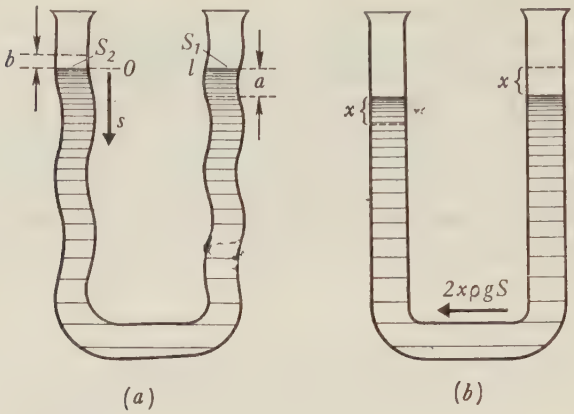

FIG. 352

of the displacement and of the velocity). At the extreme position the total energy is equal to the potential energy. For the horizontal vibrations of a load of mass m under the action of a spring with stiffness factor k the potential energy is equal to $\frac{kx^2}{2}$ (see Sec. 31). Consequently,

$$E_{\text{total}} = \frac{kx^2}{2} = \frac{kA^2}{2} \tag{125.2}$$

On comparing (125.1) with (125.2) we arrive at formula (124.25) for the natural frequency: $\omega^2 = \frac{k}{m}$.

The determination of the natural frequency by means of the comparison of the kinetic and potential energies is one of the simplest methods. The examples below demonstrate the application of the method.

Example 1. Let us determine the frequency of the oscillation of a fluid in communicating vessels of curved shape (see Fig. 352*a*). For the rectilinear cylindrical communicating vessels with constant cross-section area (Fig. 352*b*) the natural frequency can be found quite simply. When the free levels of the fluid in the right and in the left vessels depart by a distance of x from the equilibrium position the fluid undergoes the action of a restoring force of magnitude $2x\rho gS$ arising because of the unbalanced amounts of the fluid where l is the total length of the whole amount of the fluid in the communicating vessels, ρ is the density of the fluid, and S is the cross-section area. This force acts on the whole mass ρSl of the fluid and makes it move. All the particles of the fluid receiving the same displacements x, the equation of motion of the fluid has the form

$$\rho Sl\ddot{x} = -2x\rho gS$$

On cancelling by ρS we obtain

$$\ddot{x} + \frac{2g}{l}x = 0 \tag{125.3}$$

It follows that the circular frequency of the vibration is

$$\omega = \sqrt{\frac{2g}{l}} \tag{125.4}$$

When the communicating vessels are curved and the cross-section area is variable the displacements of different particles of the fluid are different and therefore the simple equation (125.3) no longer applies.

To investigate the natural oscillations of the fluid in the communicating vessels shown in Fig. 352*a* it is expedient to use the law of conservation of energy under the assumption that the whole mass of the fluid is in a harmonic vibration with some frequency ω and with a very small amplitude, the displacement being dependent on the cross-section area of the fluid. The greater the cross-section area S, the smaller is the displacement of the particle of the fluid in that cross section, and vice versa. Let the cross-section area S be a known function $S(s)$ of the coordinate s reckoned along the axis (see Fig. 352*a*). The mass of the whole fluid is $m = \int \rho S\, ds$ where the integral is taken over the whole length l and ρ is the density of the fluid. We assume that the amplitude of the vibration is so small that the cross-section area can be considered practically invariable as the particle of the fluid in the cross section receives a displacement not exceeding twice the amplitude. Then denoting by S_1 and S_2 the cross-section areas at the levels of the free surface of the fluid in the right-hand and in the left-hand vessel respectively we can write

$$S_1 a = S_2 b \tag{125.5}$$

where a is the amplitude of the vibration in the right-hand vessel and b is the amplitude in the left-hand vessel. Let the coordinate s be reckoned so that the value $s = 0$ corresponds to the position of the free surface in the left-hand vessel when the fluid is in equilibrium, the corresponding position of the free surface in the right-hand vessel being $s = l$. Then the amplitude A of the vibration in the section s with area $S(s)$ can be found from the condition

$$aS_1 = AS(s)$$

It follows that the amplitude of the velocity v in the section with coordinate s is

$$v = \omega A = \omega a \frac{S_1}{S(s)}$$

The moment the particles of the fluid pass through their equilibrium positions the total energy of the fluid becomes equal to its kinetic energy:

$$E_{\text{total}} = E_{\text{kin}} = \int_0^l \frac{\rho S v^2\, ds}{2} = \frac{\rho \omega^2 a^2 S_1^2}{2} \int_0^l \frac{ds}{S} \tag{125.6}$$

When the quarter period elapses after that moment the total energy becomes equal to the potential energy, the latter being equal to the work which should be performed in order to lift the volume $S_1 a = S_2 b$ of the fluid (see Fig. 352*a*) by a distance of

$$\frac{a+b}{2}$$

Indeed, the displacement of the fluid from the equilibrium position can be thought of as if all the particles remained in the invariable positions except those in the volume $S_1 a$ which goes into the volume $S_2 b$. The potential energy is therefore expressed as

$$E_{\text{pot}} = \rho S_1 a \frac{a+b}{2} g$$

By (125.5), we have $b = \frac{S_1}{S_2} a$ and therefore

$$E_{\text{pot}} = \frac{\rho S_1 a^2 g}{2} \left(1 + \frac{S_1}{S_2}\right) \tag{125.7}$$

On equating (125.6) to (125.7) we obtain

$$\frac{\rho \omega^2 a^2 S_1^2}{2} \int_0^l \frac{ds}{S} = \frac{\rho S_1 a^2 g}{2} \left(1 + \frac{S_1}{S_2}\right)$$

whence

$$\omega^2 = \frac{g \left(1 + \frac{S_1}{S_2}\right)}{S_1 \int_0^l \frac{ds}{S}} \tag{125.8}$$

For $S = \text{const}$ expression (125.8) goes into formula (125.4) determining the circular frequency of the vibration of the fluid in the communicating vessels with constant cross-section area.

Example 2. Let us determine the natural frequency of the oscillation of a load suspended from a heavy elastic spring. Such a system is shown in Fig. 348*a*; the weight of the spring is assumed to be of the same order as the weight of the load. Formula (124.24) expressing the natural period of the oscillation no longer applies to this case because it was derived under the assumption that the weight of the spring was negligible. To determine the period in the case of a homogeneous heavy spring we can use the law of conservation of energy. We suppose that the load is in a harmonic oscillatory motion with small amplitude a and frequency ω. Then the coil of the spring which is at a distance of y from the point of suspension in its equilibrium state oscillates with amplitude

$$A = \frac{y}{l} a$$

where l is the length of the whole spring in the equilibrium state. Let the number of the coils of the spring be N; then the amplitude of the ith coil (the coils are counted downwards from the point of suspension) is

$$A_i = \frac{ia}{N}$$

When the load passes through the equilibrium position the kinetic energy of the spring E_{kin}^0 is expressed as the sum

$$E_{\text{kin}}^0 = \frac{1}{2} \sum_{i=1}^{N} \frac{m_{\text{spr}}}{N} \omega^2 A_i^2 = \frac{m_{\text{spr}}}{2N} \frac{\omega^2 a^2}{N^2} \sum_{i=1}^{N} i^2 = \frac{m_{\text{spr}} \omega^2 a^2}{2N^3} \frac{N(N+1)(2N+1)}{6} \tag{125.9}$$

where m_{spr} is the mass of the spring.

If $N \gg 1$ then

$$E_{\text{kin}}^0 \approx \frac{1}{2} \frac{m_{\text{spr}}}{3} \omega^2 a^2$$

In this case the total kinetic energy of the system "the load and the spring" when it passes through the equilibrium is equal to

$$E_{\text{kin}} = \frac{1}{2} m a^2 \omega^2 + \frac{1}{2} \frac{m_{\text{spr}}}{3} a^2 \omega^2 = \frac{1}{2} \left(m + \frac{m_{\text{spr}}}{3}\right) a^2 \omega^2$$

The moment the spring receives the maximum elongation its potential energy becomes

$$E_{\text{pot}} = \frac{ka^2}{2}$$

where k is the stiffness factor of the spring. The equality $E_{\text{pot}} = E_{\text{kin}}$ implies

$$ka^2 = \left(m + \frac{m_{\text{spr}}}{3}\right) a^2\omega^2$$

whence

$$\omega^2 = \frac{k}{\left(m + \frac{m_{\text{spr}}}{3}\right)} \tag{125.10}$$

We see that in order to take into account the weight of the spring and to obtain a more precise formula for the natural period of the vibration of the load suspended from the spring we must add 1/3 of the mass of the spring to the mass of the load. It is evident that when the mass of the spring is very small in comparison with that of the load the second term in the denominator in formula (125.10) can be neglected, which gives us the previous result concerning the case of a weightless spring. If the number of the coils of the spring is not very large then, when determining the frequency of the vibration, we should use formula (125.9).

The energy comparison method can readily be applied to the determination of the natural frequency of a given system whose different points oscillate with different amplitudes when the amplitudes are connected by a definite condition which makes it possible to determine the amplitudes of all points knowing the amplitude of one of them.

126. Damped Natural Oscillations

The simplest experiments with vibrating bodies, such as a pendulum, an areometer floating in a fluid, show that the oscillations arising after an initial perturbation are gradually damped and the oscillating body eventually returns to the state of equilibrium. This is due to the fact that the motion of any body is accompanied by the action of friction forces, which results in a gradual transformation of the initial mechanical energy imparted to the body into heat. That is why a real oscillation process can only be regarded as an undamped harmonic vibration when the energy loss is very small during the time of observation. In the general case the dependence of the friction forces on the velocity is rather complex. However, when the velocity is small in its absolute value we can assume with a sufficient accuracy that the friction forces are proportional to the velocity of the motion (see Sec. 39). Therefore, the equation which describes the motion of a load suspended from a weightless spring (see Sec. 124) and takes into account the friction force can be written in the form

$$m\ddot{x} = -kx - h\dot{x} \tag{126.1}$$

where $h\dot{x}$ is the friction force and h is a constant coefficient. The solution of equation (126.1) has the form

$$x = Ae^{-\delta t}\cos(\omega_1 t + \varphi) \tag{126.2}$$

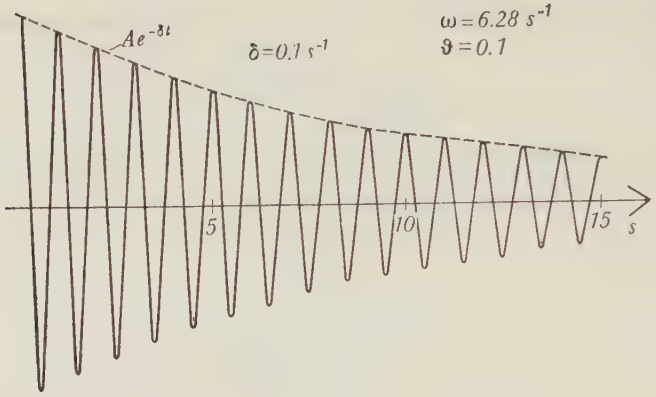

FIG. 353

where A and φ are constants depending on the initial conditions,

$$\delta = \frac{h}{2m} \tag{126.3}$$

and

$$\omega_1 = \sqrt{\frac{k}{m} - \frac{h^2}{4m^2}} \tag{126.4}$$

The coordinate x describing the motion is represented as the product of the exponential function $e^{-\delta t}$ (which tends to zero as $t \to \infty$) by the periodic function $\cos(\omega_1 t + \varphi)$ whose period is

$$T_1 = \frac{2\pi}{\omega_1}$$

The quantity T_1 can be called conditionally as the *period of the damped vibration* described by formula (126.2).

The motion of the load is a *damped* (*harmonic*) *oscillation*; its graph is shown in Fig. 353. As time t increases the amplitude of the oscillation reduces; the graph does not fall outside the region bounded by the curves $\pm Ae^{-\delta t}$. The coefficient $\delta = \frac{h}{2m}$ characterizing the rate of damping is called the *damping factor*. It is equal to the ratio of the coefficient h to twice the mass of the vibrating body.

Let us take two loads of identical spherical shape and different masses, for instance made of lead and cork, suspend them from identical springs, observe their vibrations and measure the time intervals during which the amplitudes of the vibrations decrease a definite number of times. The mass of the lead ball being approximately 50 times that of the cork ball, the damping factor of the latter ball is approximately 50 times as great as that of the former. Therefore, the time interval during which the amplitude of the lead load decreases, say twice, is about 50 times as great as the corresponding time interval for the other load.

A process described by a formula of type (126.2) starts at a certain initial instant and, theoretically, it lasts indefinitely. Therefore, in order to estimate practically the (conditional) duration of a process of this kind the notion of the *relaxation time* is introduced by which is meant the quantity $\tau = \frac{1}{\delta}$ having the

dimension of time. During time τ the deviation of the system in question from the equilibrium position in the vibration process decreases e times ($e \approx 2.73$).

The damping factor δ and the relaxation time τ do not characterize completely the vibration system because for a given time τ the system may have a different number of vibrations depending on its period. Therefore, to estimate the damping in the system and its connection with the number of the vibrations the *logarithmic decrement* (or, simply, the *decrement*) is used instead of the damping factor. The decrement is defined as the dimensionless quantity

$$\vartheta = \frac{T}{\tau} = \delta T$$

where the quantity T entering into this expression is equal to the period T_1 of the damped vibration.

The quantity

$$\frac{1}{\vartheta} = \frac{\tau}{T} = N$$

which is the reciprocal of the decrement indicates the number of vibrations of the system for which the amplitude of the vibration decreases e times. For instance, if the decrement is equal to 1/10 this means that the amplitude becomes approximately three times as small during 10 vibrations.

The magnitude of the decrement ϑ can be found experimentally as follows. Let x_1 be the deviation from the equilibrium state measured at an instant t_1. Then at time $t_2 = t_1 + T_1$ (where T_1 is the period of the damped vibration) the deviation will be

$$x_2 = x_1 e^{-\delta T_1}$$

Indeed, we have

$$x_1 = Ae^{-\delta t_1} \cos(\omega_1 t_1 + \varphi)$$

and

$$x_2 = Ae^{-\delta t_1 - \delta T_1} \cos(\omega_1 t_1 + \varphi + 2\pi) = x_1 e^{-\delta T_1}$$

because

$$\omega_1 (t_1 + T_1) = \omega_1 t_1 + \omega_1 T_1 = \omega_1 t_1 + 2\pi$$

Since $\vartheta = \delta T_1$ the ratio of the deviations is

$$\frac{x_2}{x_1} = e^{-\vartheta}$$

whence

$$\ln \frac{x_1}{x_2} = \vartheta \qquad (126.5)$$

Formula (126.5) explains why the quantity ϑ is called the logarithmic decrement. Since the time interval between two successive extreme deviations of one sign from the equilibrium position is equal to T_1 we can say that *the decrement ϑ is equal to the natural logarithm of the ratio of the magnitudes of two successive extreme deviations having one sign.*

Let x_N be the deviation at time $t_N = NT_1$, that is the deviation appearing after N vibrations following the deviation x_1. Then, arguing as above, we obtain

$$\frac{x_N}{x_1} = e^{-N\vartheta}$$

whence

$$\ln \frac{x_1}{x_N} = N\vartheta, \quad \text{that is} \quad \vartheta = \frac{1}{N} \ln \frac{x_1}{x_N} \tag{126.6}$$

In Fig. 353 is shown the experimental curve obtained in the recording of the vibration of a pendulum consisting of a ball suspended from a thread. This curve makes it possible to find the logarithmic decrement of the pendulum which turns out to be equal to 0.0159. The decrement of a pendulum with the same

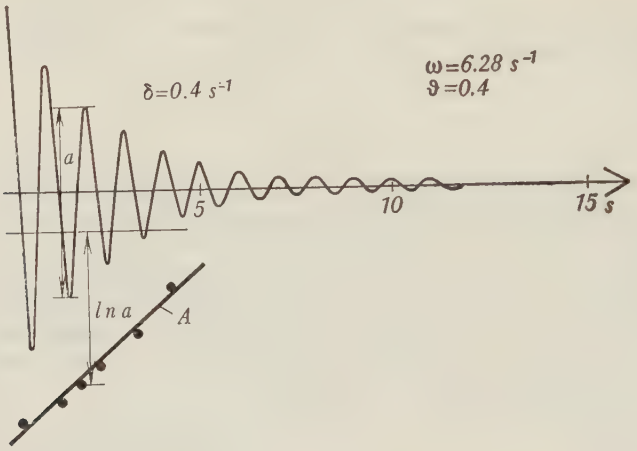

FIG. 354

length and with a ball of the same diameter whose mass is four times as small as the mass of the former pendulum is equal to 0.4. The curve obtained in the recording of the oscillation of the latter pendulum is shown in Fig. 354, the corresponding values of the damping factors being indicated in Figs. 353 and 354.

The data obtained in the recording of a damped vibration readily confirm the validity of the assumptions made in the derivation of formula (126.2). For instance, taking the diagram in Fig. 354 we set the vertical line segments proportional to the logarithms of the absolute values of the amplitudes of the vibration for each period T_1 and draw a line through the ends of these segments (in Fig. 354 the segments are set downwards from a definite horizontal level). This line is denoted A in the figure and it turns out to be a straight line.

Since all the points of line A are in one straight line we see that the theoretical assumptions laid in the foundation of the derivation of formula (126.2) describing the law of the damped vibrations prove valid in the case under consideration. When the data of an experiment of the type we have described are processed as has been described it is usually known that the mass of the vibrating body remains constant and that the restoring force is proportional to the displacement from the equilibrium position (the latter fact can easily be checked experimentally). The investigation of the character of the friction forces is a more intricate problem. If it turns out that the points of line A constructed in the above manner do not lie in a straight line this means that the friction force in

the vibration process is not proportional to the velocity and obeys some other more complex law. In the case when the points of line A are in one straight line the measurement of the angle of the inclination of this line to the axis of abscissas makes it possible to determine the decrement.

When a vibration process is accompanied by the sliding of one body on the surface of another body without lubrication the character of the vibration is essentially affected by the force of dry friction. The force of dry friction is almost constant in its magnitude, its direction being opposite to that of the velocity.

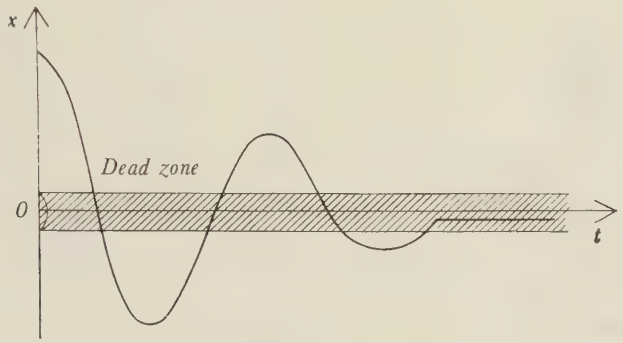

FIG. 355

When the force of dry friction dominates the force of viscous friction the curve obtained in the recording of the vibration is of the type shown in Fig. 355. The shaded region in the figure represents the "dead zone": when the body falls into this zone the elastic restoring force is less than the friction force and the body therefore remains at rest and no vibration arises.

127. Forced Oscillation and Resonance

In contrast to a natural oscillation, a *forced oscillation* arises under the action of a periodic external force. For instance, when an electromagnet whose winding is supplied by an alternating electric current J is placed near a steel plate clamped in a bench vice (see Fig. 356) the plate undergoes a forced periodic vibration. The vibration can be detected by watching the light ray falling on the screen after it has been reflected from the mirror attached to the plate.

The frequency of a forced oscillation always coincides with that of the external force. If we change the frequency of the electric current supplying the electromagnet the frequency of the oscillation of the plate also changes. Using a stroboscopic tube we can readily check that the frequencies of the alternating electric current and of the vibrating plate coincide.

To this end we observe the vibration of the plate illuminated by a neon-filled gas discharge tube whose luminous intensity oscillates together with the oscillation of the alternating electric current J (see Fig. 357). The plate is then seen as if it were motionless because during every period of the oscillation the plate is illuminated exactly once in the same position. If the oscillations of the plate and of the alternating current were not synchronous, that is if the frequencies of the plate and of the current did not coincide, we should see a blurred picture of the vibrating plate.

Forced vibrations often arise in machines with rotating or periodically moving components. For instance, a uniformly rotating flywheel (see Fig. 358) always produces forced vibrations of the shaft and of the bearings supporting the shaft. The matter is that the flywheel is always slightly unbalanced because its centre of gravity O' does not lie exactly on the axis passing through the centres of the bearings. The rotation of the flywheel is therefore accompanied by the appearance of the centrifugal force of magnitude $F = m\rho\omega^2$* which causes the vibration of the shaft, the number of vibrations per second being equal to the number of

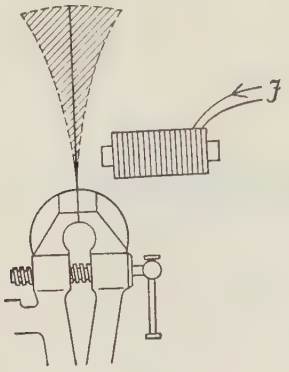

FIG. 356

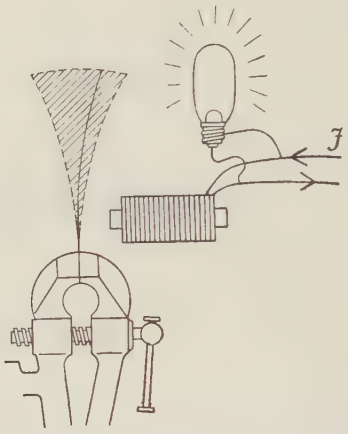

FIG. 357

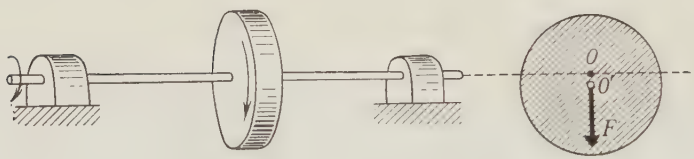

FIG. 358

revolutions per second (here ρ is the distance between the centre of gravity of the flywheel O' and the axis of the bearings, m is the mass of the flywheel, and ω is the angular speed of rotation). Indeed, as is readily seen, the projection of this centrifugal force on the horizontal is a harmonic force with frequency ω.

The vibration of the foundation of a high-power electric motor is another example of a forced oscillation caused by the unbalance of the rotor of the motor. One more example is the vibration produced by an internal-combustion engine whose piston is in a periodic reciprocal motion, which causes a periodic force producing the vibration.

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* Here by the centrifugal force is meant the reaction force applied to the shaft which (according to the third law of dynamics) corresponds to the centripetal force.

It turns out that the amplitude of a forced oscillation depends not only on the magnitude of the external force but also on its frequency. The amplitude grows sharply when the frequency of the external force is close to the natural frequency of the system on which the force acts.

Taking a system of this kind we can vary gradually the frequency of the external force (the amplitude of the force is kept invariable) and measure the corresponding amplitude of the forced oscillation. The diagram representing the results of the measurements can be drawn as is shown in Fig. 359 where along the axis of abscissas is set the ratio of the frequency of the external force p to the natural frequency ω and along the axis of ordinates is set the dimensionless quantity $\frac{Bk}{F_0}$ (B is the amplitude of the forced oscillation, F_0 is the amplitude of the periodic external force and k is a characteristic of the elastic properties of the system; see Sec. 128. This diagram is called the *resonance characteristic* (or the *resonance curve*). The phenomenon in which the amplitude of the vibration increases sharply when the frequency of the external force approaches the natural frequency is called *resonance*.

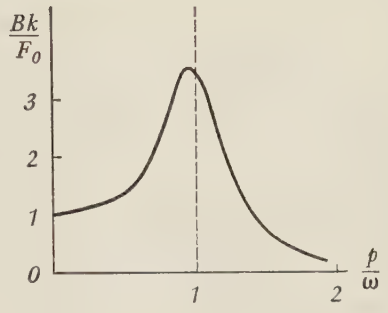

FIG. 359

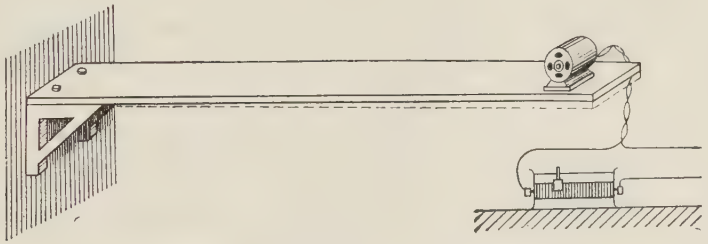

FIG. 360

Let us consider an example demonstrating resonance. The scheme of the experiment is shown in Fig. 360: an electric motor is attached to one end of a board whose other end is rigidly fixed. The angular speed of the rotor of the motor is gradually increased with the aid of a rheostat. When the speed is low the board vibrates very slightly. As the angular speed is increased the amplitude of the vertical vibration of the board sharply increases and can easily be observed for a certain range of the values of the speed. The further increase of the speed of the rotation of the motor leads to a decrease of the amplitude of the vibration. The moment the amplitude increases resonance sets in. If we stop the motor and strike it so that the whole system (the board together with the motor) is in a natural vibration the measurement shows that the resonant frequency is close to the natural frequency.

It should be noted that in this experiment the amplitude of the external force is not constant and increases proportionally to the square of the number of revolutions per second of the rotor because this external force is the centrifugal force whose magnitude is proportional to the square of the angular velocity. Nevertheless, the amplitude of the vibration of the board sharply reduces as the number of the revolutions per second becomes greater than the natural frequency (which is close to the resonant frequency).

128. Dependence of the Amplitude of a Forced Oscillation on the Frequency

Let us analyse the physical nature of the increase of the amplitude in a resonance phenomenon.

It turns out that, qualitatively, a resonant vibration of a system resembles very much its natural vibration, and the external force only supports the vibration. As in a natural vibration, the restoring force in a resonant vibration produces the necessary acceleration of the vibrating mass while the external force only balances the force of friction. On the contrary, when the frequency of the external force differs considerably from the resonant frequency its role no longer reduces to balancing the force of friction, which results in the decrease of the amplitude of the vibration.

To elucidate what has been said, let us consider the vibration of a simple pendulum under the action of an external harmonic force and subject the phenomenon to a thorough theoretical analysis. Let the mass m of the pendulum be acted upon by a horizontal force $F = F_0 \cos pt$ where F_0 is the amplitude of the force and p is its circular frequency. To set the equation of motion of the pendulum we must add the force F to the sum of the restoring force $-mg\frac{x}{l}$ (see (124.6)) and the friction force $-h\dot{x}$ (see (126.1)). The equation describing the motion of the load of the pendulum for small angles of deviation from the equilibrium position therefore has the form

$$m\ddot{x} = -\frac{mg}{l}\,x - h\dot{x} + F_0 \cos pt \qquad (128.1)$$

where x is the (linear) displacement of the load.* On denoting $\frac{h}{2m}$ by δ (δ is the damping factor) and $\sqrt{\frac{g}{l}}$ by ω (ω is the natural frequency) we rewrite the equation of motion in the form

$$\ddot{x} + 2\delta\dot{x} + \omega^2 x = \frac{F_0}{m}\cos pt \qquad (128.2)$$

The experiment shows that the forced oscillation of the pendulum is a harmonic oscillation with frequency p. The coordinate x as function of time t can be written in the form

$$x = B \cos (pt + \varphi) \qquad (128.3)$$

---

* To describe the forced vibrations of a load suspended from a spring whose natural oscillation is described by equation (126.1) it suffices to replace the restoring force $-\frac{mg}{l}x$ in equation (128.1) by the restoring force $-kx$ of the spring.

where B and φ are the amplitude and the initial phase of the oscillation, respectively. Our immediate aim is to determine the amplitude B and the initial phase φ of the forced oscillation. On differentiating (128.3) twice and finding $\dot{x}$ and $\ddot{x}$ we substitute the expressions obtained and (128.3) in (128.2), which results in

$$B\{(\omega^2-p^2)\cos(pt+\varphi)-2p\delta\sin(pt+\varphi)\}=\frac{F_0}{m}\cos pt$$

On transforming the last relation using the well-known trigonometric formulas we obtain

$$\left\{B[(\omega^2-p^2)\cos\varphi-2p\delta\sin\varphi]-\frac{F_0}{m}\right\}\cos pt+$$
$$+B[-(\omega^2-p^2)\sin\varphi-2p\delta\cos\varphi]\sin pt=0 \qquad (128.4)$$

Relation (128.4) involves a sum of two harmonic terms and can be written in the form

$$a\cos pt+b\sin pt=0 \qquad (128.5)$$

where a and b are constants.
It is obvious that for relation (128.5) to be satisfied for any t it is necessary and sufficient that both constants a and b should be equal to zero. Therefore (128.4) implies

$$\left.\begin{aligned}B[(\omega^2-p^2)\cos\varphi-2\delta p\sin\varphi]&=\frac{F_0}{m}\\(\omega^2-p^2)\sin\varphi+2\delta p\cos\varphi&=0\end{aligned}\right\} \qquad (128.6)$$

The quantities B and φ can now be found from the two simultaneous equations (128.6). The initial phase φ is readily found from the second equation:

$$\varphi=\arctan\frac{2\delta p}{p^2-\omega^2} \qquad (128.7)$$

We now divide the first equation (128.6) by B and multiply it by $\sin\varphi$; the second equation is multiplied by $\cos\varphi$ and then subtracted from the former, which yields

$$\frac{F_0}{mB}\sin\varphi=-2\delta p \qquad (128.8)$$

Let us substitute $\sin\varphi=-\frac{2\delta pmB}{F_0}$ in the second equation (128.6), which results in

$$\cos\varphi=\frac{mB}{F_0}(\omega^2-p^2) \qquad (128.9)$$

On squaring equalities (128.8) and (128.9) and adding them together we obtain

$$\frac{m^2B^2}{F_0^2}(\omega^2-p^2)^2+\frac{4\delta^2p^2m^2B^2}{F_0^2}=1 \qquad (128.10)$$

Finally, after a transformation, the amplitude of the forced oscillation B is found from relation (128.10):

$$B=\frac{F_0}{m}\frac{1}{\sqrt{(\omega^2-p^2)^2+4\delta^2p^2}} \qquad (128.11)$$

This formula shows that the amplitude of the forced oscillation is proportional to the amplitude F_0 of the external force and that it depends in a complicated manner on the frequency p. When the damping is small (that is when δ is small relative to ω) the square root in the denominator in formula (128.11) has a minimum in the vicinity of the value $p = \omega$ and therefore for a value of the frequency p close to ω the amplitude of the vibration B assumes its maximum.

If the load of the pendulum is acted upon by a constant force of magnitude F_0 it receives a static displacement B_{st}. The displacement B_{st} is readily found from the equality

$$\frac{mg}{l} \cdot B_{st} = F_0$$

which yields

$$B_{st} = \frac{F_0 l}{mg} \tag{128.12}$$

The same result can be obtained by putting $p = 0$ in formula (128.11) and taking into account that the natural frequency of the pendulum is given by the formula $\omega = \sqrt{\frac{g}{l}}$. For $p \to \infty$, that is for a very high frequency, the amplitude of the vibration tends to zero.

Let us consider in more detail the relationship between the various forces taking part in the forced oscillations as the frequency p varies; this will elucidate the cause of the change of the amplitude of the vibration depending on frequency.

Equation (128.1) expresses the fact that the product of the mass by the acceleration is equal to the sum of the following three forces: the restoring force, the force of friction, and the external force. In the forced vibration process all the three forces are in harmonic oscillations with frequency p.

Let us rewrite the basic relationship between the forces expressed by equation (128.4) in the form

$$mB(\omega^2 - p^2)\cos(pt + \varphi) - 2\delta pmB \sin(pt + \varphi) = F_0 \cos pt \tag{128.13}$$

It can readily be verified that the work of the force equal to the sum of the restoring force $\frac{mg}{l} B \cos(pt + \varphi)$ (which can be written as $m\omega^2 B \cos\left(pt + \varphi\right.$ because $\left.\omega^2 = \frac{g}{l}\right)$ and the product $m\ddot{x} = -mp^2 B \cos(pt + \varphi)$ of the mass of the load by the acceleration during the period of the forced oscillation is equal to zero (this sum is equal to the first term on the left-hand side of (128.13)). The second term on the left-hand side of (128.13) is the force of friction. Hence, equation (128.13) shows that the external force F is balanced by the resultant of the friction force and the above mentioned sum of the two forces.

When the frequency p is very small ($p \to 0$, i.e. $p \ll \omega$) the force of friction and the product of the mass by the acceleration are very small and therefore in this case the external force $F_0 \cos pt$ is mainly balanced by the restoring force $mB\omega^2 \cos(pt + \varphi)$, which implies that $\varphi \approx 0$. Consequently, for small frequencies the vibrations of the external force and of the displacement x are cophasal, and the system oscillates as if it were almost inertia-free, the force of friction were absent and only the restoring force were present.

When the frequency is very high ($p \to \infty$, i.e. $p \gg \omega$) the term $-mBp^2 \cos(pt + \varphi)$ predominates over all the other terms on the left-hand side of (128.13). The product of the mass by the acceleration becomes much greater than the restoring force and the friction force. Practically, in this case only the external

force specifies the acceleration and we can therefore approximately write

$$-mBp^2 \cos(pt + \varphi) \approx F_0 \cos pt \tag{128.14}$$

whence $\varphi \approx 180°$ and $B \approx \frac{F_0}{mp^2}$. The same result is obtained from (128.11) when $p \gg \omega$ and $p \gg \delta$. Thus, for large frequencies of the forced oscillations the displacement x and the external force F are in opposite phases and the pendulum oscillates as if the external force $F = F_0 \cos pt$ were applied to a free particle of mass m.

Summing up, we can say that for large frequencies the dominant role is played by the mass and for small frequencies by the restoring force.

We now consider the case of intermediate values of the frequency of the external force. For $p = \omega$ the first term on the left-hand side of (128.13) is equal to zero and the external force is balanced by the friction force solely:

$$-hpB \sin(pt + \varphi) = F_0 \cos pt \tag{128.15}$$

For equality (128.15) to hold for any t it is necessary and sufficient that $\varphi = -90°$, that is $x = B \cos(pt - 90°)$. This means that in a resonant vibration (for $p = \omega$) there is always a phase lag of 90° between the displacement and the external force. In resonance the dominant role is played by the friction force. If we neglected the force of friction, that is if we put $h = 0$, this would imply the incorrect conclusion that the amplitude of the vibration B turns into infinity for $p = \omega$. (This incorrect conclusion would follow for $h = \delta = 0$ and $p = \omega$ both from formula (128.11) and from equality (128.13); obviously, it is physically inconsistent.) In reality, when the friction force is sufficiently small (that is $\delta \neq 0$ and $p \approx 0$) the amplitude of the resonant vibration becomes very large but not infinite. For $p = \omega$, by (128.13), the amplitude B_{res} of the resonant vibration is expressed by the formula

$$B_{\text{res}} = \frac{F_0}{2\delta m\omega} = \frac{F_0}{h\omega} \tag{128.16}$$

Formula (128.16) shows that for small values of h the amplitude of the resonant oscillation can be very large. Resonant oscillations with large amplitude can be very dangerous for vibrating parts of machines and other constructions; to avoid the possible rupture they must be designed so that no resonance sets in.

The work performed by the external force during the period of the forced oscillation is always equal to the work of the friction forces during the period while all the other forces perform zero work. In the case of stationary forced oscillations the work of the external force is expended on heat.

According to (128.12), for low frequencies the amplitude of the forced oscillation B is given by the approximate relation

$$B \approx B_{\text{st}} = \frac{F_0 l}{mg} = \frac{F_0}{k_0} \tag{128.17}$$

where $k_0 = \frac{mg}{l}$. It is specified by the amplitude F_0 of the external force and by the coefficient k_0 characterizing the restoring force and is independent of the force of friction. For forced oscillations of a load of mass m suspended from a spring with stiffness factor k (as was mentioned at the beginning of this section, in this case we should replace $k_0 = \frac{mg}{l}$ by k in (128.1)) the quantity k is independent of m and, consequently, when the frequency is low the amplitude B does not depend on the mass m either.

According to (128.14), for high frequencies of the forced oscillations the amplitude is given by the approximate formula

$$B \approx \frac{F_0}{mp^2} \tag{128.18}$$

Formula (128.18) indicates that in this case B is inversely proportional to p^2; the amplitude depends on the frequency of the oscillation and on the mass and is almost independent of the restoring force and of the force of friction.

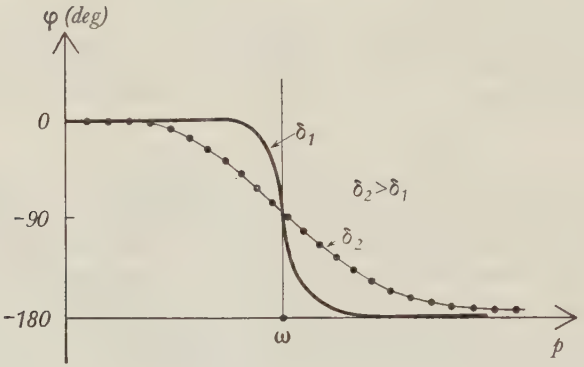

FIG. 361

As was already mentioned (see (128.16)), the amplitude of the resonant oscillation depends on the coefficient h characterizing the friction and on the natural frequency ω (it is inversely proportional to $h\omega$).

It should be once again stressed that *the amplitude of a forced oscillation is always proportional to the amplitude of the harmonic external force.*

Formula (128.7) shows that the phase shift (also called the phase displacement) between the coordinate x and the external force F depends on frequency; this dependence is represented approximately by the diagram in Fig. 361 (where φ is measured in degrees). For low frequencies ($p \ll \omega$) the oscillations of the displacement x and of the force F are cophasal, for resonant oscillations ($p \approx \omega$) there is a phase lag of 90° between the displacement and the force, and for very high frequencies ($p \gg \omega$) the oscillations of the displacement and of the force are in opposite phases. What has been said is in a complete agreement with the results of the previous analysis of the roles of the forces taking part in a forced oscillation process. Thus, all the features of the forced oscillations are specified by the relation between p and ω.

For a very small friction force ($\delta \approx 0$) the resonance peak (see the resonance curve in Fig. 359) becomes sharper and the amplitude increases fast in the vicinity of the resonant frequency. The phase of the oscillations also varies sharply near the resonant frequency when the damping factor is small (see Fig. 361).

The laws governing the forced oscillations have been derived for the special case of a simple pendulum. It is however evident that they also apply to any vibration system which can be described by an equation reducible to form (128.2). Examples of this kind are the vibrations under the action of a harmonic external force of a load suspended from a spring, of an areometer immersed in a fluid, of a torsional oscillation of a body (such as the pendulum of a watch).

129. Vibration of a Shaft with Disk

The construction of many machines (for instance, a steam turbine) involves rotating shafts with disks (e.g. a shaft carrying a flywheel). A simple model of this kind is depicted in Fig. 362 where there is a vertical needle which can rotate and on which a disk is fixed. Such a disk is always slightly unbalanced, that is its centre of mass is not exactly on the axis of the needle. For the sake of visuality the unbalance is strengthened by a small load of mass m' placed on the peri-

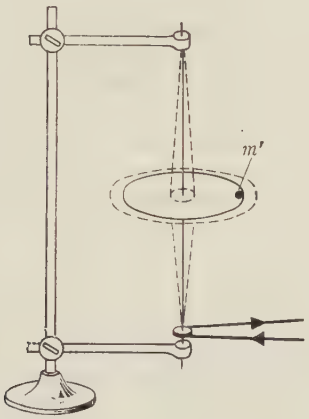

FIG. 362

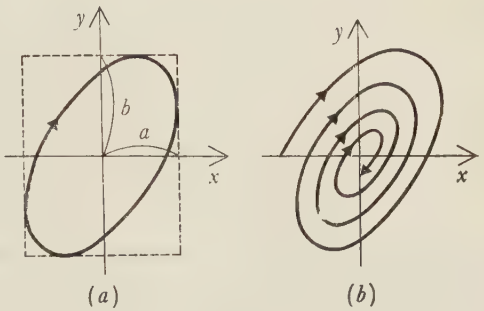

FIG. 363

phery of the disk (see Fig. 362). If we strike the needle in a direction perpendicular to it a natural vibration sets in whose frequency ω depends on the flexural rigidity of the needle and on the mass M of the disk. Let us assume that when the disk is given a displacement x in a direction perpendicular to the needle there arises the restoring elastic force kx; then the natural frequency of the oscillation of the needle with the disk resulting from the bending of the needle is given by the equality

$$\omega = \sqrt{\frac{k}{M}} \tag{129.1}$$

In a natural oscillation with constant frequency ω the disk can move in any direction lying in the plane passing through the centre of the disk and perpendicular to the needle. On choosing a coordinate system with axes x and y in that plane we can resolve an arbitrary natural oscillation of the disk into two oscillations with frequency ω in the directions of the x-axis and y-axis; in the general case the centre of the disk moves in an ellipse as shown in Fig. 363. In the state of equilibrium the centre of the disk coincides with the origin in the xy-plane (which is perpendicular to the axis of the needle). The harmonic oscillation of the centre along the x-axis is described by an equation of the form

$$x = a \cos \omega t \tag{129.2}$$

and along the y-axis by an equation

$$y = b \cos (\omega t + \varphi) \tag{129.3}$$

where a, b, and φ are constants, and the frequency of the oscillations ω is the same in both equations (φ is the phase shift between the coordinates x and y). On eliminating ωt from (129.2) and (129.3) we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2\cos\varphi}{ab} xy = \sin^2\varphi \tag{129.4}$$

which is the equation of an ellipse. Hence, the superposition of the two harmonic oscillations with the same frequency in the two mutually perpendicular directions results in a motion along an ellipse lying within a rectangle with sides $2a$ and $2b$ (see Fig. 363*a*). The shape of the ellipse depends on the phase displacement between the constituent oscillations. For $\varphi = 90°$ and $\varphi = 270°$ the semimajor and semiminor axes of the ellipse lie on the coordinate axes; for $\varphi = 0°$ and $\varphi = 180°$ the ellipse degenerates into a straight line going along the diagonal of the rectangle with sides $2a$ and $2b$. In case $a = b$ and $\varphi = 90°$ the ellipse turns into a circle.

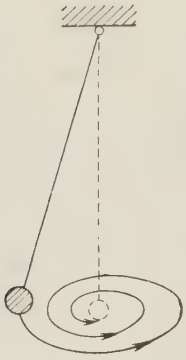

FIG. 364

However, the real motion is not the one described by formulas (129.2) and (129.3) because there always exists damping which causes the centre of the disk to move in a spiral (Fig. 363*b*). When the damping is small the spiral is close to an ellipse or, in a special case, to a circle.

The trajectory of a load suspended from a thread (Fig. 364) is of the same shape when the load is displaced from the equilibrium, given an initial jerk and then allowed to oscillate on its own. This is the *conical pendulum*. In the absence of friction (damping) its thread is always on the surface of a cone; in the presence of damping the load of the pendulum moves in a spiral. In the former case the motion of the load of the conical pendulum can be regarded as a superposition of two simultaneous harmonic vibrations with a common (constant) frequency in two mutually perpendicular directions.

Let us come back to the forced oscillation of the rotating disk on the needle. If we suppose temporarily that there is no oscillation and that the disk of mass M rotates with angular speed p then the centrifugal force (see the footnote on page 440) is equal to

$$F_c = MR_0p^2 \tag{129.5}$$

where R_0 is the distance from the centre of gravity of the disk to the axis of rotation, the force F_c acting on the axis of rotation of the disk. The direction of this force permanently changes: it rotates with angular velocity p all the time during the rotation of the disk. Let the centre of gravity be on the y-axis at the initial instant $t = 0$; then the force acting along the x-axis is

$$F_c \sin pt$$

and the force acting along the y-axis is

$$F_c \cos pt$$

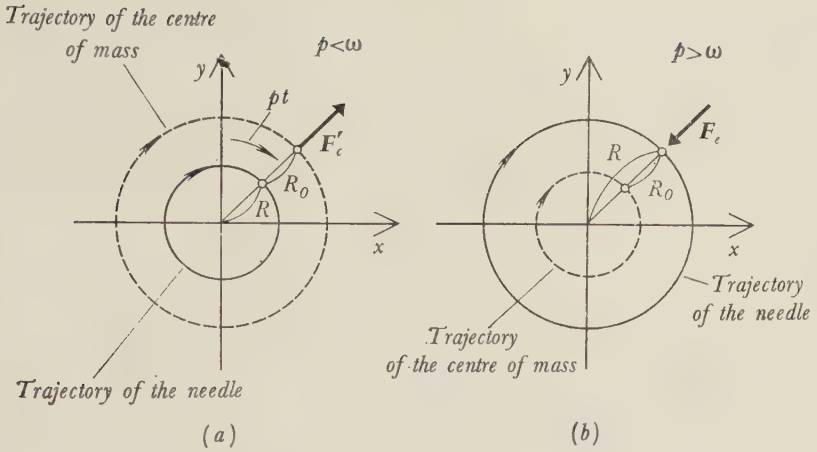

FIG. 365

Under the action of these forces there simultaneously arise two vibrations of the disk: one along the x-axis and the other along the y-axis. Since the amplitudes of these forces are equal and the elastic properties of the needle are the same in all the directions the vibrations along the two mutually perpendicular directions have equal amplitudes. This means that under the action of the centrifugal force the centre of the needle will be in a rotational motion along a circle with angular velocity p. When the angular speed is very small the oscillation of the needle is such that the displacement of the centre of gravity of the disk is greater

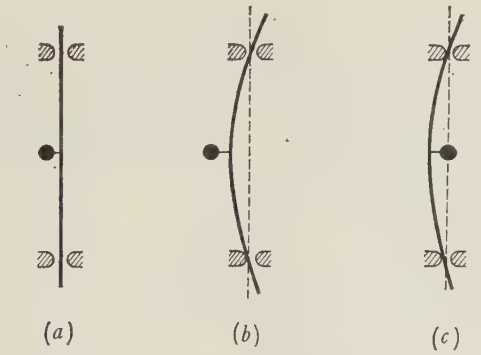

FIG. 366

than that of the centre of the needle (coinciding with the geometrical centre of the disk). In this case the centre of the needle moves in a circle of radius R as is shown in Fig. 365*a*. The needle undergoes bending and its centre departs by a distance of R from the axis of rotation; the centre of gravity of the disk is in this case farther from the origin and is placed approximately on the same radius. The centre of the disk departs from the axis of rotation owing to the bending of

the needle and therefore the centrifugal force depends on the deflection of the centre of the needle, that is on the amplitude of the oscillation of the needle. When the frequency is very small, i.e. $p \ll \omega$, the needle bends until the centrifugal force is balanced by the elastic force caused by the bending of the needle.

When the angular velocity of the disk is very high (that is $p \gg \omega$) the centre of the needle moves in a circle as shown in Fig. 365*b*; in this case the deviation of the centre of the needle is always greater than that of the centre of the disk. The distance R_0 between the centre of gravity of the disk and the geometrical centre of the disk is supposed to be very small. Therefore, the amplitude of the oscillation of the needle is also very small. As the angular velocity is increased the centre of mass approaches the origin and the amplitude of the oscillation of the needle becomes closer to R_0. If the angular velocity of the rotation is very high the oscillation of the needle is very small, which can easily be observed when the disk shown in Fig. 362 is rotated very fast. The validity of our conclusions can readily be checked experimentally by observing the rotating disk when it is illuminated with the aid of a synchronous stroboscopic tube. On increasing R_0 and the mass m we readily see that the centre of the needle starts moving in a circle around the centre of gravity of the disk.

For the sake of visuality this experiment can be modified in the following way. The disk is removed from the needle and instead of it a small load is attached on one side of the needle as shown in Fig. 366*a*. When the velocity of rotation is small (smaller than the resonant frequency ω) the load bends the needle as shown in Fig. 366*b*. When the velocity of the rotation becomes large the needle is bent so that the load is nearer to the axis of rotation than the centre of the needle (Fig. 366*c*). Thus, the greater the velocity of the rotation, the nearer is the load to the axis of rotation passing through the centres of the bearings.

When the angular speed of the rotation p is close to the natural frequency ω and resonance sets in, the amplitude of the resonant oscillation of the needle sharply increases and the needle can even rupture. However, if the angular speed is increased sufficiently fast so that the zone of resonance is passed in a short time we can see that the needle makes a slight movement as p becomes close to ω and then, for $p \gg \omega$, it is in a vibration with a very small amplitude.

For $p < \omega$ and $p > \omega$ the amplitude of the oscillation of the disk can easily be computed. Let us consider the motion of the disk relative to a coordinate system rotating with angular velocity p. The disk is at rest in this system and its equilibrium is determined by the balance of the centrifugal force and the force of the elastic deformation of the needle.

When the velocity of the rotation is small ($p < \omega$) the centrifugal force is

$$F'_c = Mp^2(R + R_0)$$

(see the notation in Fig. 365) and the restoring elastic force is

$$F_e = kR$$

On equating these two forces we receive the following expression for the amplitude of the oscillation of the needle:

$$R = \frac{R_0}{\frac{k}{Mp^2} - 1} = \frac{R_0}{\frac{\omega^2}{p^2} - 1}$$

where (according to (129.1)) $\sqrt{\frac{k}{M}} = \omega$ is the natural frequency of the oscillation of the disk on the needle.

When the velocity of the rotation is large ($p > \omega$) the centrifugal force is

$$F_c^* = Mp^2(R - R_0)$$

(see again the notation in Fig. 365) while the elastic force is expressed by the same formula as in the case $p < \omega$.

On equating the two forces we find the expression

$$R = \frac{R_0}{1 - \frac{k}{Mp^2}} = \frac{R_0}{1 - \frac{\omega^2}{p^2}}$$

For $p = \omega$ (this is the resonance condition) the amplitude becomes infinite, which makes no sense from the physical point of view. In this case the force of friction, however small, must be taken into account; then the calculations give a finite value of the amplitude (which can be very large).

The results of the analysis of the oscillation of the disk we have presented can be applied to technical problems connected with the design and construction of various types of turbines and other machines involving rotating parts. A rotating shaft with a massive disk can be considered a model of a turbine. When the velocity of the rotation becomes close to the natural frequency the resonant vibrations may lead to a rupture if the parameters of the machine are chosen inappropriately. For instance, if the shaft is made very hard this increases the natural frequency ω but since the angular velocity p of a modern turbine is very large the rupture can nevertheless occur when p becomes close to ω. It turns out that it is preferable to take a thinner shaft whose stiffness is not very large. In this case, as the machine starts working, the resonant zone is passed in a sufficiently short time after which p becomes much greater than ω and the danger of the rupture is thus avoided. A construction of this type is simpler and less expensive.

130. Transient Processes and Compound Vibrations. Harmonic Analysis

In foregoing sections we studied stationary forced oscillations. In practice the amplitude of the external force is usually varied. The natural frequency of the system to which the force is applied can also change. In all such cases there always appear damped natural vibrations and therefore a stationary oscillation regime can only set in after some time, that is a stationary harmonic oscillation of the system in question appears some time after the external force or the natural frequency has changed. This means that during some period after a change of the external force there is a superposition of natural vibrations and forced vibrations, which results in a compound motion referred to as a *transient process.*

As an example, see Fig. 367*b* where is shown the experimental curve obtained in the recording of the oscillation of a pendulum arising after a harmonic force F (see Fig. 367*a*) has started to act on the pendulum, the frequency of the force coinciding with the natural frequency of the pendulum. We see that the stationary forced harmonic oscillation sets in only after some time during which a transient process takes place (see Fig. 367*b*).

Immediately after the initial instant there arise natural oscillations such that they reduce the amplitude of the forced oscillation. Then, after the natural vibrations are damped, the motion of the pendulum becomes a purely forced oscillation. The smaller the damping of the natural oscillations, the longer the duration of the transient process.

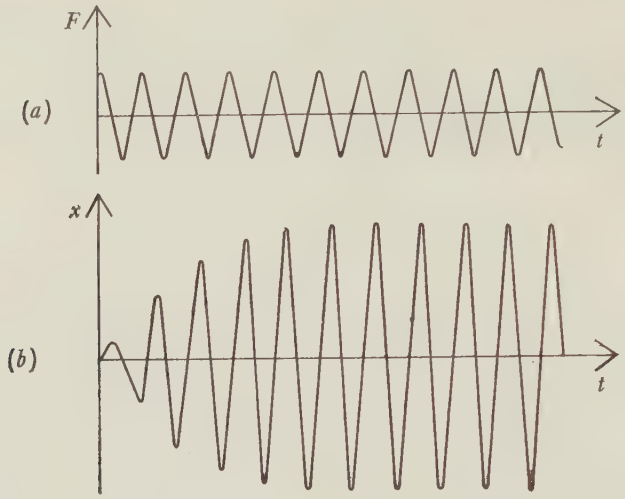

FIG. 367

Experiments show that when a vibration system, for instance, a pendulum is acted upon not by one but by several simultaneous harmonic forces with different frequencies each of the forces produces a forced oscillation whose frequency coincides with the frequency of that force. Thus, the resultant oscillation is compound and non-harmonic. Each of the forces produces an oscillation independently of whether the other forces are present, and the character of this oscillation is as if there were no other forces at all.

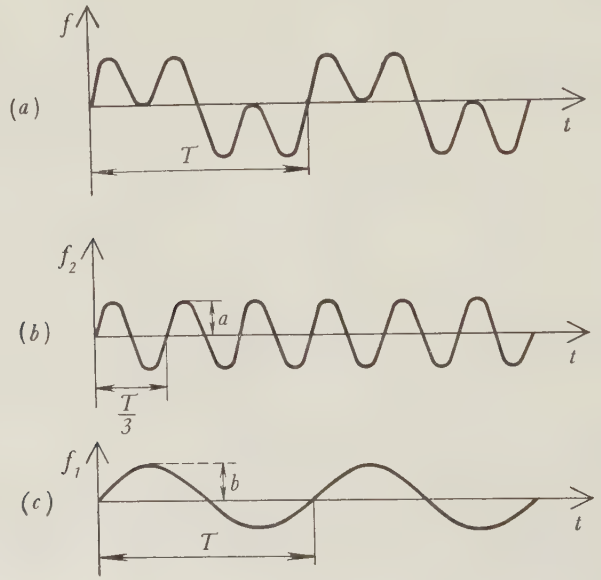

FIG. 368

As is proved in mathematics, every periodic function $f(t)$ with period T can be represented as a sum of harmonic functions whose periods are equal to $\frac{T}{n}$ where n is an integer which, in the general case, runs over all the natural numbers 1, 2, 3, ... For instance, the periodic force $f(t)$ whose graph is shown in Fig. 368*a* is representable as the sum of the two harmonic forces whose graphs are shown in Fig. 368*b* and *c*:

$$f = b \sin \frac{2\pi}{T} t + a \sin \frac{6\pi}{T} t \qquad (130.1)$$

The forces $f_1 = b \sin \frac{2\pi}{T} t$ and $f_2 = a \sin \frac{6\pi}{T} t$ are called the *harmonics* of the force f. The harmonic f_1 whose period coincides with that of the force f is called the *fundamental* (or the *first*) *harmonic*. Under the action of each of the harmonics there arises a forced harmonic oscillation, the resultant motion being the superposition of these oscillations. The force f_1 produces oscillation with the (*fundamental*) frequency $p_1 = \frac{2\pi}{T}$ and the force f_2 produces oscillation with frequency $p_2 = \frac{6\pi}{T}$.

If the frequency of one of the harmonics coincides with the natural frequency, say $p_2 = \omega$, the oscillation arising under the action of the force f_2 predominates over the oscillation produced by the fundamental harmonic. In this case the system on which the force f acts serves as a resonator selecting the oscillation (whose frequency is approximately equal to ω) corresponding to the force f_2 from the compound oscillation produced by the whole force f.

The sharper the resonance peak, the smaller the damping of the natural oscillation of the resonator. The natural oscillation of a resonator with a sharper peak is closer to a harmonic vibration and such a resonator responds more effectively to the oscillation with a frequency close to the natural frequency. The oscillations corresponding to the other harmonics of the external force are also present but their amplitudes are very small when the damping is small. This is the *selectivity property* of the resonator which is widely used in many technical devices.

If we have a set of resonators with different natural frequencies which are subject to the action of a force involving harmonics with various frequencies the oscillations of the resonators make it possible to perform the frequency analysis of the force, that is to determine the frequencies of the harmonics of that force.

For instance, let us consider the device shown in Fig. 369. Here we have a number of pendulums of different lengths suspended from the horizontal bar AA'. The bar AA' is connected with the post BB' by a light rod CC'. The post BB' carries two pendulums *1* and *2*. After pendulums *1* and *2* are made to swing their oscillation produces the vibration of the post BB' from which they are suspended. These vibrations are transmitted through the rod CC' to the bar AA' which starts to vibrate slightly under this action. In some time all the pendulums suspended from AA' begin to oscillate, their motion being a forced oscillation. The observation shows that pendulums *1'* and *2'* whose frequencies are close to the frequencies of pendulums *1* and *2*, respectively, have the greatest amplitudes.

The action of the rod CC' swings the horizontal bar AA' with a force which is a sum of two harmonics whose frequencies are equal to the natural frequencies of pendulums *1* and *2*. The vibration of the bar AA' makes the pendulum suspended from it to oscillate. All these pendulums start to oscillate with the two frequencies; the amplitudes of pendulums *1'* and *2'* are considerably great-

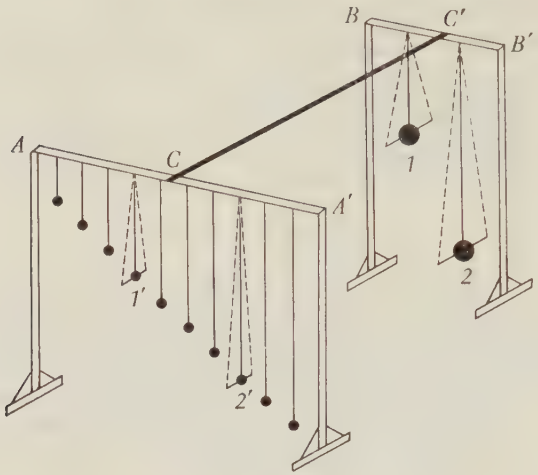

FIG. 369

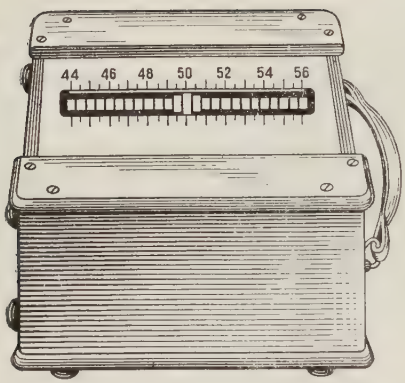

FIG. 370

er than those of the other pendulums. Pendulum *2′* mainly responds to the lower frequency (close to the frequency of pendulum *2*) while pendulum *1′* mainly responds to the higher frequency (close to that of pendulum *1*). Imagine that we do not see the post BB' with its pendulums and the rod CC' and can only observe the pendulums suspended from the bar AA'. Nevertheless, the oscillation of the pendulums suspended from AA' makes it possible to draw certain conclusions concerning the oscillation of the rod CC'. The set of the pendulums which serve as resonators plays the role of a *harmonic* (*frequency*) analyser which enables us to investigate the compound oscillation of the post BB'.

The properties of the resonators are widely used in practical applications for the construction of various instruments meant for measuring and analysing vibrations.

Examples of such instruments are a reed-frequency meter for an alternating electric current (Fig. 370) and a reed tachometer. Both instruments are based on the properties of resonators. The former is meant for measuring the frequency of an alternating electric current and the latter for determining the angular speed (the number of revolutions per second) of a rotating shaft. The basic element of each of them is a set of resonators with various natural frequencies. A model of this kind is shown in Fig. 371. Here a plate D carries a number of reeds a, b, c, . . . of different lengths with small masses attached to their ends. The dimensions and the materials of the reeds and the masses attached to them are chosen so that the natural frequencies of the reeds are equal to definite values ω_1, ω_2, ω_3, . . .

The vibration whose frequency ω_0 should be measured is transmitted to the plate D carrying the reeds which start to oscillate. As a result, the reed whose natural frequency is closer to the frequency ω_0 oscillates with the greatest amplitude which can readily be determined by observing the blurred picture of the end of that reed. In the reed-frequency meter the electric current supplies an electromagnet placed near the plate D and making the latter to oscillate. In Fig. 370 is shown schematically the vibration of reed g; near each reed is usually marked its natural frequency. If the natural frequency of reed g is 50 Hz (that

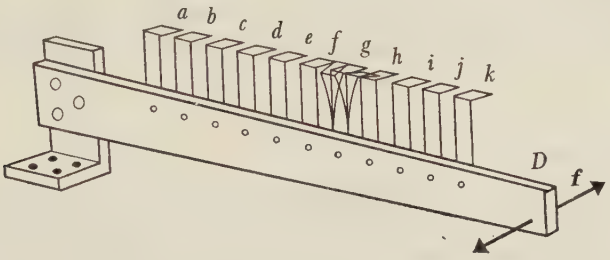

FIG. 371

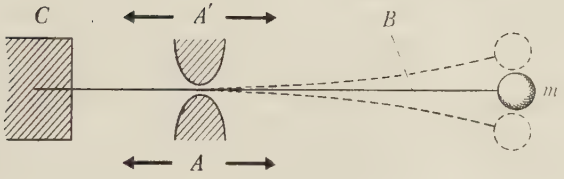

FIG. 372

is 50 vibrations per second) this means that the frequency of the alternating current subjected to the measurement is close to 50 Hz.

In the tachometer such a plate D is connected with a rod (the feeder). When the feeder touches a machine whose shaft rotates with a definite angular velocity, the plate D starts oscillating with a frequency coinciding with that of the oscillation of the body of the machine which, as a rule, is equal to the frequency of the rotation of the shaft.

The work of this device can be demonstrated if we connect the plate with the reeds with the suspension of a gyroscope. When the gyroscope is first made to rotate fast and then decelerates gradually we can observe that the reeds start to oscillate in turn with a considerable amplitude and then stop oscillating one after another.

Resonance phenomena are the foundation of the construction of many other instruments meant for measuring the frequency of a vibration. For instance, in Fig. 372 we see a scheme of a reed-frequency meter whose main element is the reed B carrying a mass m on one end, the other end being rigidly fixed in the instrument case C. The reed B has a clamp AA' placed at a certain distance from the rigid fixing; this distance can be varied. When the instrument touches a vibrating body whose frequency should be measured the reed starts to oscil-

late. Then the clamp is moved along the reed until a position is found in which the reed has the maximum amplitude of the oscillation. The natural frequencies of the reed corresponding to various positions of the clamp AA' are known and are marked on the scale of the instrument, which makes it possible to determine the frequency which we want to measure.

When we deal with a compound oscillation involving harmonic vibrations with various frequencies the frequency meter detects all these frequencies provided that they are not too close to one another.

The sensing element of all instruments of this kind is a resonator (or a set of resonators) whose frequency can easily be measured.

Resonators play a very important role in acoustics, optics, radio engineering, etc. because of their selectivity making it possible to perform the frequency analysis.

131. Auto-oscillation

In certain conditions a stable periodic vibration may arise in a system which is not subject to an external periodic action. An example of this kind is the fiddle stick and the string (considered as one system); the motion of the fiddle stick makes the string to oscillate periodically. If we gave the string a deviation from the equilibrium position in the absence of the fiddle stick and then allowed it to move on its own the natural oscillation of the string would set in; this natural oscillation would be damped very soon by the friction forces. When the fiddle stick excites the string the forces in this vibration system change in such a way that they support the vibration. The work of these forces compensates for the energy losses connected with the work of the friction forces which inevitably appear when the string oscillates. Thus, the conditions in which the string oscillates under the action of the fiddle stick are such that there appears a definite periodic force supporting the oscillation; in the absence of the oscillation the force with which the fiddle stick acts upon the string would remain constant.

A system in which a periodic oscillation arises in the absence of any external periodic action is referred to as a *self-oscillating* or an *auto-oscillating system*, the oscillation process being called an *auto-oscillation* (or a *self-excited vibration*).

As a simple example of this kind, let us consider the auto-oscillation of a pendulum with a sliding bearing on a rotating shaft (see Fig. 373*a*). The friction forces arising in the bearing of the pendulum create a moment of forces F_{fr} acting on the pendulum*. Let us consider the work of the moment M_{fr} during the periodic oscillation of the pendulum. Since the process we deal with is periodic it suffices to consider a time interval equal to the period of the oscillation. Let the oscillating pendulum pass through the equilibrium position at the beginning of this time interval and let it move at the initial instant as shown in Fig. 373*b*. During the first half of the period when the shaft and the pendulum rotate in opposite directions the work of the moment M_{fr} of the friction forces is equal to the energy lost by the pendulum. During the second half of the period when, conversely, the pendulum and the shaft rotate in one direction the work of the moment M_{fr} is expended on the increase of the energy of the pendulum (see Fig. 373*c*).

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* It is supposed that the absolute value of the angular velocity of the rotation of the shaft remains all the time greater than that of the pendulum during the oscillation process. This implies that the direction of the moment M_{fr} ap lied to the pendulum is always preserved.

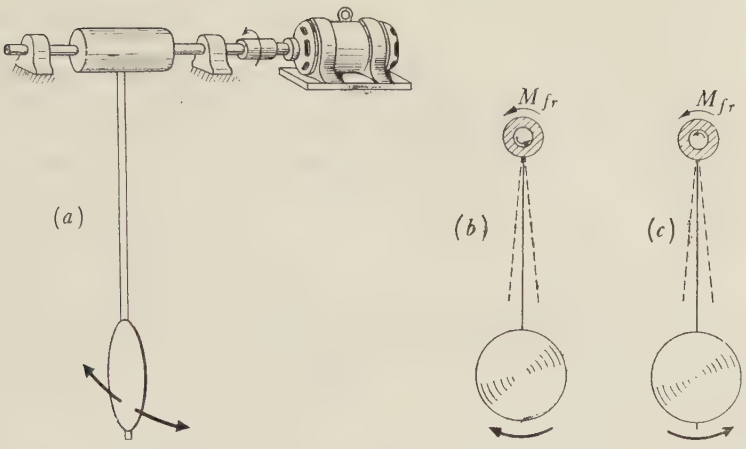

FIG. 373

In certain conditions the force of dry friction is almost independent of the velocity of sliding; then the total work of the friction forces during the whole period is equal to zero. In this case the friction between the pendulum and the shaft introduces no damping in the oscillation process.

In case the friction force between the shaft and the pendulum depends on the velocity of sliding the character of the phenomenon changes.

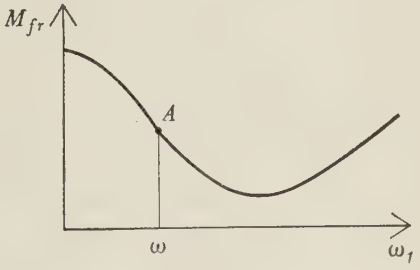

FIG. 374

Suppose that the friction force increases together with the velocity of sliding. Then in the state shown in Fig. 373*b* the moment of the friction forces is greater than in the state shown in Fig. 373*c*; consequently, during the period of the oscillation the pendulum loses some energy and the oscillation is damped. The mechanical energy of the pendulum is lost and the friction between the pendulum and the rotating shaft increases the damping of the oscillation.

The course of the process may essentially change when the friction force reduces as the velocity of sliding increases. Such a case can be realized practically for a certain range of variation of the velocity of sliding by introducing a small amount of a lubricant. For a pendulum in a state of rest a typical diagram representing the dependence of the friction force moment on the angular velocity of the rotation of the shaft is shown in Fig. 374.

Let us consider the case when the velocity of the rotation of the shaft corresponds to the abscissa of the point A in Fig. 374. Then the energy of the pendulum increases during the period of the oscillation, which can readily be confirmed by an argument analogous to the one used in the case of a constant moment M_{fr}. In this case the oscillating pendulum receives a definite amount of energy during the period and if this amount exceeds the energy loss connected with the air friction the amplitude of the pendulum will increase.

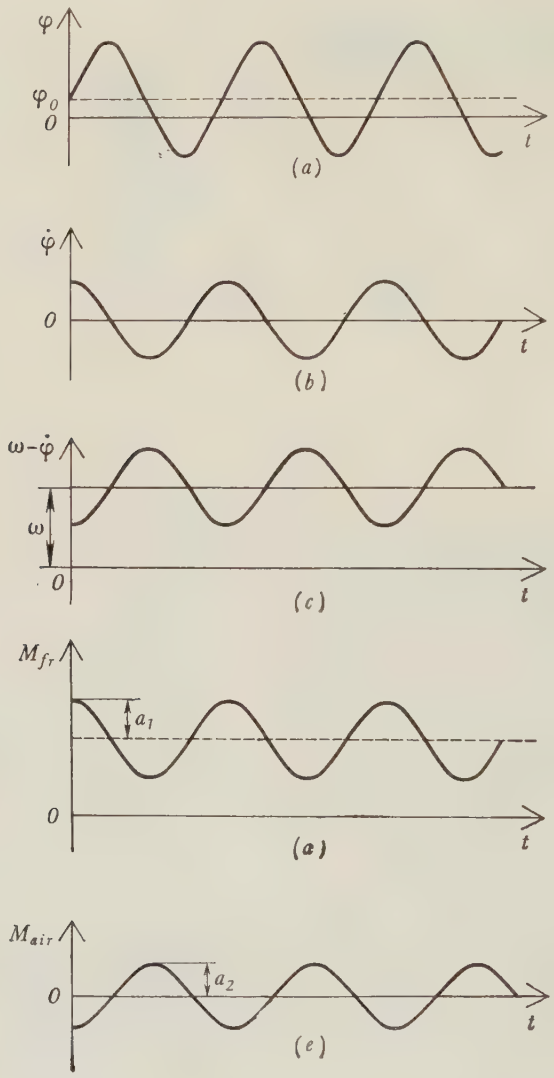

FIG. 375

The character of the oscillation is demonstrated by the graphs shown in Fig. 375. Graph *a* represents the oscillation of the angle φ of deviation of the pendulum from the equilibrium position; graph *b* shows the time variation of the angular velocity $\dot{\varphi}$ of the rotation of the pendulum; graph *c* describes the variation of the relative velocity (the velocity of sliding) $\omega_1 = \omega - \dot{\varphi}$. The velocity ω_1 oscillates about the constant value ω (the angular velocity of the shaft). The diagram in Fig. 375*d* represents the variation of the moment M_{fr} of the friction forces between the shaft and the bearing of the pendulum for the case when the friction force reduces as the velocity of sliding increases. Fig. 375*e* shows the dependence of the moment of air friction forces M_{air} on

time t. The oscillation of the angle φ and that of M_{air} are always in opposite phases. It is evident that if the amplitude a_1 exceeds the amplitude a_2 the amplitude of the oscillation of the pendulum grows.

As the amplitude of the oscillation of the pendulum increases the amplitude a_1 of the moment of the friction force between the pendulum and the shaft grows slower than the amplitude a_2 of the air friction moment M_{air} and when the amplitude of the oscillation of the pendulum attains a certain value the amplitudes a_1 and a_2 become equal. Since then the stationary oscillation (the auto-oscillation) of the pendulum sets in.

The rotating shaft supplies the pendulum with energy necessary to compensate for the loss of the energy which is transformed into heat in the auto-oscillation process. The energy is transmitted from the shaft rotated by the electric motor by means of the force of sliding friction. Our considerations obviously show that the frequency of the auto-oscillation is determined by the natural frequency of the pendulum. Experiments show that in other cases the frequency of an auto-oscillation process is also close to the natural frequency of the resonator entering into the vibration system.

An auto-oscillation system in which there arises a vibration which is close to a harmonic oscillation always involves a resonator (e.g. the pendulum in the above example) and an energy source (e.g. the electric motor rotating the shaft in the above example). When the resonator oscillates it acts on the source of energy in such a way that the force exerted by the source on the resonator becomes periodic and supports the oscillation of the resonator. In such a system there is always a *feed-back* between the energy source and the resonator which produces a periodic vibration of the force exerted by the energy source. In the above example the feed-back is created by the oscillation of the velocity of sliding as the friction force between the shaft and the pendulum varies periodically and supports the oscillation of the pendulum. For the auto-oscillation to arise an initial excitation, however small, is needed; the oscillation process described above starts when the pendulum is given a deviation from the equilibrium position after which it begins to oscillate.

The oscillation of the pendulum on the rotation shaft is an example of an almost harmonic auto-oscillation. There also exist auto-oscillation phenomena which are periodic but non-harmonic. An example of this kind is the auto-oscillation in the hinges of a door caused by the force of dry friction in the hinges and causing the squeak of the door.

Another typical example of non-harmonic auto-oscillations is the auto-oscillation produced by an alternating energy source. In Fig. 376 we see a scheme of a model of this kind. The vessel shown in the figure can be filled with water from the tap A and there is a syphon pipe built in the wall of the vessel. The constant velocity of the water pouring from the tap can be regulated in such a way that a periodic oscillation of the water level in the vessel sets in. Suppose that at the initial instant there is no water in the vessel. After the tap has been turned on the water level starts to rise gradually; the moment the water achieves the level marked by the letter B in Fig. 376 it starts flowing out through the syphon pipe. The water flow carries away the air from the pipe and the water fills the whole syphon pipe. Then the water flows out of the vessel through the syphon pipe. The velocity of the water pouring from the tap A can be chosen so that it is much less than the velocity of the flow of the water through the syphon pipe. Then the altitude of the water level will decrease rather fast until the free surface of the water achieves the level marked by the letter C in the figure. Now the air enters the syphon pipe through its end placed at level C, the water pours

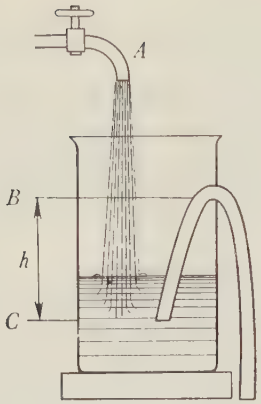

FIG. 376

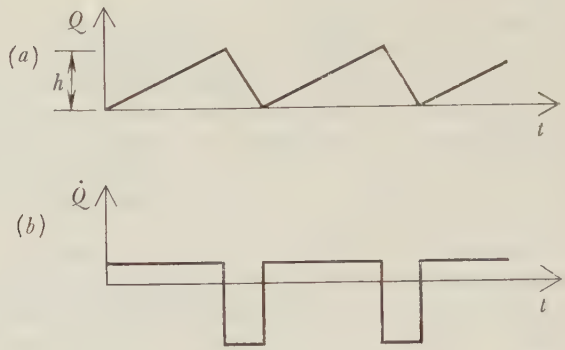

FIG. 377

out of the pipe and the water flow through the syphon pipe stops. Next the process we have described repeats and thus there arises a periodic vibration of the altitude of the water column (between the levels C and B) which may last indefinitely. The altitude Q (reckoned from level C) will vary in a way shown by the diagram in Fig. 377*a*. The variation of the velocity $\dot{Q}$ of the water level is represented in Fig. 377*b*.

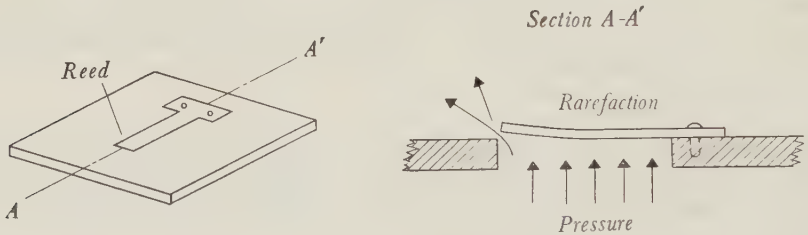

FIG. 378

The graphs in Fig. 377 show that the vibration process is not harmonic. The velocity of the motion of the water level changes particularly sharply at the moments when the syphon pipe starts and stops working. However, the oscillation in this example is purely periodic and is supported by the uniform flow of the water from the tap. The role of a feed-back device in this example is played by the syphon pipe which periodically allows the water to flow from the vessel and stops the flow.

The oscillation of the water level in the vessel is an example of a non-harmonic *relaxation* auto-oscillation. Generally, in a relaxation oscillation process the motion or its velocity is described by a discontinuous function or by a function which is close to a discontinuous one (the graphs in Fig. 377 represent functions of this type).

Here we also mention some other examples of auto-oscillations supported by a jet of air or water. The sound produced by an accordion results from the auto-oscillation of a reed (a very thin elastic plate) placed in the air jet (see Fig. 378). The oscillation of the air in a reed-pipe is supported by the air stream incident on the sharp edge of an opening (Fig. 379). Here the role of a resonator is played by the volume of the air within the pipe. Other musical instruments of this kind (the reeds) are based on the same principle.

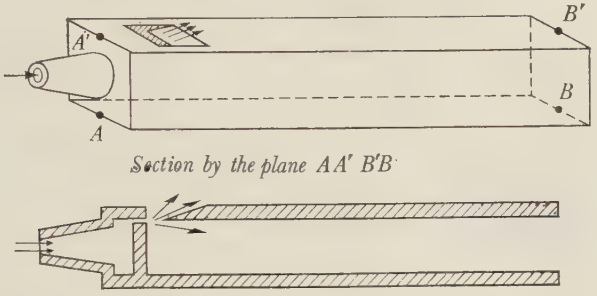

FIG. 379

132. Natural Oscillations of a System with Many Degrees of Freedom

In the foregoing sections we considered vibrations of a single body (e.g. a load suspended from a thread or from a spring, a body immersed in a fluid). When we considered the oscillation of a fluid in communicating vessels the situation was such that the harmonic oscillation of one particle of the fluid determined uniquely the motion of all the other particles. In all processes of this kind it suffices to know the variation of one coordinate in order to describe completely the whole process. In such cases we say that the system in question possesses *one degree of freedom*.

We now consider an example of another type: suppose that two loads suspended as shown in Fig. 380 are made to oscillate vertically. In the oscillation process the lower spring undergoes a deformation and therefore the displacement x_1 of the upper load is not equal to the displacement x_2 of the lower load. Hence, the oscillation is characterized by simultaneous variations of two coordinates, namely x_1 and x_2. If we jerk sharply the lower load it starts to oscillate and almost immediately excites the other load which also begins to oscillate. The motion is transmitted from the lower load to the upper one through the spring between them which undergoes alternating extension and compression. Hence, to describe the process we should take into account the simultaneous variations of the two coordinates x_1 and x_2 specifying the positions of the loads. A system of this kind is said to have *two degrees of freedom*.

Similarly, two coupled pendulums connected with the aid of a light spring (see Fig. 381 where the spring is denoted as a) form a system with two degrees of freedom (it is supposed that the pendulums can only oscillate in the vertical plane passing through their points of suspension). The oscillation of each of the pendulums is not completely independent and is connected with the oscillation of the other pendulum by a definite law.

We can also take a system of three, four, etc. pendulums joined with springs. A system of this kind possesses three, four, etc. degrees of freedom respectively, because it involves three, four, etc. pendulums oscillating simultaneously. If one or several pendulums entering into such a system are made to oscillate the resultant oscillation process is rather complicated and, generally speaking, the oscillation of each of the pendulums is non-harmonic.

We now investigate in more detail the oscillation process in a simple system of this kind consisting of two identical pendulums connected with a spring (see

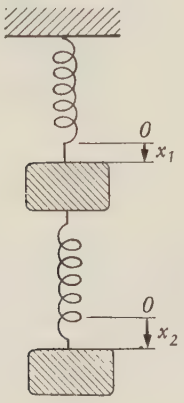

FIG. 380

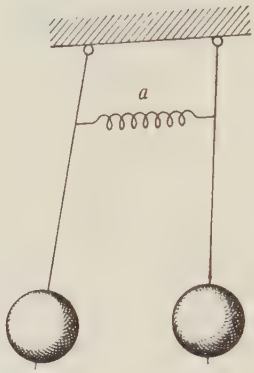

FIG. 381

Fig. 381). Let us give an initial displacement to one of the pendulums while the other is kept fixed and then simultaneously allow both pendulums to move on their own. Suppose that we have a recording device making it possible to obtain curves representing the variations of the coordinates x_1 and x_2 of the first and the second pendulum. Then it turns out that these curves are of the type shown in Fig. 382.

At the initial stage of the process the first pendulum oscillates approximately as if the other pendulum were fixed and the spring between the pendulums compresses and extends periodically. Then the force exerted by the spring on the second pendulum starts to swing it gradually. Since a part of the energy of the first pendulum is transmitted to the other pendulum the amplitude of the oscillation of the first pendulum gradually decreases and the amplitude of the second pendulum increases.

The energy transfer from the first pendulum to the second one lasts for some time (which we denote $\tau/2$) until the first pendulum stops oscillating while the second pendulum oscillates with an amplitude which is almost equal to the amplitude of the first pendulum at the beginning of the process (provided that the energy loss connected with the friction is small). In the further course of the process the roles of the pundulums interchange: the second pendulum starts to swing the first one and the part of the process described above is repeated in the reverse order and in just the same way because the pendulums are identical. The increase and the decrease of the amplitudes of the pendulums will alternate and the pendulums will exchange their energies periodically with period τ. An oscillation process of this kind is referred to as *beating* (or *beats*) and the time inter-

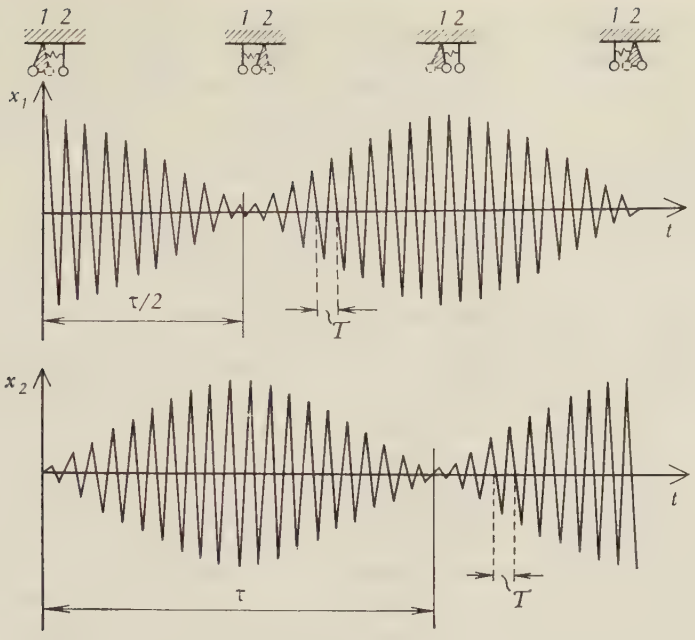

FIG. 382

val τ is called the *beat period*. In the course of the process the mechanical energy is periodically transferred (almost completely) from one pendulum to the other and vice versa. Since in reality there are always friction forces the process lasts until the mechanical energy is completely transformed into heat and the pendulums stop oscillating.

Now let us excite the oscillation of the pendulums in an arbitrary way and record the oscillation process. In the general case the amplitudes of oscillation of each pendulum are not constant and their increase and decrease alternate. As

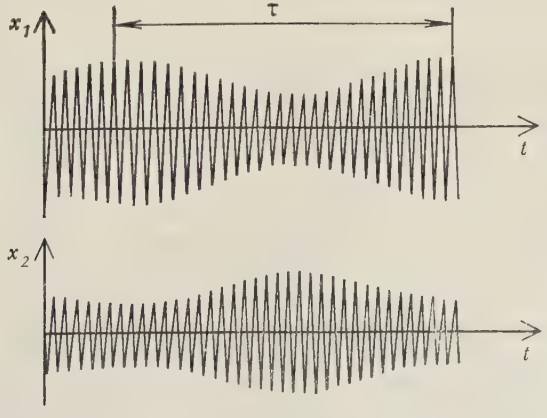

FIG. 383

before, when the amplitude of the first pendulum increases the amplitude of the second pendulum decreases and vice versa. The amplitude of the oscillation of the first or of the second pendulum must not necessarily reduce up to zero; in the general case it varies between a certain minimum value and a certain maximum value as is shown schematically in Fig. 383. The time interval during which the energy of one pendulum is transferred to the other pendulum, that is the time during which the amplitude of the oscillation of the pendulum varies from its maximum to its minimum, is again equal to the same quantity $\tau/2$. Irrespective of the way the vibrations of the pendulums are excited the beat period remains the same; only the difference between the maximum and the minimum of the amplitude of the oscillation of the pendulums depends on the initial conditions in which the oscillation process starts (cf. Figs. 382 and 383).

In the general case the oscillation of each pendulum is non-harmonic. Each pendulum oscillates and its amplitude varies periodically with the constant beat period τ. The range of the variation of the amplitude in the beating process (the *depth of modulation*) depends solely on the way the oscillation has been excited. It is evident that there can be special excitation conditions under which the depth of modulation is very small so that the vibrations of the pendulums become very close to harmonic oscillations. Based on the symmetry of the pendulums in the system in question, we can readily indicate the following two cases when the initial excitation causes both pendulums to oscillate purely harmonically: in the first case the pendulums are given equal initial displacements in one direction and then allowed to move on their own and in the second case the initial deviations of the pendulums coincide in their magnitudes and are opposite in the directions.

The excitation of the first type makes both pendulums oscillate harmonically as if there were no spring connecting them at all because during the oscillation the length of the spring remains invariable. Under the assumption that the weight of the spring is negligibly small the period of the vibrations of the pendulums coincides with the natural period T_2 of each pendulum.

After the excitation of the second type the pendulums oscillate in opposite phases and the spring undergoes a deformation so that its extension and compression alternate while its midpoint does not move and behaves as if it were fixed. Both pendulums are in similar conditions and oscillate harmonically with a period T_1. It is evident that the period T_1 must be less than T_2 because in this case the restoring force of the spring is added to the restoring force of each pendulum.

However, if the initial displacements of the pendulums slightly differ there arise beating phenomena with a small depth of modulation in the vibrations of both pendulums.

133. Theoretical Analysis of Beating

To elucidate the character of the phenomenon of beating let us subject to a theoretical analysis a superposition of two harmonic vibrations with different frequencies (it should be noted that the superposition of two vibrations with coinciding frequencies always results in a harmonic oscillation with the same frequency).

Let a point oscillate in a definite direction according to a sinusoidal law described by a function $A \sin(\omega t + \varphi_1)$ relative to a reference system which itself oscillates harmonically in the same direction, the oscillation of the system being described by a function $B \sin(\omega t + \varphi_2)$. Then the oscillation of the given point

in this direction relative to a fixed reference system is determined by the corresponding coordinate x whose variation is expressed as

$$x = A \sin(\omega t + \varphi_1) + B \sin(\omega t + \varphi_2) = (A \cos \varphi_1 + B \cos \varphi_2) \sin \omega t + \\ + (A \sin \varphi_1 + B \sin \varphi_2) \cos \omega t = C \sin(\omega t + \varphi) \quad (133.1)$$

where

$$C = \sqrt{(A \cos \varphi_1 + B \cos \varphi_2)^2 + (A \sin \varphi_1 + B \sin \varphi_2)^2}$$

and

$$\varphi = \arctan \frac{A \sin \varphi_1 + B \sin \varphi_2}{A \cos \varphi_1 + B \cos \varphi_2}$$

The oscillation resulting from the superposition of two harmonic vibrations having the same direction but different frequencies is of another character and is not harmonic. Let

$$x_1 = A \sin(\omega_1 t + \varphi_1) \text{ and } x_2 = B \sin(\omega_2 t + \varphi_2)$$

be the coordinates describing the oscillation of the point relative to a moving reference system and the oscillation of the reference system itself, respectively. Then the resulting oscillation is

$$x = x_1 + x_2 = A \sin(\omega_1 t + \varphi_1) + B \sin(\omega_2 t + \varphi_2) = \\ = A \sin(\omega_1 t + \varphi_1) + A \sin(\omega_2 t + \varphi_2) + (B - A) \sin(\omega_2 t + \varphi_2) = \\ = 2A \cos\left(\frac{\omega_1 - \omega_2}{2} t + \frac{\varphi_1 - \varphi_2}{2}\right) \sin\left(\frac{\omega_1 + \omega_2}{2} t + \frac{\varphi_1 + \varphi_2}{2}\right) + \\ + (B - A) \sin(\omega_2 t + \varphi_2) \quad (133.2)$$

The first term in the rightmost sum in formula (133.2) does not represent a harmonic oscillation because it is equal to the product of two harmonic factors the first of which has the frequency $\frac{1}{2}(\omega_1 - \omega_2)$ and the other the frequency $\frac{1}{2}(\omega_1 + \omega_2)$. If the difference between the frequencies ω_1 and ω_2 is small (relative to both ω_1 and ω_2) the frequency $\frac{1}{2}(\omega_1 + \omega_2)$ is approximately of the order ω_1 (or ω_2) while the frequency $\frac{1}{2}(\omega_1 - \omega_2)$ is very small in comparison with ω_1 (or ω_2).

In this case the first term of the resultant oscillation (133.2) can be thought of as an almost harmonic oscillation with frequency $\omega_1 \approx \omega_2$ whose amplitude $\left(\text{here we apply the term "amplitude" to the factor } 2A \cos\left(\frac{\omega_1 - \omega_2}{2} t + \frac{\varphi_1 - \varphi_2}{2}\right)\right)$ varies slowly. The graph of this oscillation coincides with the one shown in Fig. 382 for the function $x_1(t)$. We can speak of it as pure beating, the beat period being

$$\tau = \frac{2\pi}{\omega_1 - \omega_2} \quad (133.3)$$

The second term in the rightmost sum in formula (133.2) represents a harmonic oscillation with frequency ω_2. The superposition of the pure beating and the harmonic oscillation results in a beating phenomenon of a general type in which the amplitude of the oscillation varies periodically with period τ but does not necessarily achieve a zero value; the graph of a vibration of this type is shown in

Fig. 383. When the two constituent oscillations have different frequencies and equal amplitudes $A = B$ only pure beating is present. In case $A \neq B$ we have a beating phenomenon of a general type shown in Fig. 383, the beat frequency (see formula (133.3)) being always equal to

$$\nu_b = \frac{1}{\tau} = \frac{\omega_1 - \omega_2}{2\pi} = \nu_1 - \nu_2$$

where $\nu_1 = \frac{\omega_1}{2\pi}$ and $\nu_2 = \frac{\omega_2}{2\pi}$ are the frequencies of the harmonic oscillations whose superposition results in the beats. We see that the beat frequency is equal to the difference between the frequencies of the constituent vibrations and does not depend on the amplitudes and the initial phases of these vibrations.

The time interval between two successive instants when the zero value is passed in one direction (that is in the direction of increase or decrease) is equal to

$$T = \frac{4\pi}{\omega_1 + \omega_2}$$

(see Fig. 382).

The phenomenon of beating can also be interpreted in the following way. When two harmonic vibrations in one direction with slightly differing frequencies are added together the maximum departure from the equilibrium is achieved at an instant t_1 when the vibrations are in one phase. In the further course of the resultant oscillation the amplitude decreases and attains its minimum value at an instant $t_2 > t_1$ when the constituent vibrations are in opposite phases. Then the amplitude of the oscillation starts to increase again until at an instant $t_3 > t_2$ the constituent vibrations are in one phase; at time t_3 the amplitude again attains its maximum, and so on. It is evident that the beat period τ is equal to

$$\tau = t_3 - t_1 \tag{133.4}$$

For definiteness, let us suppose that the phase difference between the constituent vibrations at time t_1 is equal to zero, that is

$$\omega_1 t_1 + \varphi_1 - \omega_2 t_1 - \varphi_2 = 0 \tag{133.5}$$

Then at time t_3 the phase difference is equal to

$$\omega_1 t_3 + \varphi_1 - \omega_2 t_3 - \varphi_2 = 2\pi \tag{133.6}$$

Performing the termwise subtraction of (133.5) from (133.6) we obtain

$$(\omega_1 - \omega_2)(t_3 - t_1) = 2\pi$$

whence, taking into account (133.4), we find the beat period:

$$\tau = \frac{2\pi}{\omega_1 - \omega_2} \tag{133.7}$$

The results of the theoretical analysis of the superposition of two harmonic vibrations with different frequencies indicate that the natural oscillation of the system of two identical pendulums coupled as was described above is representable as a superposition of two harmonic vibrations, the difference between the frequencies of the vibrations being equal to the beat frequency.

134. Natural Frequencies of Coupled Pendulums

Experiments show and theoretical calculations confirm that for any excitation of a system of coupled pendulums there arises the phenomenon of beating of one definite character.* This means that the oscillation of each of the pendulums is representable as a sum of two harmonic vibrations with the frequencies

$$\omega_1 = \frac{2\pi}{T} + \frac{\pi}{\tau} \quad \text{and} \quad \omega_2 = \frac{2\pi}{T} - \frac{\pi}{\tau}$$

The frequencies ω_1 and ω_2 are determined by the physical parameters of the pendulums (that is ω_1 and ω_2 depend on the lengths of the pendulums, the masses of their loads, the stiffness of the spring, the positions of the points at which the ends of the spring are attached to the pendulums, etc.) and are independent of the initial conditions after which the oscillation of the system sets in. That is why ω_1 and ω_2 are referred to as the *natural frequencies* of the system of the two pendulums. The way in which the system is excited (that is the initial conditions) only specifies the magnitudes of the amplitudes and of the initial phases of the harmonic vibrations of the first and the second pendulum. Therefore, the initial conditions also specify the depth of modulation in the beating process; when the vibration with one frequency has a very small amplitude relative to the amplitude of the vibration with the other frequency the depth of modulation is also very small, which means that in this case the resultant vibration is close to a harmonic oscillation with a large amplitude.

If the initial conditions are chosen so that both pendulums oscillate with the same frequency (without beating) the frequency of the vibrations of the pendulums coincides with one of the natural frequencies ω_1 or ω_2. That is why the harmonic oscillation of the system of two pendulums after they are given initial deviations of the same magnitude and in the same direction, that is in the same phase (see Sec. 132), is nothing but the natural oscillation of the system with its smaller natural frequency ω_2. Similarly, the oscillation with the greater frequency ω_1 is observed after the pendulums are given initial displacements of the same magnitudes and opposite directions (opposite phases). Thus, the natural vibrations of both pendulums with the smaller frequency are the cophasal harmonic vibrations and the natural vibrations with the greater frequency are harmonic vibrations in opposite phases.

A more thorough investigation shows that any natural oscillation of the system of two coupled pendulums arising after any initial conditions is representable as a sum of vibrations in one phase of the two pendulums with the frequency ω_2 and vibrations in opposite phases with the frequency ω_1. This general conclusion can be confirmed by the following argument.

Let us suppose that the first pendulum has an initial displacement x_{10} and the second pendulum has an initial displacement x_{20} (it is assumed that the initial velocities of the pendulums are equal to zero). The displacements are considered positive when the pendulums are shifted in one direction from the equilibrium and negative when the directions of the displacements are opposite. It is always possible to choose a displacement a of both pendulums in one direction and a displacement b in opposite directions so that the given initial deviations x_{10} and x_{20} are the result of the superposition of the displacements a and b. Indeed, the sought-for quantities a and b are uniquely determined by the two simulta-

* The beat period τ and the time interval T (see Sec. 133) have the same values for any initial conditions.

neous equations

$$x_{10}=a-b \quad \text{and} \quad x_{20}=a+b$$

On solving this system of equations we find

$$a=\frac{x_{10}+x_{20}}{2} \quad \text{and} \quad b=\frac{x_{20}-x_{10}}{2}$$

The initial deviation of the pendulums in one direction by a distance of a produces vibrations of both pendulums with frequency ω_2 in one phase while their deviation in opposite directions by b results in vibrations of the pendulums with frequency ω_1 in opposite phases. The superposition of these two motions results in the oscillation of the system corresponding to the given initial displacements x_{10} and x_{20}, which is what we intended to show. In this way we can describe the general character of the natural oscillation of a system consisting of two identical coupled pendulums.

In a more general situation when the coupled pendulums are different it can also be shown that an arbitrary oscillation of the system specified by arbitrary initial conditions is also representable as a sum of two harmonic vibrations with some frequencies ω_1 and ω_2 which are the natural frequencies of the given system. The oscillation regimes in which both pendulums oscillate with a common frequency coinciding with one of the natural frequencies can be excited by choosing appropriate initial conditions; however, for pendulums of different lengths and masses the theoretical computation of the necessary initial data is a rather complex task which we do not consider here.

135. Natural Frequencies of Three Connected Pendulums

Three connected pendulums form a system with three degrees of freedom. The natural oscillations of this system are still more complex than the oscillations of two coupled pendulums. Such a system possesses three natural frequencies and its arbitrary natural oscillation is representable as a sum of three harmonic vibrations.

The vibrations of the system of three identical connected pendulums shown in Fig. 384 in which all the pendulums oscillate with one of the natural frequencies can readily be observed in an experiment. As in the case of two coupled pendulums, when the initial conditions are chosen so that the three pendulums oscillate harmonically with a common frequency this frequency coincides with one of the natural frequencies.

In Fig. 384 are shown three cases in each of which one of the natural oscillations arises (it is supposed that the natural frequencies of this system of three connected pendulums are different: $\omega_1 > \omega_2 > \omega_3$). It is evident that the initial conditions shown in Fig. 384*a* and *b* generate harmonic vibrations of the pendulums with frequencies ω_3 and ω_2, respectively.*

In the case shown in Fig. 384*a* all the pendulums oscillate in one phase with a common frequency (ω_3) with which every pendulum would oscillate separately without being connected with the other pendulums. The springs do not take part in this process because they undergo no deformation.

In the case shown in Fig. 384*b* the right-hand and left-hand pendulums oscillate in opposite phases, the forces of the springs exerted on the middle pendulum

* In the case shown in Fig. 384*b* the amplitude of the oscillation of the middle pendulum is equal to zero; we can therefore say that this pendulum oscillates with frequency ω_2 and zero amplitude.

balance each other and therefore this pendulum remains at rest. The frequency ω_2 of this oscillation process is greater than the frequency ω_1 in the former case because in the latter case the additional force arising owing to the deformation of each spring is added to the restoring force of the corresponding pendulum (as is known, for a small amplitude the restoring force of an isolated simple pendulum is equal to $mg\alpha$ where mg is the force of weight of the pendulum and α is the angle of deviation of the pendulum).

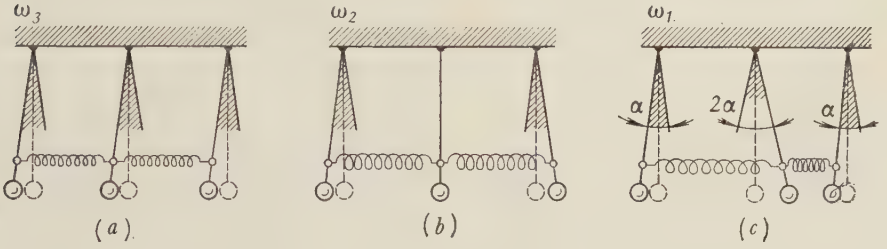

FIG. 384

The appearance of a harmonic vibration in the system after the excitation corresponding to the initial condition shown in Fig. 384*c* is explained in a more complicated way. The restoring force characteristic of the middle pendulum is twice the corresponding restoring force of the left-hand or of the right-hand pendulum because the deviation of the middle pendulum is measured by the angle α while the deviations of the right-hand and the left-hand pendulum are equal to $\frac{\alpha}{2}$. The additional force resulting from the actions of the two springs on the middle pendulum is twice as large as the force with which the right and the left springs act on the right-hand and on the left-hand pendulum respectively because one of the springs receives an elongation proportional to $\frac{3\alpha}{2}$ while the other spring undergoes a compression of the same magnitude (the springs are identical and have the same stiffness, the forces exerted by them being proportional to their deformations). Finally, the coefficients characterizing the restoring forces of the pendulums are equal because the masses and the lengths of the pendulums are the same. It follows that the resultant force acting on each pendulum is proportional to its deviation from the equilibrium position with the same proportionality coefficient and hence the periods of the vibrations of the pendulums coincide.

The oscillations of the system of the three pendulums generated by the initial conditions shown in Fig. 384*a*, *b*, and *c* consist of three coherent harmonic vibrations of the three pendulums with one of the natural frequencies ω_1, ω_2 and ω_3, respectively.

As in the case of two pendulums, an arbitrary natural oscillation of the system of the three connected pendulums is representable as a superposition of three vibrations each of which corresponds to the coherent harmonic vibrations of the pendulums with one of the three natural frequencies ω_1, ω_2, and ω_3. Every combination of the coherent vibrations with frequencies ω_i ($i = 1, 2, 3$) is called a *normal mode of vibration* or, simply, a *normal mode* or a *principal oscil-*

lation of the whole system. Thus, any natural oscillation of the system is a sum of three normal modes.

For a vibration system consisting of an arbitrary number of different pendulums or other oscillating bodies the situation is completely analogous. In the general case of arbitrary initial conditions the corresponding complex oscillation of the system reduces to a superposition of the normal modes. The number of the different normal modes and the number of the natural frequencies coincide with the number of the degrees of freedom of the whole system consisting of the oscillating bodies.

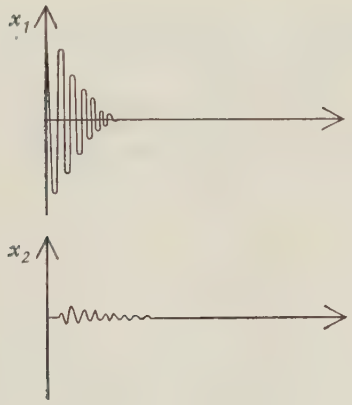

FIG. 385

The analysis of the oscillation of a composite system we carried out in Secs. 134 and 135 was based on the assumption that the friction forces were negligibly small. When the action of small friction forces is taken into account the course of the oscillation of the system during a finite time interval comparable with the period is not changed considerably. However, for a sufficiently long time interval the oscillation will be damped. The presence of large friction forces must of course essentially change the character of the natural oscillations. For instance, when two coupled pendulums connected by a spring with very small stiffness undergo the action of sufficiently large friction forces the diagrams representing the vibrations of the pendulums can be of the type shown in Fig. 385. As is seen from this figure, the oscillation of the first (excited) pendulum can be damped very fast before the excitation is transferred to the other pendulum which is not excited initially (cf. Fig. 382).

When the friction forces become very large the motion of the pendulums is no longer oscillatory. The pendulum which is given an initial deviation will slowly approach the equilibrium position and will produce a slow motion of the same type of the other pendulums. The pendulums which are not strongly connected with the excited one practically remain at rest.

136. Forced Oscillations in Composite Systems

Let us consider the oscillation of a system of two coupled pendulums (Fig. 386) under the action of a harmonic force F with frequency p applied to one of the pendulums. When a stationary oscillation regime arises each of the pendulums is in a forced harmonic vibration with the excitation frequency p. As in the case of a forced vibration of a system with one degree of freedom, the amplitudes of the two pendulums depend on the frequency. This dependence becomes particularly noticeable when the damping in the system is small. The forced vibrations of both pendulums with the maximum amplitudes is observed when the frequency p of the external force F coincides with one of the natural frequencies of the system of the two coupled pendulums. In this case we speak of a *resonant vibration* of the system. Similarly, in a system consisting of n pendulums reso-

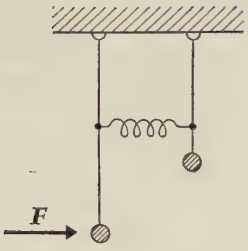

FIG. 386

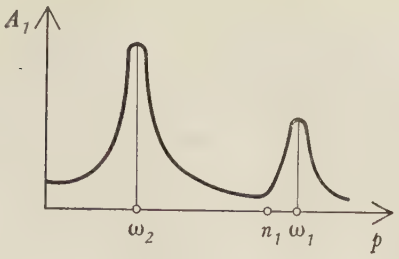

FIG. 387

nance sets in when the frequency of the external force coincides with one of the n natural frequencies of the system. The resonance curve representing the dependence of the amplitude A_1 of the longer pendulum to which the external force $F = a \cos pt$ is applied (see Fig. 386) on the frequency p is shown in Fig. 387 (it is assumed that the amplitude of the external force F remains constant as the frequency p is varied). We see two resonance peaks in Fig. 387 in the vicinity of the natural frequencies ω_1 and ω_2. An essential role in the oscillation process is also played by the minimum amplitude corresponding to a certain value n_1

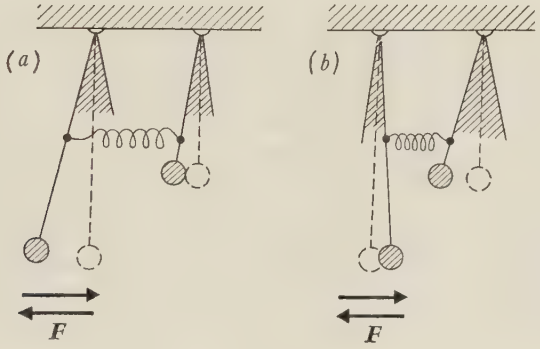

FIG. 388

of the frequency p (see the figure). The importance of this minimum will be discussed later. In the resonant oscillation (when $p = \omega_1$ or $p = \omega_2$) the action of the external force is balanced by the action of the friction forces. If the friction forces are very small and the external force is also sufficiently small the forced resonant oscillations in the system resemble the normal modes of vibration because the frequencies of the forced oscillations are equal to the natural frequencies.

In the resonance arising when $p = \omega_2$ the relationship between the amplitudes and the phases is of the type shown in Fig. 388*a*. Both pendulums oscillate in one phase but the angle of the deviation of the longer pendulum is greater than that of the shorter one. The frequency ω_2 is close to the natural frequency of the longer pendulum characterizing this pendulum when it oscillates separate-

ly without being connected with the other pendulum. When the pendulums are coupled and are in the vibration regime shown in Fig. 388*a* the longer pendulum oscillates with a frequency which is close to its own natural frequency. It makes the shorter pendulum follow the motion of the longer pendulum (the natural frequency of the shorter pendulum is greater than that of the longer). The force of the spring exerted on the shorter pendulum* varies periodically so that it is cophasal with the oscillation of that pendulum. The spring receives the maximum elongation in the position shown in Fig. 388*a*; after a half-period time the spring undergoes the maximum compression.

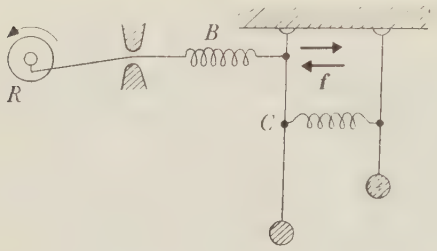

FIG. 389

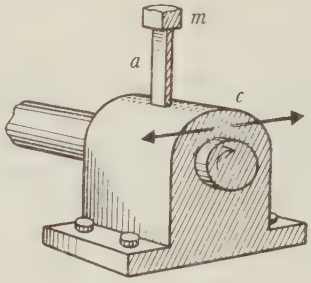

FIG. 390

In the resonance characterized by the relation $p = \omega_1$ the oscillation of the pendulums is close to the normal mode with frequency ω_1. This oscillation regime is shown schematically in Fig. 388*b*. The vibrations of the pendulums are in opposite phases and their common frequency ω_1 is close to the natural frequency of the shorter pendulum (characterizing it when it oscillates separately). In this case we can say that the shorter pendulum excites the longer one because the latter oscillates with frequency ω_1 exceeding its natural frequency. That is why the force exerted by the spring on the longer pendulum and the displacement of this pendulum vary periodically in opposite phases. When the external force F is small and the frequency p differs essentially from the resonant frequencies the amplitudes of the pendulums are very small so that they practically do not move.

The phenomena taking place in the forced oscillation of a system of two pendulums can easily be demonstrated using the device shown schematically in Fig. 389. The spring B must be very light and its stiffness must be sufficiently small while the radius R of the driving crank to which one of the ends of the spring B is attached must be much greater than the displacement of the other end of the spring during the oscillation process.

The case when the frequency p of the external force assumes the value n_1 (see Fig. 387) is of a particular interest because the amplitude of the vibration of the longer pendulum becomes very small. This phenomenon can be used for damping undesirable vibrations. The experiment and the theoretical analysis show that the frequency n_1 is equal to the frequency of the free vibration of the shorter pendulum (in the absence of the external excitation) coupled with the longer pendulum with the aid of a spring when the latter pendulum is at rest.

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* This force can be considered as an external force acting on the shorter pendulum.

In the case $p = n_1$ the shorter pendulum oscillates with frequency n_1 so that the force of the spring applied at the point C (see Fig. 389) balances the action of the external force f; that is why the longer pendulum does not move.

The application of coupled oscillators for damping undesirable vibrations is possible when these vibrations have a constant frequency. For instance, to avoid horizontal vibrations of the bearing c of a machine depicted in Fig. 390 when the shaft supported by the bearing rotates making a constant number of revolutions n, a rod a carrying a mass m on its end is attached to the bearing. The mass m and the elasticity of the rod a must be such that the natural frequency of the rod is equal to n. When the machine works the vibration of the rod may achieve a large amplitude while the bearing remains almost at rest, which damps the undesirable vibrations. Such a device is known as a *dynamic damper*.

The construction of dynamic dampers used for eliminating dangerous torsional vibrations of the crankshafts in the internal combustion engines is based on the same principle.

In the general case when a periodic external force is applied to a system with many degrees of freedom possessing many natural frequencies (whose number coincides with that of the degrees of freedom) the resonant vibrations can arise for every natural frequency. Therefore to avoid the undesirable consequences of the resonant vibrations the frequency of the external force must not coincide with any of the natural frequencies. When a machine is designed this requirement becomes particularly important if it is known that the friction forces are very small. On the contrary, when the friction forces are sufficiently large the resonant vibrations are not dangerous because they are damped very fast after they have been excited.

CHAPTER 15

Vibrations in a Continuous Medium

137. Waves

As was shown in Sec. 120, any variation of the pressure or the density at a certain place in a continuous medium is transmitted with a definite velocity to neighbouring regions of the medium, which results in analogous variations of the pressure or the density in these regions; in this way a pressure wave or a density wave propagates in the medium. For instance, the vibration of the particles of the air caused by the vibration of the vocal cords of a man is transmitted from one particle to another so that a sound wave propagates in the air.

In a homogeneous medium the sound wave generated by a sound source propagates in a similar manner in all directions. At a sufficiently large distance from the sound source the particles of the air whose vibration starts at a given instant owing to the propagation of the initial perturbation lie on a surface of an almost spherical shape. That is why the waves of this kind are called *spherical waves.* A surface on which lie the particles of the homogeneous medium oscillating in the same manner is referred to as a *wave surface* or a *wave front.* It is evident that a sphere at whose centre a sound source with very small dimensions is placed is a wave surface of the spherical wave generated by the source.

FIG. 391

The waves propagating on the free surface of a fluid are a well-known example of waves. The waves on the water surface travelling after a stone has been thrown into the water are an example of the *circular waves.* When a body immersed into the water oscillates harmonically with a given frequency its oscillation generates periodical circular waves propagating from that body. In this case the moving wave crests and wave troughs on the water surface are of circular form (Fig. 391).

The simplest example of a wave motion is a wave propagating in one direction. For instance, such is the wave propagating in the air within a pipe along its axis which is generated by the vibration of the piston (see Fig. 392). The condensation and rarefaction waves propagate from the piston in one direction and

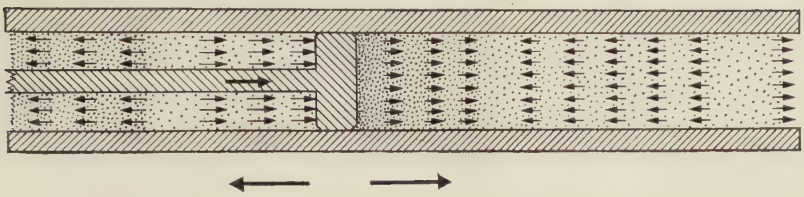

FIG. 392

all the particles of the air taking part in the wave oscillatory motion move along the axis of the pipe. A similar wave propagating in one direction can be generated in an elastic rod by striking one of its ends in the axial direction. In these cases the wave surfaces are the planes perpendicular to the axis of the pipe and to the axis of the rod. That is why the waves of this kind are termed *plane waves.*

A wave in which the particles oscillate along the direction of the propagation of the wave is called a *longitudinal wave* (an example of a longitudinal wave is seen in Fig. 392). A wave in which the particles move in the direction transverse

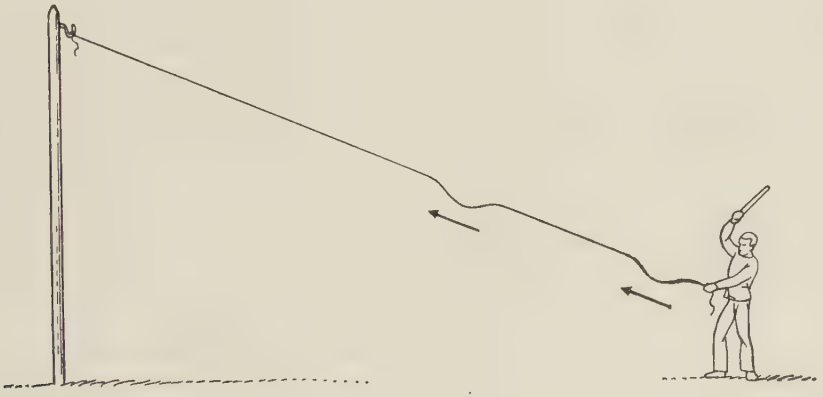

FIG. 393

to the direction of the propagation of the wave is referred to as a *transverse wave* (the example in Fig. 391 shows a transverse wave).

A transverse wave can propagate along a rope, a rubber cord, a string, etc. when they are taut (see Fig. 393). After a taut rubber cord is struck sharply we see that the wave disturbance propagates along it from the place where it has been struck. When the propagating disturbance passes a given part of the cord the particles in this part oscillate in the direction perpendicular to the cord.

A simple example of a wave motion is a wave propagating along a taut string. Let us consider in more detail the propagation of an initial perturbation of the shape of a crest (see Fig. 394) along the string. The crest moves along the string with a definite velocity c which depends on the properties of the string and on its tension, the wave form remaining invariable during the propagation process. The quantity c is called the *wave velocity* (or the *phase velocity* of the wave).

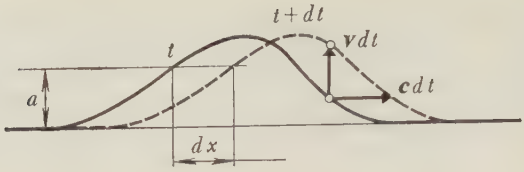

FIG. 394

FIG. 395

One should distinguish between the phase velocity c of the wave and the velocity v of the motion of the given particle of the string (see Fig. 394). After an infinitesimal time interval dt the initial disturbance shown in the continuous line in Fig. 394 passes a distance dx and the string takes the form shown in the dotted line in the figure. The vertical line segment $v\,dt$ is equal to the displacement of the given particle of the string while the horizontal line segment $c\,dt$ is equal to the path of the wave crest during time dt. The velocity v of a particle of the string varies with time; it depends on the wave form and is different for differ-

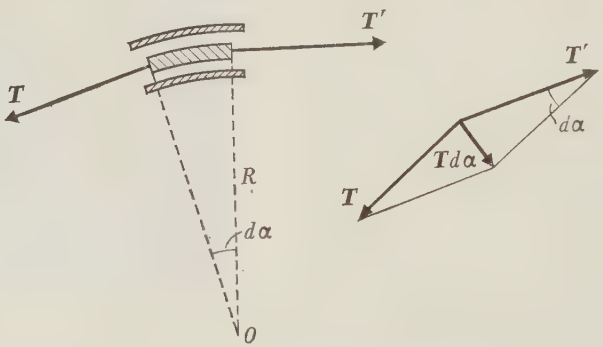

FIG. 396

ent particles while the wave propagation velocity c is constant and is the same for all the parts of the string. The quantity dx in the figure is the distance between two neighbouring particles of the string which have a common transverse displacement a at time instants t and $t + dt$. It is evident that $c = \frac{dx}{dt}$. Let the wave form of the taut string at a given instant t be as shown in Fig. 395. The wave form at the given instant does not provide any information on the direction of the propagation of the wave; such a wave can move either to the right or to

the left. The wave form only indicates that, for instance, the particles a and a' are acted upon by forces F and F' directed upwards while the particle b is acted upon by a force directed downwards (see Fig. 395). These forces are caused by the tension of the string and by its curvature; the greater the curvature of the string at a given place, the greater is the transverse force acting on the corresponding particle. At the points of inflection (for instance, at the points c and c') the transverse force turns into zero.

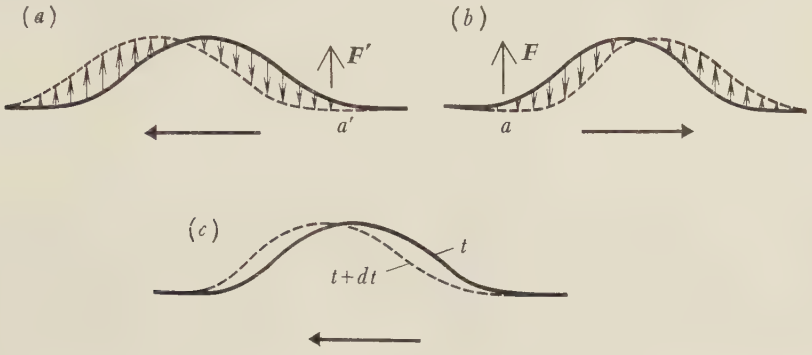

FIG. 397

The appearance of the transverse forces can be explained as follows. Let us isolate mentally a small part of the string (see Fig. 396). The remaining parts of the string exert on the isolated part tensile forces T and T' perpendicular to the cross sections of the string. The resultant F of these forces is directed almost perpendicularly to the string. Hence, the curvature of the string at a given place specifies both the magnitude and the direction of the forces acting on the corresponding part of the string. However, the form of the string at a single time instant t provides no information about the direction of the propagation of the wave. The forces acting on the given part of the string only indicate in which way the velocity of the transverse motion of that part varies.

In case the velocities of motion of the points of the string are known for a given instant we can determine the direction of the propagation of the wave. Let us suppose that the velocities of the particles of the string are as shown by the arrows in Fig. 397*a*. Then the force F' acting on the particle a' will decelerate it and the velocity of the particle will decrease. It is evident that this distribution of the velocities corresponds to a wave propagating to the left.

Knowing the velocities of the points of the string at an instant t we can construct the wave form of the string at time $t + dt$. We see such a construction in Fig. 397*c* showing the wave form at time $t + dt$ for the disturbance having at time t the form depicted in Fig. 397*a*. It is also possible to solve the reverse problem, that is to find the velocities of the various points of the string at a given instant when a wave of a known form propagates along the string without distortion in a definite direction with a known phase velocity. Figure 397*b* represents the distribution of the velocities of the string for a wave propagating to the right. If we bent the string so that it takes the form shown in the figure and simultaneously gave the points of the string the initial velocities shown in the figure the wave pulse of the given form would propagate along the string to the right.

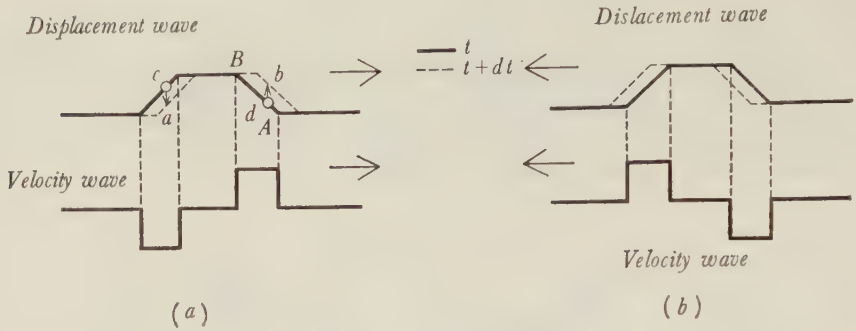

FIG. 398

The connection between the displacements of the particles in a wave and their velocities can easily be demonstrated for the wave disturbance (perturbation) of the form of a trapezoid shown in Fig. 398. In Fig. 398*a* we see two successive states of the string along which this wave pulse propagates to the right for time t (the continuous line) and for time $t + d$ (the dotted line). During time interval dt the point c of the string moves to the position a while the point d moves to the position b. The line segments ca and db are obviously proportional to the corresponding velocities; all the points of the string on the left slant side of the

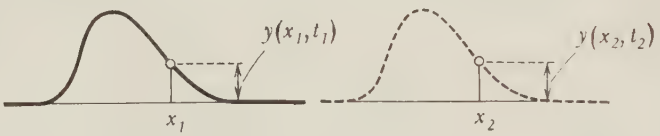

FIG. 399

trapezoid have the same velocity directed downwards and the points on the right slant side have the same velocity directed upwards. The velocity of a given point of the string changes in a jump-like manner when a corner point of the wave disturbance passes through it; for instance, the velocity of the point A at time instant t increases in a jump-like manner while the velocity of the point B reduces up to the zero value. In Fig. 398*b* an analogous construction is shown for the wave propagating to the left.

A wave of an arbitrary form propagating along the string to the right without distortion can be represented analytically in the general form

$$y(x,\ t) = f\left(t - \frac{x}{c}\right) \qquad (137.1)$$

where the function $y\ (x,\ t)$ is equal to the displacement of the point of the string having a coordinate x (reckoned along the string to the right) at time t. The displacement y is a function of the two variables x and t; the function f of the binomial $t - \frac{x}{c}$ represents the form of the string at an instant t as the coordinate x varies along the string.

We can readily show that a wave of an arbitrary form described by a function $f\left(t-\frac{x}{c}\right)$ propagates to the right with the phase velocity c. Indeed, let the form of the wave at an instant t_1 be as shown in the continuous line in Fig. 399. During the time interval from t_1 to t_2 ($t_2 > t_1$) the wave form receives a displacement (without distortion) to the right and the form of the string becomes as shown in the dotted line in the figure. We see that the particle with coordinate x_2 has a displacement $y(x_2, t_2)$ at time t_2 equal to the displacement $y(x_1, t_1)$ of the point with the coordinate x_1 ($x_1 < x_2$) at the earlier instant t_1. It is evident that the arguments of the function f must be equal for t_1 and t_2:

$$t_1-\frac{x_1}{c}=t_2-\frac{x_2}{c}$$

It follows that

$$x_2-x_1=c(t_2-t_1)$$

which means that the wave propagates with the phase velocity c in the direction of the positive x-axis.

General expression (137.1) applies to any plane wave propagating along the positive direction of the x-axis; in such a wave a point with coordinate x receives at time t a departure of magnitude $y(x, t)$ from its equilibrium position. It can similarly be shown that the equation

$$y(x, t)=f\left(t+\frac{x}{c}\right)$$

describes a wave propagating with phase velocity c along the negative x-axis.

The phase velocity of a wave propagating along a taut string can be found by using the following argument. Let a wave pulse of an invariable form propagate along the string. Imagine that a thin rigid tube shaped as the wave form of the string is put on the string and is moved along the string with the velocity c of the propagation of the wave (see Fig. 396). Then the tube exerts no forces on the string. Similarly, if the tube is fixed and the string is pulled through it with constant velocity c in the direction opposite to the direction of the propagation of the wave along the string the tube and the string do not exert any forces on each other either. For the wave propagates along the moving string with the same velocity c and its form remains fixed in space. If the (imaginable) tube were removed the wave crest would remain in the same position. Using this scheme we shall find the value of the velocity c for which the curved rigid tube exerts no action on the taut string pulled through it. Let R be the radius of curvature of the tube at a given place. The element of the string of length $R\,d\alpha$ mentally cut out of the string is acted upon by forces T and T' applied to its ends. These forces have equal magnitudes but their lines of action form an angle of $\pi - d\alpha$. For a small value of the angle $d\alpha$ the resultant of these forces is approximately equal to

$$T\,d\alpha$$

and is directed towards the centre of curvature O. If we assume that the tube exerts no forces on the string then the centripetal acceleration of the element of the string of length $R\,d\alpha$ moving with velocity c along a circle of radius R is only produced by the resultant of the forces T and T'. Therefore we have

$$T\,d\alpha=\rho R\,d\alpha\,\frac{c^2}{R} \qquad (137.2)$$

where $\rho R\, d\alpha$ is the mass of the element of the string and ρ is the *line density* of the string, that is the mass of an element of the string having unit length.

From (137.2) we find the expression for the phase velocity of the propagating wave:

$$c = \sqrt{\frac{T}{\rho}} \tag{137.3}$$

Thus, the velocity of propagation of the wave is equal to the square root of the ratio of the tensile force T to the line density ρ of the string. Let us check that the dimensions of the terms in formula (137.3) are correct. Indeed, we have

$$[T] = \frac{\text{kg}\cdot\text{m}}{\text{s}^2}, \quad [\rho] = \frac{\text{kg}}{\text{m}}, \quad \text{and} \quad \left[\frac{T}{\rho}\right] = \frac{\text{m}^2}{\text{s}^2}$$

In the derivation of the formula for the velocity of propagation of the wave we assumed that the string was absolutely flexible, that is we regarded the string as possessing no flexural rigidity. When a real string is stretched (for instance, a string made of a steel wire possessing a considerable flexural rigidity) the magnitudes of the forces producing the bending moment and the magnitudes of the tensile forces are of the same order. Therefore, the wave propagation process becomes more complex for such a string: the wave disturbance will undergo a distortion and, in the general case, the various wave forms will have different velocities of propagation.

138. Sinusoidal Plane Sound Wave

Let us consider a sound wave propagating in the air within a pipe (see Fig. 392) when the piston exciting the wave oscillates periodically with a frequency lying in the range of the acoustical frequencies (approximately from 16 Hz to 20 000 Hz), the amplitude of the oscillation of the piston being very small. The wave generated in this way is a sinusoidal plane sound (acoustic) wave propagating from the piston along the pipe.

Let the harmonic oscillation of the piston be described by an equation of the form $y_0(t) = A \cos \omega t$ where y_0 is the displacement of the piston and A is its amplitude. The particles of the air adjoining the piston have the same displacement as the piston itself. The particles of the air which are at a distance of x from the particles adjoining the piston (when the piston is at rest) oscillate with a time lag of magnitude $\tau = \frac{x}{c}$ when the piston oscillates. It takes the wave the time $\tau = \frac{x}{c}$ to pass the distance x. The oscillation of these particles can therefore be described by the equation

$$y(x,\ t) = A \cos \omega (t - \tau) = A \cos \left(\omega t - \frac{\omega x}{c}\right) \tag{138.1}$$

Formula (138.1) gives an analytical representation of a *travelling* (*progressive*) sinusoidal plane wave. For any instant t this formula indicates the displacement from the equilibrium position for a particle lying at a distance of x from the piston. The departure (displacement) y $(x,\ t)$ of the particle from the equilibrium is a function depending both on x (x is the coordinate of the particle when it is in the equilibrium position) and on time t. All the particles are in harmonic vibrations with amplitude A and frequency ω but the phases of the particles with different coordinates x are different. It is obvious that the wave front is a plane perpendicular to the x-axis.

Similarly, the equation

$$y = A\cos\omega\left(t + \frac{x}{c}\right)$$

describes a sinusoidal wave propagating along the negative x-axis.

The displacement diagram for the time $t = \frac{T}{4}\left(T = \frac{2\pi}{\omega}\right)$ representing the displacements of the points in the sinusoidal wave described by formula (138.1)

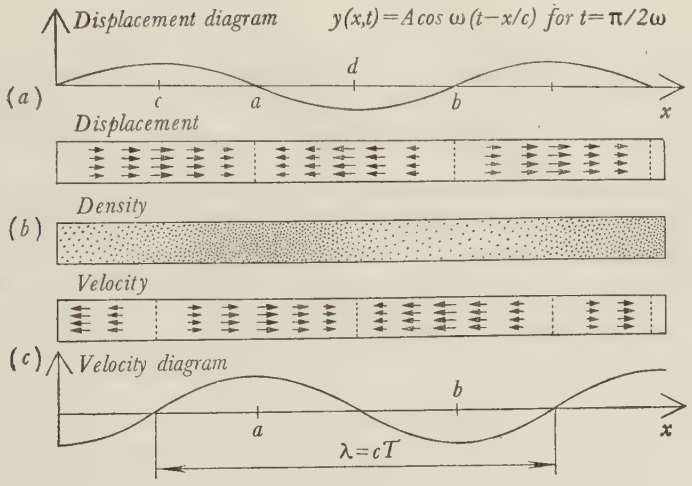

FIG. 400

is shown in Fig. 400*a*. The positive direction of the y-axis corresponds to the displacements of the points in the direction of the increase of the coordinate x.

It is evident that at the point a there is a maximum compression and at the point b a maximum rarefaction. Behind the point a the particles receive a forward displacement and in front of the point a a backward displacement; in the vicinity of the point b the situation is reverse (see Fig. 400*b*). At the points c and d the density does not change since the particles in the vicinity of this point have almost the same displacement.

The particles located at the points a and b have the maximum velocities; at the point a the velocity is directed forward and at the point b backward (Fig. 400*c*). The velocity wave is described by the equation

$$v(x,\ t) = \frac{\partial y}{\partial t} = -A\omega\sin\left(\omega t - \frac{\omega x}{c}\right) \qquad (138.2)$$

The diagram representing this distribution of the velocity for time $t = \frac{T}{4}$ is shown in Fig. 400*c*.

The shortest distance between two points in the wave which oscillate in one phase is called the *wave length* (we denote it λ). The phase difference for two points spaced at a distance of s is equal to

$$\varphi_s = \frac{\omega s}{c} = \frac{2\pi s}{cT} \qquad (138.3)$$

where $T = \frac{2\pi}{\omega}$ is the period of the harmonic vibrations of the particles in the sinusoidal wave. It follows that for two points oscillating in one phase and lying at a distance of λ from each other we have

$$\varphi_\lambda = 2\pi = \frac{\omega\lambda}{c} = \frac{2\pi\lambda}{cT} \tag{138.4}$$

whence

$$\lambda = cT \tag{138.5}$$

Formula (138.5) enables us to state an alternative definition of the wave length: the wave length λ is equal to the distance which the wave passes during a time interval equal to the period T of the vibration. It is evident that the distance between two successive points of maximum compression (or rarefaction) is also equal to λ.

Sinusoidal displacement and velocity waves of the same type can propagate along a long taut homogeneous string; in this case $y\ (x, t)$ is the transverse displacement of the point with coordinate x at time t.

The analysis of the processes in a sound wave (see Fig. 400) shows that a displacement wave is always accompanied by a density wave; the latter is obviously connected with the propagation of the corresponding waves of the pressure, of the velocity, of the acceleration, etc. In other words, all the quantities involved are harmonic functions of time (that is they oscillate harmonically) and their oscillations propagate in space with the phase velocity c. The wave processes in a taut string can be described analogously. Thus, we can say that a wave motion is the propagation in space of the variations of the pressure and the other quantities characterizing the physical state and the motion of the particles of the medium.

The propagation of a travelling wave is of course connected with the transfer of energy from one particle to another. We see that a wave motion is one of the forms of the propagation of energy in space.

139. Energy of Sound Wave

A particle of the medium in which a sound wave propagates oscillates and possesses both the kinetic energy and the strain (potential) energy. The particles which are at a given instant in the places with the maximum compression or rarefaction possess the maximum strain energy (the potential energy of deformation) and the maximum kinetic energy because the velocities of motion in these places also have maximum values (see Fig. 400*c*). The particles which are in the places where the density is constant possess neither the potential nor the kinetic energy.

The kinetic energy of the particles in unit volume of the medium (the *density* of the kinetic energy) is given by the expression

$$E_{\text{kin}} = \frac{(\rho_0 + \rho)\, v^2}{2} \tag{139.1}$$

where ρ_0 is the density of the medium in the absence of the wave motion, ρ is the increment of the density arising because of the compression or rarefaction in the propagating wave, and v is the particle velocity. The variation of the

density in a sound wave is usually very small and therefore ρ is negligibly small in comparison with ρ_0. On neglecting ρ we obtain from (139.1) the approximate formula

$$E_{\text{kin}} \approx \frac{\rho_0 v^2}{2} \tag{139.2}$$

The substitution of the expression of the velocity v given by formula (138.2) in (139.2) yields the expression for the density of the kinetic energy in a harmonic wave:

$$E_{\text{kin}} = \frac{1}{2}\rho_0 \omega^2 A^2 \sin^2\left(\omega t - \frac{\omega x}{c}\right) \tag{139.3}$$

To determine the density of the strain energy, that is the amount of the potential energy (connected with compression or rarefaction of the medium) contained in unit volume, it is necessary to find the increment p of the pressure. Let the pressure in the state of rest be p_0. The variations V and p of the volume and of the pressure are connected by the relation

$$(p_0 + p)(V_0 + V)^{\varkappa} = p_0 V_0^{\varkappa} \tag{139.4}$$

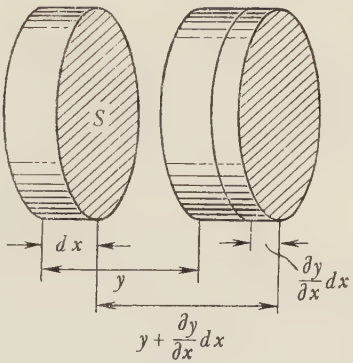

FIG. 401

characteristic of adiabatic processes (see Sec. 105). Here V_0 is the volume of a given particle in the state of rest (the increment V of the volume appears owing to the wave propagation). It follows from (139.4) that

$$p = -\varkappa \frac{p_0}{V_0} V \tag{139.5}$$

(in the derivation of (139.5) we take into account that V is small relative to V_0, use the relation

$$(V_0 + V)^{\varkappa} = V_0^{\varkappa}\left(1 + \frac{V}{V_0}\right)^{\varkappa} \approx V_0^{\varkappa}\left(1 + \varkappa \frac{V}{V_0}\right)$$

for small values of $\frac{V}{V_0}$ and neglect the terms of the second order of smallness relative to the quantities $\frac{p}{p_0}$ and $\frac{V}{V_0}$).

Let us determine the variation of the volume in the wave. To this end we consider the volume $V_0 = S\,dx$ where S is the cross-section area of the pipe in which the sound wave propagates (see Fig. 401). Because of the wave displacements the particles contained in the volume V_0 are expanded or compressed and occupy a new volume

$$V_0 + V = S\left(dx + \frac{\partial y}{\partial x} dx\right)$$

(see Fig. 401). It follows that

$$V = S \frac{\partial y}{\partial x} dx \tag{139.6}$$

On substituting (139.6) in (139.5) we obtain the expression for the variation of the pressure arising in the propagating wave:

$$p = -\varkappa \frac{p_0}{V_0} S \frac{\partial y}{\partial x} dx = -\varkappa \frac{p_0}{S\,dx} S \frac{\partial y}{\partial x} dx = -\varkappa p_0 \frac{\partial y}{\partial x} \tag{139.7}$$

We see that the increment of the pressure at a given place is proportional to minus the derivative of the displacement y with respect to the coordinate x. Since the sound velocity in the medium is $c = \sqrt{\varkappa \frac{p_0}{\rho_0}}$ (see Sec. 120) expression (139.7) for the variation p of the pressure can be written in the form

$$p = -\rho_0 c^2 \frac{\partial y}{\partial x} \tag{139.8}$$

Consequently, the variations of the pressure produced by the propagating wave (determined by formula (138.1)) describing the displacements of the particles in the sinusoidal acoustic wave are represented by the formula

$$p(x,\ t) = -\rho_0 c^2 \frac{A\omega}{c} \sin\left(\omega t - \frac{\omega x}{c}\right) = -\rho_0 A\omega c \sin\left(\omega t - \frac{\omega x}{c}\right) \tag{139.9}$$

Hence, the oscillation of the pressure and the oscillation of the velocity of the particle are always cophasal (cf. (138.2) and (139.9)). In the places where the density of the kinetic energy achieves its maximum value at a given instant t the density of the strain energy also attains its maximum.

The potential energy (the strain energy) is equal to the work which must be expended on the increase (decrease) of the pressure of the air by a small quantity p. We can also say that this work is expended on the corresponding increase (decrease) of the volume V_0 by an amount of V. For small variations of the pressure and of the volume $\left(\text{that is for } \frac{p}{p_0} \ll 1 \text{ and } \frac{V}{V_0} \ll 1\right)$ it is always allowable to assume that the pressure variation p is proportional to the volume variation V (see (139.5)). In this case the work expended on the increment V of the volume V_0 is equal to $-\frac{pV}{2}$.* Therefore the density of the potential energy can be written in the form

$$E_{\text{pot}} = -\frac{pV}{2V_0} \tag{139.10}$$

The substitution of the value of V given by formula (139.6) and of the value of p given by (139.8) in (139.10) results in the expression for the density of the strain energy:

$$E_{\text{pot}} = \frac{1}{2} \rho_0 c^2 \left(\frac{\partial y}{\partial x}\right)^2 \tag{139.11}$$

Consequently, the formula describing the variation of the potential energy in the wave has the form

$$E_{\text{pot}} = \frac{1}{2} \rho_0 c^2 \left[\frac{\partial}{\partial x} A \cos\left(\omega t - \frac{\omega x}{c}\right)\right]^2 = \frac{1}{2} \rho_0 A^2 \omega^2 \sin^2\left(\omega t - \frac{\omega x}{c}\right) \tag{139.12}$$

---

* Indeed, the variation of the pressure is equal to $p = -aV$ where a is a constant. The work is therefore equal to

$$-\int_0^p p\,dV = \frac{1}{a} \int_0^p p\,dp = \frac{p^2}{2a} = -\frac{pV}{2}$$

The comparison of (139.12) with the expression of the kinetic energy (see (139.3)) shows that for any instant t for every point x the densities of the kinetic and the potential energy of a particle in a progressive acoustic wave are equal to each other. Therefore the density of the total energy transferred by the wave is

$$E = E_{\text{kin}} + E_{\text{pot}} = \rho_0 A^2 \omega^2 \sin^2 \left(\omega t - \frac{\omega x}{c} \right) \tag{139.13}$$

During a small time interval Δt the wave propagates through a distance equal to $c\ \Delta t$. Consequently, the amount of energy transferred through a unit plane

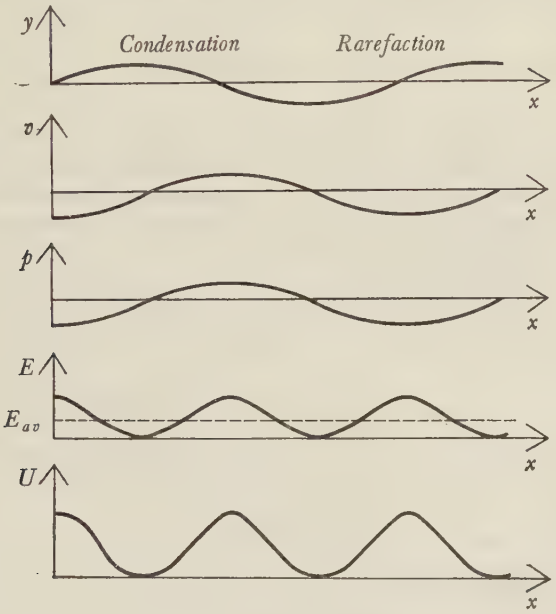

FIG. 402

area perpendicular to the direction of the propagation of the wave is equal to

$$\Delta U_e = Ec\ \Delta t \tag{139.14}$$

The ratio of the amount of energy passing during time Δt across unit area to Δt is called the *energy flux*. In the case under consideration the energy flux is expressed by the formula

$$U_e = \frac{\Delta U_e}{\Delta t} = Ec = \rho_0 A^2 \omega^2 c \sin^2 \left(\omega t - \frac{\omega x}{c} \right) \tag{139.15}$$

The energy flux is usually represented by a vector indicating the direction of the propagation of the energy, the length of the vector being equal to the amount of energy passing through unit area in unit time. This is the *energy flux vector*.

In Fig. 402 we see the graphs representing for a given instant the space distribution of the displacement, of the velocity and of the pressure in the wave and also the distribution of the energy density and of its flux along the direction of the propagation of the wave. As is seen from Fig. 402, the average energy den-

sity* equals half the maximum value of the density of the total energy (see (139.13)):

$$E_{\text{av}} = \frac{1}{2} \rho_0 A^2 \omega^2 \tag{139.16}$$

Consequently, the mean density of the total energy is equal to the maximum value of the density of the kinetic energy (which coincides with the maximum value of the density of the potential energy).

We have investigated the motion of the particles of the air and the propagation of energy in a plane sound wave along the pipe. Since the diameter of the pipe can be quite arbitrary it follows that all we have established applies to any plane harmonic progressive (travelling) sound wave.

140. Plane Waves in a Gas and in a Homogeneous Elastic Medium

The front of a plane wave is a plane surface normal to the direction of the propagation of the wave. We considered plane sinusoidal waves propagating in a cylindrical pipe. However, in such a pipe there can also exist progressive plane waves of any other form as in the case of a string. In this section we shall derive the general equations describing arbitrary plane waves of type (137.1) propagating in a gas or in an elastic medium.

Let us consider a cylindrical volume of gas shown in Fig. 403. Denoting by dx the altitude of this cylinder and using the notation of Secs. 138 and 139 we can write the equality

$$pS - \left(p + \frac{\partial p}{\partial x}\, dx\right) S = \rho_0 S\, dx\, \frac{\partial^2 y}{\partial t^2}$$

expressing the second law of dynamics. This equality implies

$$-\frac{\partial p}{\partial x} = \rho_0 \frac{\partial^2 y}{\partial t^2} \tag{140.1}$$

Using expression (139.8) we find

$$\frac{\partial p}{\partial x} = -\rho_0 c^2 \frac{\partial^2 y}{\partial x^2}$$

The comparison of the last expression with (140.1) leads to

$$c^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} \tag{140.2}$$

Relation (140.2) is the *wave equation* describing the propagation of the plane waves in a gas.

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* By the average (mean) density of the total energy can be meant either the mean value of E taken with respect to time t for a given point $x = \text{const}$ or the mean value with respect to the spatial coordinate x for $t = \text{const}$. The former is computed according to the formula $E^t_{\text{av}} = \frac{1}{T} \int_0^T E\, dt$ for $x = \text{const}$ and the latter according to the formula $E^x_{\text{av}} = \frac{1}{\lambda} \int_0^\lambda E\, dx$ for $t = \text{const}$ where T is the period and λ is the wave length. It is obvious that $E^x_{\text{av}} = E^t_{\text{av}} = E_{\text{av}}$.

It can easily be verified that any function $y(x, t)$ of the form

$$y(x, t) = f\left(t \pm \frac{x}{c}\right)$$

is a solution to this equation. Indeed, we have

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} f'' \quad \text{and} \quad \frac{\partial^2 y}{\partial t^2} = f''$$

where f'' is the second derivative of f with respect to the argument $t \pm \frac{x}{c}$. The function f can be quite arbitrary (it is only required that it should be differentiable twice). It specifies a wave form propagating without distortion with the constant phase velocity c. All the particles lying on the wave front (which is a plane perpendicular to the x-axis) corresponding to an arbitrary value of the coordinate x receive the same displacement y, are under the same pressure p, have the same density ρ, etc. at any given instant t.

In an elastic medium there can also exist waves in which disturbances of deformation, of stresses, of displacements, etc. propagate.

We shall suppose that the continuous medium occupying some region in space is homogeneous and isotropic. The laws governing the propagation of elastic waves are rather complex even when the substance is homogeneous and isotropic. When a non-homogeneity of the deformation is caused in some place there appear non-homogeneous stresses; they produce deformations and displacements of adjoining particles of the medium which, in their turn, transmit the motion to other particles, and so on. As a result, there appear elastic waves propagating in the medium.

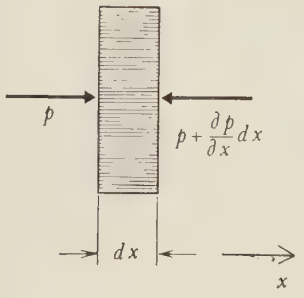

FIG. 403

To set the equation of motion of a particle of an infinitesimal volume dv under the action of non-homogeneous stresses let us determine the force exerted on this volume by the surrounding medium. The stresses at every point depend on the coordinates x_1, x_2, and x_3 of the point and on time t. Let us find the component of the force (acting on the particle) along axis *1* (axis *1* corresponds to the coordinate x_1; see Fig. 404). This component is equal to the sum of all forces acting on the particle dv in the direction of the x_1-axis. According to Sec. 85, this sum can be written in the form

$$dF_1 = \left(\sigma_{11} + \frac{\partial \sigma_{11}}{\partial x_1} dx_1\right) dx_2\, dx_3 - \sigma_{11}\, dx_2\, dx_3 +$$

$$+ \left(\sigma_{12} + \frac{\partial \sigma_{12}}{\partial x_2} dx_2\right) dx_3\, dx_1 - \sigma_{12}\, dx_3\, dx_1 + \left(\sigma_{13} + \frac{\partial \sigma_{13}}{\partial x_3} dx_3\right) dx_1\, dx_2 - \sigma_{13}\, dx_1\, dx_2 =$$

$$= \left(\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3}\right) dx_1\, dx_2\, dx_3$$

Consequently, the force density along axis *1* is

$$f_1 = \frac{dF_1}{dv} = \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} \tag{140.3}$$

For the other two axes we find in just the same manner

$$f_2 = \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} \quad \text{and} \quad f_3 = \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} \tag{140.4}$$

By the second law of dynamics, the force acting on the volume dv equals its mass $\rho\, dv$ times the acceleration. Therefore, we have

$$f_1\, dv = \rho\, dv\, \frac{\partial^2 s_1}{\partial t^2}, \quad f_2\, dv = \rho\, dv\, \frac{\partial^2 s_2}{\partial t^2} \quad \text{and} \quad f_3\, dv = \rho\, dv\, \frac{\partial^2 s_3}{\partial t^2} \tag{140.5}$$

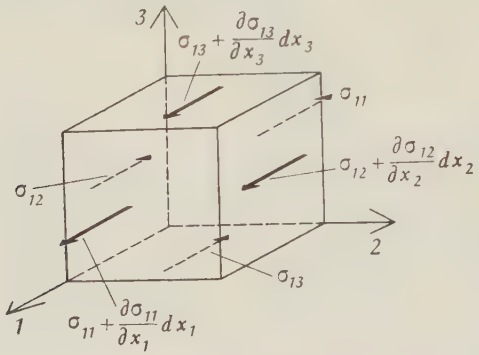

FIG. 404

where ρ is the density of the substance and the quantities s_1, s_2, and s_3 are the components of the displacement vector of the particle located originally at the point (x_1, x_2, x_3).

To derive the sought-for equation we must express f_1, f_2, and f_3 in terms of the components s_1, s_2, and s_3 of the displacement vector $\boldsymbol{s}$. In the general case the structure of equations (140.5) is complex and the analysis of the equations is rather intricate. That is why we limit ourselves to the simplest case of a plane wave.

For definiteness, let us consider a plane wave propagating along the x_1-axis; in this case all the quantities involved depend only on the coordinate x_1 and on time t and all the derivatives with respect to x_2 and x_3 are equal to zero. From (140.3) and (140.5) it follows that

$$f_1 = \frac{\partial \sigma_{11}}{\partial x_1} = \rho\, \frac{\partial^2 s_1}{\partial t^2} \tag{140.6}$$

By (87.15), we have $\sigma_{11} = 2G\left(\varepsilon_1 + \frac{\mu}{1-2\mu} 3\varepsilon\right)$ where $\varepsilon_1 = \frac{\partial s_1}{\partial x_1}$ and $3\varepsilon = \frac{\partial s_1}{\partial x_1} + \frac{\partial s_2}{\partial x_2} + \frac{\partial s_3}{\partial x_3}$.

Therefore

$$\frac{\partial \sigma_{11}}{\partial x_1} = 2G\left[\frac{\partial \varepsilon_1}{\partial x_1} + \frac{\mu}{1-2\mu} \frac{\partial}{\partial x_1} (3\varepsilon)\right] = 2G\left[\frac{\partial^2 s_1}{\partial x_1^2} + \frac{\mu}{1-2\mu} \frac{\partial^2 s_1}{\partial x_1^2}\right]$$

We can now rewrite (140.6) in the form

$$2G\left(1 + \frac{\mu}{1-2\mu}\right) \frac{\partial^2 s_1}{\partial x_1^2} = \rho\, \frac{\partial^2 s_1}{\partial t^2} \tag{140.7}$$

The equation we have obtained is the wave equation describing the propagation of a longitudinal elastic wave because the particles taking part in the wave motion receive displacements s_1 only along the x_1-axis in the direction of which the wave propagates. Equation (140.7) belongs to type (140.2) and therefore the phase velocity of the displacement wave described by the variation of the quantity s_1 as function of the coordinate x_1 and time t is given by the formula

$$c_1 = \sqrt{\frac{2G}{\rho}\left(1+\frac{\mu}{1-2\mu}\right)} = \sqrt{\frac{E(1-\mu)}{\rho(1+\mu)(1-2\mu)}} \tag{140.8}$$

We see that the velocity c_1 depends on E, ρ and μ.

For the displacements s_2 and s_3 along the directions of axes *2* and *3* we can similarly derive from (140.4) and (140.5) the equations

$$f_2 = \frac{\partial \sigma_{21}}{\partial x_1} = \rho \frac{\partial^2 s_2}{\partial t^2} \quad \text{and} \quad f_3 = \frac{\partial \sigma_{31}}{\partial x_1} = \rho \frac{\partial^2 s_3}{\partial t^2} \tag{140.9}$$

By (87.6), we have $\sigma_{12} = G\gamma_{12} = G\left(\frac{\partial s_1}{\partial x_2} + \frac{\partial s_2}{\partial x_1}\right)$ whence

$$\frac{\partial \sigma_{21}}{\partial x_1} = G \frac{\partial^2 s_2}{\partial x_1^2}$$

On substituting the last expression in equation (140.9) we obtain for s_2 the equation

$$G \frac{\partial^2 s_2}{\partial x_1^2} = \rho \frac{\partial^2 s_2}{\partial t^2} \tag{140.10}$$

For s_3 we similarly obtain

$$G \frac{\partial^2 s_3}{\partial x_1^2} = \rho \frac{\partial^2 s_3}{\partial t^2}$$

The phase velocity of the waves described by the last two equations is

$$c_2 = \sqrt{\frac{G}{\rho}} = \sqrt{\frac{E}{2(1+\mu)\rho}} \tag{140.11}$$

The waves with phase velocity c_2 described by the last two equations are transverse because the displacements s_2 and s_3 are normal to the direction of the propagation of the waves along the x_1-axis.

For the longitudinal waves in a rectilinear elastic rod the wave velocity is

$$c = \sqrt{\frac{E}{\rho}} \tag{140.12}$$

Indeed, let the axis of the rod coincide with axis *1*; then $\sigma_{22} = \sigma_{33} = 0$ and from the general equality (87.1) it follows that $\varepsilon_1 = \frac{1}{E}[\sigma_{11} - \mu(\sigma_{22} + \sigma_{33})]$ whence

$$\sigma_{11} = E\varepsilon_1, \quad \text{that is} \quad \sigma_{11} = E \frac{\partial s_1}{\partial x_1}$$

Therefore, equation (140.6) takes the form

$$E \frac{\partial^2 s_1}{\partial x^2} = \rho \frac{\partial^2 s_1}{\partial t^2} \tag{140.13}$$

It turns out that for such waves the phase velocity is less than the propagation velocity of the longitudinal waves in the medium (with parameters E, ρ, and μ) discussed earlier and is greater than the phase velocity of the transverse waves.

For we have

$$\frac{1-\mu}{(1+\mu)(1-2\mu)} \geqslant 1 > \frac{1}{2(1+\mu)} \tag{140.14}$$

because $\frac{1}{2} > \mu \geqslant 0$.

The dependence of the wave velocities c_1 and c_2 on Poisson's ratio μ is demonstrated by the diagrams in Fig. 405. We see that in all these cases the sinusoidal waves of any frequency propagate with the same velocity. This means that for the waves of this kind the medium is not *dispersive*, that is the wave velocity is independent of the frequency.

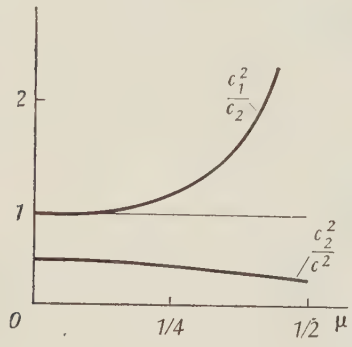

FIG. 405

It should be noted that the plane waves propagating in an unbounded spatial region we have considered here are an abstraction because in reality such waves propagate along a pipe, in a rod, etc., that is in a bounded part of space. For instance, longitudinal (or transverse) plane waves can be observed in a region near the surface of a piston in a pipe exciting the corresponding vibrations or on the water surface in a bounded basin where the waves are generated by a vibrating body immersed into the water.

When a wave generated by a point source propagates in a gas (we speak of a point source if the dimensions of the source are very small) the wave can be regarded as being plane in regions whose dimensions are sufficiently small in comparison with the distance from the source and which lie sufficiently far from the source. In this case every sufficiently small part of the spherical wave generated by the source can be considered approximately as being plane.

141. Superposition (Interference) of Waves

Experiments show that when there are several waves propagating simultaneously in a medium every particle of the medium takes part in several simultaneous vibrations and that for the acoustic waves there holds the *superposition principle*. The superposition principle states that every constituent wave propagates in the medium irrespective of the presence of the other waves, that is the course of every constituent wave process is such as if there were no other waves at all. It follows that in order to determine the resultant motion of a particle of the medium we should determine the motion of the particle in each of the waves separately and then add together all these motions.

The phenomenon of superposition of two or more wave motions is called the *interference* of waves.

Let us consider the interference of two sound waves propagating in a pipe. Suppose that in a pipe there are two waves of one frequency propagating simultaneously in opposite directions. Let one of these waves propagate along the positive x-axis and be determined by the equation

$$y_1 = A \cos \omega \left(t - \frac{x}{c}\right)$$

and let the other wave propagate in the opposite direction and be described by the formula

$$y_2 = B \cos \omega \left(t + \frac{x}{c}\right)$$

To determine the resultant wave motion we use the evident fact that at any instant t the deviation y of every point from the equilibrium position is given by the relation

$$y = y_1 + y_2$$

The wave y_2 can always be represented as a sum of two travelling waves in the form

$$y_2 = A \cos \omega \left(t + \frac{x}{c}\right) + (B - A) \cos \omega \left(t + \frac{x}{c}\right) \tag{141.1}$$

Therefore the resultant vibration $y\ (x,\ t)$ can be written as

$$y = y_1 + y_2 = A \cos \omega \left(t - \frac{x}{c}\right) + A \cos \omega \left(t + \frac{x}{c}\right) + (B - A) \cos \omega \left(t + \frac{x}{c}\right) =$$

$$= 2A \cos \frac{\omega x}{c} \cos \omega t + (B - A) \cos \omega \left(t + \frac{x}{c}\right) \tag{141.2}$$

We see that the resultant wave motion consists of two parts one of which is the *standing wave*

$$2A \cos \frac{\omega x}{c} \cos \omega t \tag{141.3}$$

while the other is a travelling wave of the form

$$(B - A) \cos \omega \left(t + \frac{x}{c}\right) \tag{141.4}$$

In case $B = A$ when the two travelling waves propagating in the opposite directions have equal amplitudes the resultant wave motion reduces to standing wave (141.3) alone.

The motion of the particles in standing wave (141.3) can be interpreted quite visually for the case $A = B$ if we perform the graphical addition of the deviations of the particles taking part in the two constituent travelling waves for equidistant time instants as it is done in Fig. 406. The dash-line in the figure represents the wave form propagating to the left, the dot-line represents the wave form propagating to the right and the continuous line shows the position of the particles in the resultant standing wave (for the sake of convenience in Fig. 406 is shown a standing wave whose initial phase is equal to $-\pi/8$).

As it follows from formula (141.3), the particles in a standing wave oscillate either in one phase or in opposite phases and that, generally, the amplitudes of different points are different. This is clearly seen in Fig. 406 where the particles located at the points O, O', O'', . . . are at rest all the time, the particles which are located at the points A, A', A'', . . . (marked by the small circles) in the state of equilibrium oscillate with the maximum amplitude when the standing wave arises and all the other particles have amplitudes ranging between zero and the maximum value. The points O, O', O'', . . . whose amplitude is equal to zero are called the *nodes* of the standing wave. The nodes lie at a distance of $\frac{\lambda}{2}$ from one another, that is at a distance equal to half the wave length.

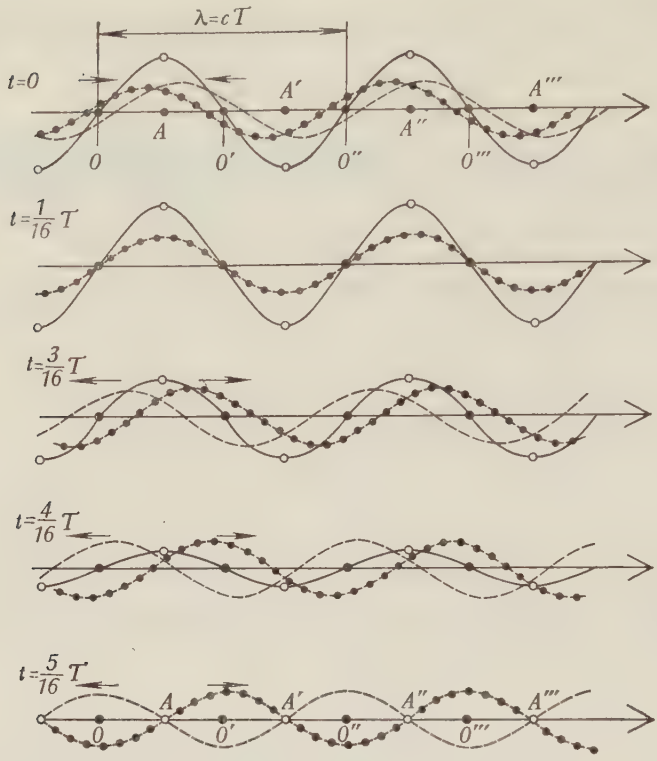

FIG. 406

Taking into account that $\lambda = cT = \frac{2\pi c}{\omega}$ we can rewrite expression (141.3) for the standing wave in the form

$$2A \cos \frac{2\pi x}{\lambda} \cos \omega t \qquad (141.5)$$

whence it is seen that the nodes of the wave have the coordinates $x = \frac{2n+1}{2} \frac{\lambda}{2}$ where n is an integral index.

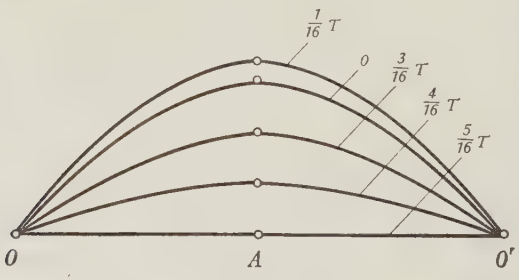

FIG. 407

The points A, A', A'', . . . are called the *antinodes* of the standing wave. The coordinates of the antinodes are $x_n = n\frac{\lambda}{2}$ where n is an integer (this follows from representation (141.5) of the standing wave).

In Fig. 407 are shown the successive positions of the particles lying between the two nodes O and O' for the time instants corresponding to the curves in Fig. 406. We see that the particles lying between two neighbouring nodes oscillate harmonically in one phase (they simultaneously attain their extreme positions, simultaneously pass through the equilibrium position, etc.) but their amplitudes are different.

Since the particles located at the nodes are permanently at rest there can be no transfer of energy through the nodes. Only those particles which lie between any two neighbouring nodes exchange their energies; as a whole, the motion of the particles in a standing wave is not connected with the transfer of energy along the wave although this wave results from the superposition of two travelling waves with equal amplitudes each of which, when considered separately, is accompanied by a transfer of energy.

When the amplitudes of the travelling waves propagating in the opposite directions are not equal the resultant wave motion consists of standing wave (141.3) and travelling wave (141.4). The amplitude of the travelling wave described by (141.4) is equal to the difference between the amplitudes of the constituent travelling waves. The quantity

$$k_\sigma = \frac{|A-B|}{B} \tag{141.6}$$

is called the *travelling-wave ratio*. When this ratio is different from zero the energy is transferred in the direction of the propagation of the wave whose amplitude is greater.

142. Reflection of Waves

Standing waves usually appear when a travelling wave is reflected from the surface bounding the medium in which the wave propagates. The same occurs when a travelling wave is incident on the interface between two different media and, generally, when the properties of the substance in which a wave propagates change sharply. In all such cases there appear *reflected waves* whose superposition with the original waves may result in the appearance of standing waves.

To demonstrate the mechanism of the phenomenon of reflection of waves we consider the reflection of a local wave disturbance propagating along a taut rubber cord (or along a string, a rope, and the like).

Let a wave pulse propagate along a taut rubber cord as was shown in Fig. 393. When this pulse achieves the fixed end of the cord (see Fig. 408) it is reflected and then travels in the backward direction. The experiment shows that the reflected pulse has the same wave form as the incident one but the oscillations of the particles in the incident wave and in the reflected wave are in opposite phases. If the incident wave disturbance is of the form of a crest the reflected pulse is shaped as a trough and vice versa.

During the period of reflection of the wave pulse the fixed end of the cord is acted upon by a force perpendicular to it. The action of this force not only makes the particles of the cord adjoining the fixing remain at rest but also generates the reflected wave in which the direction of the deviation is opposite to that of the deviation of the particles in the incident wave (see Fig. 408*a*). If, for instan-

ce, the wave form of the incident pulse is a crest convex upwards the force exerted by the rigid fixing is directed downwards and its action is equivalent to a downward impact producing a wave-form trough convex downwards and travelling in the opposite direction as shown in Fig. 408*b*.

The particles of the cord adjoining the fixing are permanently at rest and therefore the energy of the wave motion cannot be transferred through the point of fixing. Therefore, by the energy conservation law, the reflected wave pulse must be of a shape similar, to within the symmetry, to the shape of the incident

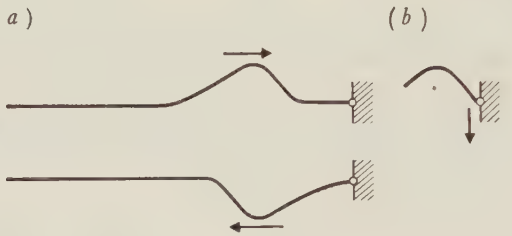

FIG. 408

wave. This qualitative argument can be confirmed by the exact theoretical computations (under the assumption that the energy loss connected with the deformation of bending of a cord or of a string is negligibly small).

The variation of the wave form during the reflection process can be easily determined if we imagine that the incident pulse (see Fig. 409) propagates without distortion further to the right of the point of rigid fixing while there simultaneously appears a reflected wave pulse propagating in the opposite direction whose shape is symmetric to the incident pulse with respect to the axis of the cord. At every instant the resultant deviation of the particles of the cord adjoining the point of fixing is equal to zero because this deviation is equal to the sum of the two deviations corresponding to the two (imaginary*) wave disturbances which have equal magnitudes and opposite directions. The real deviation of any other particle of the cord can also be found as the result of the superposition of the two waves.

The dotted lines in Fig. 409 represent the (imaginary) incident and reflected wave forms for different instants and the continuous line shows the position of the particles of the cord for the same instants. It is assumed here that the wave velocity is equal to c and that the length of the pulse is equal to a, the shapes of the incident and the reflected pulses being shown in Fig. 409 for the instants $t = 0$, $t = 2\tau$, $t = 4\tau$,' $t = 6\tau$, and $t = 8\tau$ where $2\tau = \frac{a}{5c}$.

The wave form of the pulse in Fig. 409 is such that the particles lying on the steeper side of the wave crest possess the velocity which is four times that of the particles on the slanter side. Knowing this fact we can determine not only the form of the resultant wave but also the velocities of the particles of the cord by

* To avoid misunderstanding it should be noted that the term "imaginary" is used here in a conditional sense because only those parts of the two waves which are on the right of the rigid fixing do not exist in reality while their parts on the left of the fixing are quite real and their superposition results in the appearance of the reflected wave imposed on the incident wave.

adding together geometrically the velocities of the particles corresponding to the incident and the reflected pulse. For example, it can readily be seen that the rectilinear part of the cord adjoining the point of fixing has a zero velocity during the time interval from $t = 2\tau$ to $t = 6\tau$. The velocity of the particles of the cord lying on this part becomes non-zero after the instant $t = 6\tau$. During the time interval from $t = 8\tau$ to $t = 10\tau$ (Fig. 409 does not show the form of the cord for $t = 10\tau$) there appears a second corner point moving from the former corner point towards the point of fixing. The second corner point (which moves towards the fixing) is seen in the figure showing the form of the cord for the instant $t = 8\tau$. After time $t = 10\tau$ the reflected wave takes its final form (symmetric to the original incident wave form about the axis of the cord) which propagates to the left of the fixing.

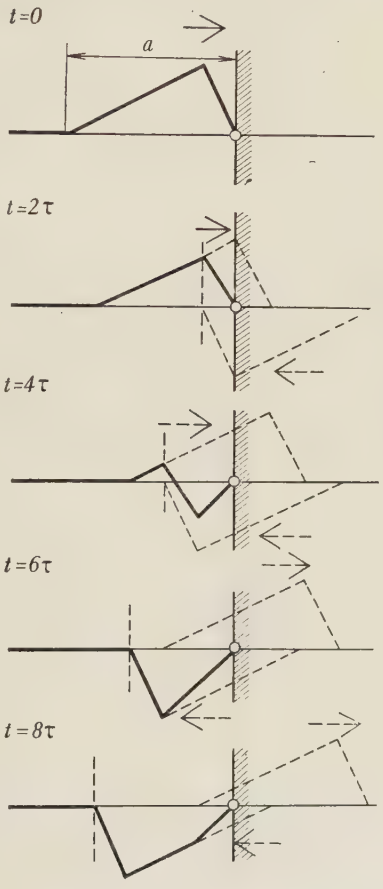

FIG. 409

The analysis of Fig. 409 shows that the vertical velocity of a particle changes at the moments when the corner points of the wave form pass through the point where the particle is located; at these instants the particle undergoes jerks which change its velocity. For the resultant of the tensile forces applied to the particle of the cord from the left and from the right has a non-zero vertical component only at these moments of time. The vertical force applied to the particle of the cord is proportional to the variation of the angle of inclination of the wave front at the given place. This analysis of the reflection of the propagating wave perturbance from the fixed end of the cord shows that during the reflection process the form of the incident pulse undergoes a deformation and that the reflected pulse propagating in the opposite direction has the similar form and only its phase changes to the opposite.

The same law governs the reflection of a sound wave from a fixed wall. Suppose that there is a travelling sound wave propagating along a pipe which is of the type of a moving local disturbance of the air. Every particle in the propagating sound pulse is compressed under the action of the neighbouring particle located behind it, receives a definite velocity, and then transmits this action to the neighbouring particle located in front of it, and so on. Near the fixed wall the situation changes: the particles adjoining the wall undergo compression but cannot receive a non-zero velocity and therefore their compression must be greater than that of the other particles. This additional compression generates the reflected wave of the opposite phase propagating from the wall in the backward

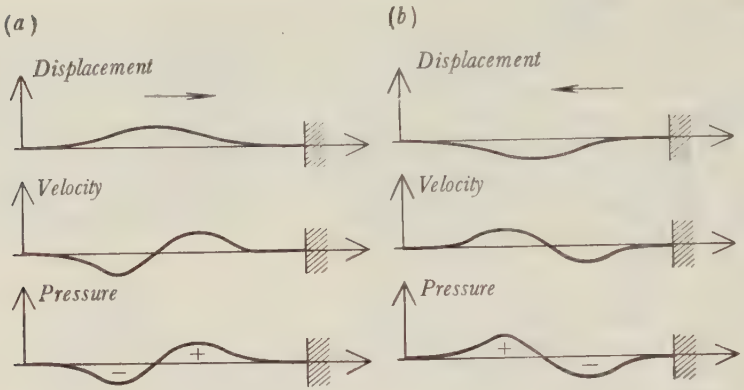

FIG. 410

direction. In Fig. 410*a* we see the graphs representing the distributions of the displacement, of the velocity and of the pressure in a sound wave pulse propagating to the right (the variations of the pressure characterize the corresponding variations of the density of the air). In Fig. 410*b* the graphs representing the same quantities are shown for the same wave pulse after it has been reflected from the fixed wall. The graphs indicate that the condensation is always located in the front part of the wave and that the velocity and the pressure in a travelling wave are always in one phase.

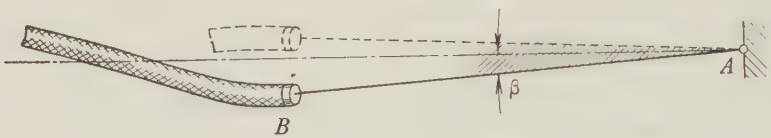

FIG. 411

The situation becomes quite different when a travelling wave is reflected from a free boundary of the medium. For instance, when a sound wave propagating along an elastic steel rod achieves its end it is reflected backwards because the density of the surrounding air is very small in comparison with the density of the steel and therefore the motion of the particles of the air cannot affect noticeably the motion of the particles of the rod. In other words, the particles of the steel adjoining the surface of the rod move as if the rod were placed in vacuum. Therefore, the energy of the wave cannot be transferred through the terminal cross section of the rod, the wave is reflected and then propagates in the opposite direction.

The reflection of a transverse wave from a free end of a rubber cord can be demonstrated in the experiment whose scheme is shown in Fig. 411. Here a sufficiently thin and light thread is attached to an end of the cord, the other end of the thread being fixed. In the experiment we observe the wave pulses propagating along the stretched cord which are excited by striking the cord. It is readily

seen that the pulses are reflected from the point B to which the thread is attached and then propagate in the backward direction in the same phase; for instance, a wave disturbance in which the particles receive downward displacements retains the direction of the displacements after it has been reflected. The end of the rubber cord to which the thread is attached can be considered free because the line density of the thread is much less than that of the cord while the tension is the same for both the cord and the thread. Therefore, only a very small part of the energy carried by the incident wave can be transferred to the thread; almost the whole energy is reflected and is carried in the backward direction by the reflected wave. The end of the thread moves under the action of the cord as if there were no other transverse forces between the points A and B: at any instant the thread retains a rectilinear form (see Fig. 411). The angle β characterizing the angular deviation of the thread from the horizontal line is negligibly small in comparison with the angle which various parts of the cord form as the wave pulse propagates along it. Therefore, the transverse component of the tensile forces in the thread is negligibly small in comparison with the transverse force connected with the bending of the rubber cord.

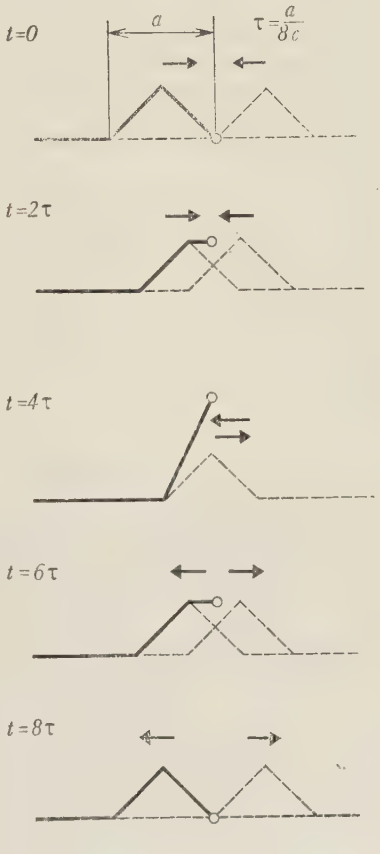

FIG. 412

The wave reflected from the free boundary and the incident wave have one phase. To account for it let us come back to the example of a steel rod along which a sound wave propagates. A particle of the rod adjoining its free end cannot be compressed because the free end of the rod exerts no force on the particle. Therefore, the action of the neighbouring particle only gives a certain velocity to the former particle and hence the displacement of the particle near the free end of the rod must be twice the displacement of any other particle. This displacement (which is twice as large as the displacement in the incident wave) generates a reflected wave propagating backwards from the free end of the rod. The amplitude of the reflected wave must be equal to the amplitude of the incident wave, which follows from the law of conservation of energy.

In Fig. 412 we see successive stages of the reflection process when a wave disturbance of a triangular form achieves the free end of the rubber cord. The moment the maximum displacement of the incident wave achieves the free end the latter receives a doubled displacement.

As was shown earlier, the superposition of two sinusoidal travelling waves with equal amplitudes and frequencies propagating in opposite directions results in a standing wave. When a sinusoidal sound wave propagating along a pipe is reflected from the closed (or open) end of the pipe there always appears a standing wave in the pipe provided that the reflection process is not connected with energy losses. Hence, in a pipe with two closed ends there may arise harmonic vibrations of the air of the form of a standing wave having nodes of displacement at the closed ends of the pipe. An analogous standing wave may arise in a taut string with rigidly fixed ends.

In a standing wave there is no propagation of energy along the wave because the energy cannot be transferred through the cross sections of the pipe (or of the string) in which the nodes of the displacement wave are located. The particles lying in such a cross section remain at rest all the time and cannot transmit energy from one particle to another. It is only in the intervals between any two neighbouring nodes that the exchange of energy between the oscillating particles may take place.

Indeed, the moment the particles pass through the equilibrium position (for instance, at the instant $t = \frac{5}{16} T$ in Fig. 407) they all possess the kinetic energy solely. The particle lying at the midpoint between two neighbouring nodes possesses the greatest velocity and, consequently, at this moment the density of the kinetic energy attains the greatest value (equal to the density of the total energy) at the midpoint and gradually reduces to zero in the directions towards nodes. The moment the particles achieve their extreme positions (for instance, at the instant $t = \frac{1}{16} T$ in Fig. 407) they have zero velocity and therefore their energy reduces to the potential energy of strain. The particle at the midpoint undergoes no deformation at this moment because we have $\frac{\partial y}{\partial x} = 0$ for this point and, by formula (139.6), the variation of the pressure also turns into zero at this point. At the same time, the particles lying closer to the nodes undergo deformation: the smaller the distance between a particle and a node, the greater is the deformation. Consequently, the particles near the nodes possess the greatest amount of the potential energy. Thus, in a standing wave the energy stored between two neighbouring nodes is transferred from the nodes to the midpoint twice during the period of the oscillation.

The particles lying between two neighbouring nodes of a standing wave oscillate as if their motion were completely independent of the motion of all the other particles. If some rigid walls were placed into the pipe (in which a standing sound wave exists) in the cross sections of the pipe where the nodes are located this would not affect the oscillation of the particles of the air in the pipe. Similarly, if a string with a standing wave were fixed at the points where the nodes of the wave are located the course of the oscillation process would continue as if there were no fixing at the nodes.

143. Natural Vibrations of a String and of the Air in a Pipe

Let us consider a string fixed at its two ends or an amount of air contained in a pipe whose both ends are closed. If an initial disturbance (perturbation) makes the particles of the string depart from the equilibrium position there arise oscillations of the string. Similarly, if the piston in the pipe produces an ini-

tial compression or rarefaction of the air (after which the piston is kept fixed) there arise longitudinal vibrations of the particles of the air in the pipe.

The vibrations appearing after these perturbations should be called *natural* vibrations (of the string or of the air in the pipe) because their course is determined solely by the forces produced by the system in question. In the general case when the initial perturbation has an arbitrary form the resultant natural vibrations can be of a rather complex character: the particles of the string or the particles of the air oscillate periodically (provided that the damping can be neglected)

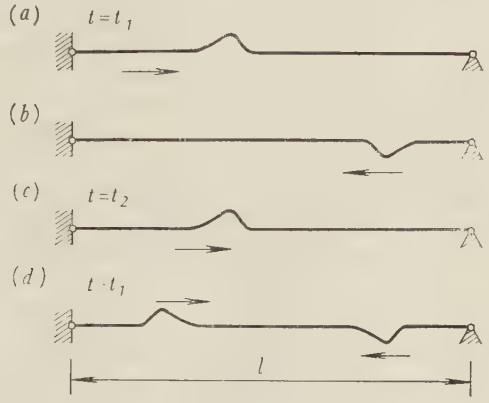

FIG. 413

but different particles take part in different vibration processes. The periodic character of the vibrations can be confirmed by the following simple argument.

Suppose that the initial disturbance of the string results in a local wave pulse propagating to the right (as is shown in Fig. 413*a* for an instant $t = t_1$). On attaining the right end of the string this wave disturbance undergoes reflection and starts to move in the opposite direction (see Fig. 413*b*). Then at an instant $t = t_2$ it is again reflected from the left end of the string after which it propagates to the right (Fig. 413*c*); at a certain instant the pulse will occupy the same position as at the initial instant $t = t_1$. The further course of the vibrations of the particles of the string will exactly repeat their motion during the time interval from t_1 to t_2, and so on. Therefore, $T = t_2 - t_1$ is the period of the oscillation of all the points of the string. It is obvious that $T = t_2 - t_1$ is equal to the time during which the wave disturbance passes a distance of $2l$, that is the doubled length of the string.* Consequently, if the wave velocity is c then the natural period of the oscillation of the string is equal to

$$T = \frac{2l}{c} \tag{143.1}$$

When the perturbation is of a more complicated nature and does not reduce to a simple wave pulse it can always be represented as a superposition of simple pulses for each of which our conclusion concerning the period of the oscillation remains valid.

* It should be stressed that our considerations are only applicable when the wave form of the pulse does not undergo distortion during the propagation and the reflection.

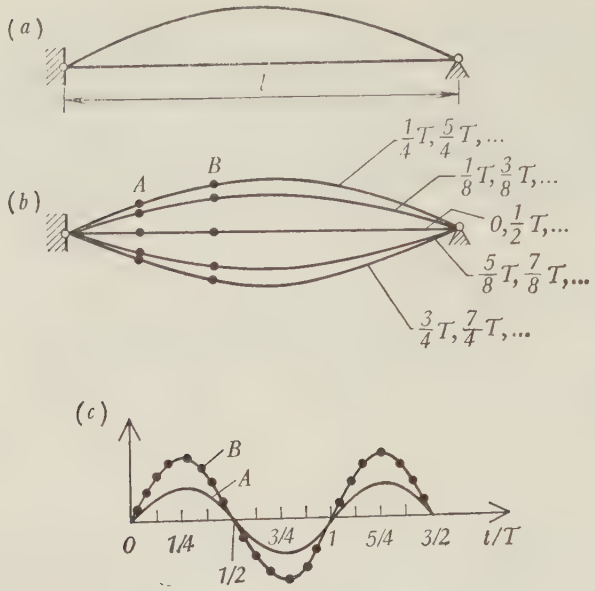

FIG. 414

Here we must make an important remark: the period T determined by formula (143.1) is the greatest period of the oscillation and there can exist some special tnitial perturbations for which the oscillation period is twice, three times, four iimes, etc. as small as T. For instance, suppose that at an initial instant $t = t_1$ the perturbation creates two pulses of the form shown in Fig. 413*d* propagating in opposite directions. These pulses have the same shape and are in opposite phases (they are symmetric with respect to the midpoint of the string). It can readily be shown that in this case every particle of the string oscillates with

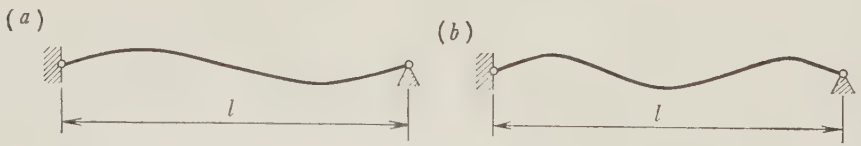

FIG. 415

period $\frac{T}{2}$. We can also easily construct initial distributions of several pulses for which the period of the oscillation of the string is equal to $\frac{T}{3}$, $\frac{T}{4}$, etc.

In the general case a natural oscillation of the particles of the string (or of the particles of the air in the pipe) is periodic but non-harmonic, that is the dependence of the coordinate of every particle on time is not described by a sinusoidal law.

However, there are some special initial perturbations of the string which generate harmonic natural oscillations. For instance, suppose that the initial depar-

ture of the string from the equilibrium position obeys a sinusoidal law as is shown in Fig. 414*a*. It can easily be proved that when the string is allowed to oscillate on its own after this initial disturbance the particles of the string move as in the case of a standing wave with wave length $\lambda = 2l$.

Indeed, let us imagine a sufficiently long string whose tension and line density are equal to the tension and the density of a shorter string of length l with its two ends fixed. If there is a standing harmonic wave in the long string such that the distance between the neighbouring nodes is exactly equal to the length l of the short string and the maximum deviation of the particles in the standing wave coincides with the maximum initial deviation of the short string the particles of the short string and of the long string will move in just the same way because these motions arise under the same initial conditions and under the action of the same forces of tension. Therefore, every particle of the short string will oscillate harmonically in one phase according to a sinusoidal law (see Fig. 414*b* and *c*).

As we know, the wave length λ and the period of the oscillation T are connected with the wave velocity c by the relation

$$T = \frac{\lambda}{c} \tag{143.2}$$

The oscillation period, the wave length, and the natural period are independent of the amplitude of the wave. From relations (143.1) and (143.2) we find that the greatest natural period (the *fundamental period*) of the string is

$$T_1 = \frac{\lambda}{c} = \frac{2l}{c}$$

where l is the length of the string (see Fig. 414*a*).

A similar argument shows that the initial deviation of the string shown in Fig. 415*a* generates a harmonic natural oscillation with period

$$T_2 = \frac{l}{c}$$

because in this case $\lambda = l$.

It is also evident that the initial perturbation of the form shown in Fig. 415*b* produces a natural harmonic oscillation with period

$$T_3 = \frac{2l}{3c}$$

because $\lambda = \frac{2l}{3}$.

Thus, a string with fixed ends can be in natural harmonic vibrations with the different periods

$$T_1 = \frac{2l}{c}, \quad T_2 = \frac{2l}{2c}, \quad T_3 = \frac{2l}{3c}, \quad \ldots, \quad T_n = \frac{2l}{nc} \tag{143.3}$$

depending on its initial state. Hence, the string possesses an infinite number of natural periods to which there corresponds an infinite number of the natural frequencies

$$\omega_n = \frac{2\pi}{T_n} = \frac{2\pi}{2l} nc = \frac{\pi nc}{l}$$

where $n = 1, 2, 3, \ldots$. When the string oscillates with an arbitrary natural frequency ω_n the particles of the string at every instant lie on a definite curve representing the corresponding wave form. For a given number n the wave form is a sinusoid whose half length is n times as small as the length of the string.

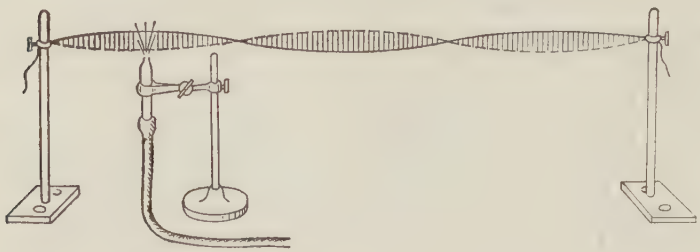

FIG. 416

When the oscillation of the string is excited in some other manner, that is when the particles of the string are given some arbitrary initial deviations from the equilibrium position and some arbitrary initial velocities, the resultant compound oscillation is representable as a superposition of a finite or infinite number of natural harmonic vibrations with different natural frequencies.

It is rather difficult to observe a pure natural oscillation of a real string with one definite frequency because the motion of the particles of the string is damped rather fast. It is therefore simpler to observe natural vibrations of a string in

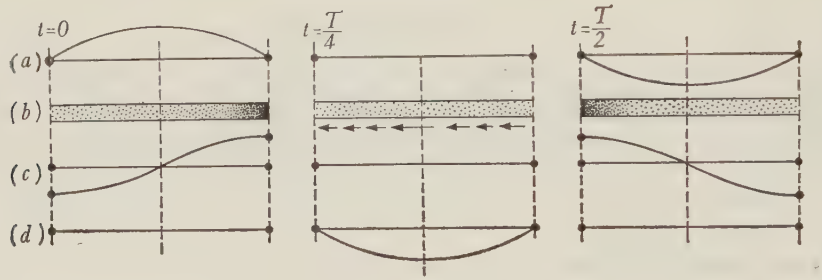

FIG. 417

an auto-oscillation regime. As was said in Sec. 131, in an auto-oscillation process the frequency of a vibration is close to a natural frequency and therefore the corresponding wave form is close to the form of a natural oscillation.

The wave form corresponding to a natural oscillation of a taut rubber cord can be demonstrated by the experiment whose scheme is shown in Fig. 416. The auto-oscillation of the taut rubber cord (which is usually a rubber pipe of a diameter of 3-5 mm) whose length is of several metres is excited by an air jet flowing out of a small nozzle (having a diameter of 1-2 mm). The action of the jet excites an auto-oscillation process in the cord with one of the natural frequencies. The cord can be made to oscillate with a given natural frequency by fixing the string at the initial instant at the point corresponding to a node of the corresponding standing wave. Practically, when the cord is sufficiently long it is possible to obtain stable vibrations with frequencies ranging from the first to the seventh natural frequencies. The way in which the natural vibrations of the strings of the violin are excited with the aid of the fiddle-stick is, in principle, analogous to what has been said about the oscillation of the rubber cord.

The natural oscillations of the air column within a pipe are quite analogous to the natural oscillations of a string; the only distinction is that the particles of the string are in a transverse vibration (that is their displacements are perpendicular to the direction of the propagation of the waves) while the particles of the air are in a longitudinal vibration (they move in the direction of the propagation of the waves).

In Fig. 417 we see the graphs representing a natural oscillation of the air in a pipe with its both ends closed for the lowest natural frequency $\omega = \frac{\pi c}{l}$ (the *fundamental frequency*). For this oscillation (called the *first oscillation mode*) Fig. 417*a* represents the graph of the displacements of the particles, Fig. 417*b* shows the variation of the density, Fig. 417*c* demonstrates the variation of the pressure, and Fig. 417*d* represents the graph of the velocity (for three different instants during one oscillation period). At the initial instant $t = 0$ all particles have zero velocity and obtain their maximum deviation to the right of the equilibrium position. After a quater-period $\left(\text{for } t = \frac{T}{4}\right)$ all particles receive their maximum velocities directed to the left; at this instant all particles pass through the equilibrium position and the pressure and the density are the same for all of them. At the instant $t = \frac{T}{2}$ all particles achieve their maximum deviations to the left of the equilibrium position, and so on.

When the particles of the air oscillate with the fundamental frequency $\omega_1 = \frac{\pi c}{l}$ they alternatively undergo compression in the direction towards the left and towards the right end of the pipe. The compression of the particles near the right end of the pipe at the instant $t = 0$ is shown in Fig. 417*a*. During the time interval from $t = 0$ to $t = \frac{T}{4}$ the compression decreases, the particles move in the direction towards the rarefaction near the left end and at the instant $t = \frac{T}{4}$ the compression disappears and the density becomes uniform. At this instant the particles have the greatest velocity; next, beginning with $t = \frac{T}{4}$, the particles move by inertia and at the instant $t = \frac{T}{2}$ the compression of the air arises near the left end of the pipe, and so on.

At the ends of the pipe there are nodes in which the particles have zero displacements and do not oscillate, their velocities being equal to zero all the time. However, these particles are located at the antinodes of the pressure and the density waves and they undergo the greatest variations of the density and of the pressure. At the same time, the particles located at the antinode of the displacement wave (at the midpoint of the pipe) oscillate with the greatest amplitude but undergo no variation of the pressure and the density because the antinode of the displacement coincides with the node of the pressure and the density.

At the instant $t = 0$ all particles possess the potential energy of compression solely, the particles near the ends of the pipe having the maximum energy. In a quater-period $\left(\text{at the instant } t = \frac{T}{4}\right)$ all particles lose their potential energy and possess only the kinetic energy. At this instant the density of the particles is as if the air were in the equilibrium state; the potential energy of compression is equal to zero and the maximum kinetic energy is attained by the particles near the midpoint of the pipe. For $0 < t < \frac{T}{4}$ the particles possess both the

kinetic and the potential energy and during this time interval the energy is transferred from the ends of the pipe (from the antinodes of the pressure) to the midpoint (where the antinode of the displacement is located).

In a pipe there can exist natural vibrations with different periods. In a pipe whose both ends are closed the possible natural vibrations are such that for each of them the length of the pipe is an integer multiple of half the wave length. In this case the first oscillation mode has two nodes of displacement, the second has three nodes of displacement, etc. Between any two neighbouring nodes of

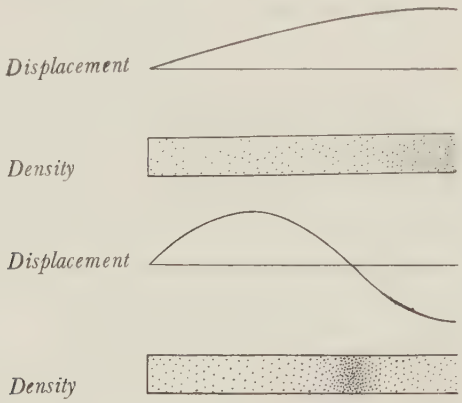

FIG. 418

displacement the course of any natural oscillation is as in the case of the first oscillation mode (see Fig. 417).

The natural vibrations in a pipe with one of its ends open are of a completely different character. In Fig. 418 we see the graphs of the displacement at the instant when the maximum deviation from the equilibrium state is attained. The upper graph corresponds to the first oscillation mode and the lower one to the second mode. In this case at the open end of the pipe the antinode of the displacement is always located and therefore the pipe is an even number of times as long as the quarter wave length. The natural frequencies of the oscillation of the air in such a pipe are in the ratio 1 : 3 : 5 : 7: . . ..

We conclude this section with the general remark that the number of natural frequencies always coincides with the number of degrees of freedom of the system in question. A system with one degree of freedom (for instance, a pendulum) possesses one natural frequency and has only one natural oscillation mode. A system with two degrees of freedom (for instance, two coupled pendulums) possesses two natural frequencies, and so on. Similarly, a string consisting of an infinite number of particles is a *system with an infinite number of degrees of freedom* and has an infinite number of natural frequencies ω_1, ω_2, ...

CHAPTER 16

Elements of Acoustics

144. Basic Phenomena

The sensation of hearing arises as a result of the action of mechanical oscillations on the human organs of hearing. A human ear can perceive oscillations of sufficient intensity whose frequencies are approximately within the limits from 16 Hz to 20 000 Hz. This is the *audio-frequency range*; mechanical vibrations with these frequencies which can be perceived by a normal ear are called *sound* or *acoustic oscillations.*

Any body which is in a mechanical vibration whose frequency belongs to the audio range is a source of sound. For instance, a string, a membrane, a plate oscillating with such a frequency generate longitudinal oscillations in the surrounding medium and are sound sources. Not only a rigid body but also a gas or a fluid can serve as a sound source. The oscillation of an amount of gas or fluid contained in a definite volume or flowing in some vessels can produce sound waves. A sound source generates oscillations of the density (or of the pressure) of the particles of the medium surrounding it and these oscillations are transmitted to the other particles of the medium so that the variations of the density and of the pressure propagate as waves in all directions.

The character of the propagation of a sound wave from a given sound source is specified both by the properties of the source and by the properties of the surrounding medium. When the dimensions of a sound source are small in comparison with the wave length λ of the sound wave generated by the source (we remind the reader that $\lambda = cT$ where c is the speed of sound in the medium and T is the oscillation period) it can be regarded as a *point source*: when placed in a homogeneous medium, such a source generates a *spherical acoustic wave.*

As an example, let us consider the sound wave with frequency 350 Hz produced by a sound source whose dimensions are of the order of 1 cm. The formula $\lambda = cT$ where $T = \frac{1}{\nu}$ (ν is the frequency) readily shows that for the frequency of 350 Hz the wave length of a sound wave in the air is approximately equal to 1 m and we can therefore regard this wave as a spherical wave propagating from a point source.

When the dimensions of a sound source are large in comparison with the wave length the wave form becomes more complicated; in the general case it can be determined by using the *Fresnel-Huygens principle.* According to this principle, every small part of the wave surface radiates, at a given instant t, an elementary spherical wave and the position of the wave surface at the following instant $t + dt$ coincides with the envelope of all these elementary waves. This enveloping wave surface results from the interference of the elementary waves. When

the surface of an oscillating body is a plane plate whose all points oscillate in one phase and when the dimensions of the plate are large in comparison with the wave length this sound source radiates a plane wave propagating in the direction perpendicular to the plate. Near the edges of the plate and at a certain distance from them the plane wave is of course distorted by the interference of the elementary waves.

145. Reflection of Sound Waves

When a sound wave is incident on the interface between two media whose mechanical properties differ it undergoes reflection. For instance, when a sound wave passes from the air into the water or from the water into the air most of the wave energy is reflected and only a small part of the energy is transmitted to the other medium.

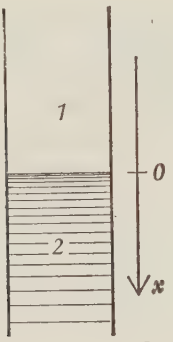

FIG. 419

Sound waves propagating in the air are also reflected from rigid bodies, for instance, from the surface of the Earth; this often results in the appearance of the well-known phenomenon of echo. In the reflection of sound waves an essential role is played by the variation of the density ρ of the medium and of the acoustic velocity c; more precisely, the character of the reflection is specified by the variation of the quantity $c\rho$. The greater the variation of $c\rho$ when a sound wave passes from one medium into another, the greater is the part of the energy of the sound wave reflected from the interface between the media.

We shall derive the laws governing the reflection of sound waves for the simplest case of a plane wave incident in the normal direction on the plane interface between two media.

Let us consider a long pipe (Fig. 419) filled with two different substances along whose axis a plane sound wave propagates. Suppose that the sound source is placed in medium *1* far from the interface and that it generates a travelling plane sound wave in which the displacements of the particles are described by the equation $y_1 = a_1 \cos\left(\omega t - \frac{2\pi x}{\lambda_1}\right)$ where a_1 is the amplitude, x is the coordinate reckoned along the axis of the pipe, ω is the frequency, λ_1 is the wave length and, y_1 is the displacement of the particle of the first medium (with coordinate x at time t) in the incident wave. Then wave reflected from the interface which is described by an equation of the form $y_2 = a_2 \cos\left(\omega t + \frac{2\pi x}{\lambda_1} + \varphi_1\right)$ will propagate in the first medium in the backward direction and the transmitted wave described by an equation $y_3 = a_3 \cos\left(\omega t - \frac{2\pi x}{\lambda_2} + \varphi_2\right)$ will propagate along the x-axis in medium *2*. Here y_2 is the displacement of the particle of the first medium (with coordinate x at time t) in the reflected wave and y_3 is the displacement of the particle of the second medium in the transmitted wave. For $x = 0$

(this is the coordinate of the interface between the two substances) the conditions

$$y_1 + y_2 = y_3 \tag{145.1}$$

and

$$p_1 + p_2 = p_3 \tag{145.2}$$

must be fulfilled for any instant t where p_1, p_2, and p_3 are the deviations of the pressure from the static pressure p_0 (i.e. from the pressure in the media when there are no sound waves) in the incident, reflected, and transmitted waves, respectively. Condition (145.1) expresses the continuity of the displacements of the particles on the interface and condition (145.2) means that the pressure is the same on both sides of the interface. On substituting $x = 0$ in the expressions of y_i ($i = 1, 2, 3$) we obtain from (145.1) the following equality:

$$a_1 \cos \omega t + a_2 \cos (\omega t + \varphi_1) = a_3 \cos (\omega t + \varphi_2) \tag{145.3}$$

According to (139.7), we have the equation $p = - \varkappa p_0 \frac{\partial y}{\partial x}$ for the pressure p in a plane wave where $\varkappa$ is the adiabatic exponent. On substituting y_1 for y and p_1 for p in this equation we obtain

$$p_1 = - \varkappa_1 p_0 a_1 \frac{2\pi}{\lambda_1} \sin \left(\omega t - \frac{2\pi x}{\lambda_1} \right)$$

The corresponding expressions for p_2 and p_3 are derived in an analogous manner. Now the substitution of these expressions in condition (145.2) results in

$$\frac{\varkappa_1}{\lambda_1} \{ - a_1 \sin \omega t + a_2 \sin (\omega t + \varphi_1) \} = - \frac{\varkappa^2}{\lambda_2} a_3 \sin (\omega t + \varphi_2) \tag{145.4}$$

Since equalities (145.3) and (145.4) must hold for any value of t we can transform (145.3) and (145.4) by using the well-known addition formulas of trigonometry and then equate the coefficients in $\cos \omega t$ and $\sin \omega t$ on both sides of these equalities to zero. This yields the following system of four simultaneous equations with the four unknowns a_1, a_2, φ_1, and φ_2:

$$\left.\begin{aligned} a_1 + a_2 \cos \varphi_1 &= a_3 \cos \varphi_2 \\ - a_1 + a_2 \cos \varphi_1 &= - \frac{\lambda_1 \varkappa_2}{\lambda_2 \varkappa_1} a_3 \cos \varphi_2 \\ a_2 \sin \varphi_1 &= a_3 \sin \varphi_2 \\ a_2 \sin \varphi_1 &= - \frac{\lambda_1 \varkappa_2}{\lambda_2 \varkappa_1} a_3 \sin \varphi_2 \end{aligned}\right\} \tag{145.5}$$

Let us solve these equations. The subtraction of the fourth equation from the second one gives

$$a_3 (1 + \Delta) \sin \varphi_2 = 0$$

where $\Delta = \frac{\lambda_1 \varkappa_2}{\lambda_2 \varkappa_1} \neq 0$ and $\Delta > 0$. The last equality can only be satisfied when $\varphi_2 = k\pi$ where $k = 0, 1, 2, \ldots$. Now from the second equation we find that $\varphi_1 = m\pi$ where $m = 0, 1, 2, \ldots$ because $a_3 \neq 0$. On subtracting the third equation from the first one we obtain $2a_1 = a_3 (1 + \Delta) \cos \varphi_2$. Since $a_3 > 0$ and $\Delta > 0$ we have $\cos \varphi_2 = + 1$ whence $\varphi_2 = 0, 2\pi, \ldots$. Consequently, the amplitude a_3 is given by the formula

$$a_3 = a_1 \frac{2}{1 + \Delta} \tag{145.6}$$

Substituting (145.6) in the first equation and taking into account that $\cos \varphi_2 = +1$ we derive the relation

$$a_2 \cos \varphi_1 = a_1 \frac{1-\Delta}{1+\Delta} \tag{145.7}$$

By what was established earlier, the quantity $\cos \varphi_1$ must be equal either to $+1$ or to -1. Therefore, it follows from (145.7) that

$$\cos \varphi_1 = +1, \quad \varphi_1 = 0 \text{ for } \Delta < 1$$

and

$$\cos \varphi_1 = -1, \quad \varphi_1 = \pi \text{ for } \Delta > 1$$

because all the other quantities involved are positive.

Let us analyse the results we have obtained. The condition $\varphi_2 = 0, 2\pi, \ldots$ means that the transmitted wave and the incident wave are in one phase, that is the interface does not affect the phase of the transmitted wave. The last two equalities imply that the phase of the reflected wave can be either equal to zero or to π, that is the reflected wave and the incident wave can be either in one phase or in opposite phases depending on whether Δ exceeds unity or is less than unity. The quantity Δ characterizes the physical properties of the media which determine the reflection process.

Since $c_1 = \sqrt{\frac{\varkappa_1 p_0}{\rho_1}}$, that is $c_1^2 \rho_1 = \varkappa_1 p_0$, the quantity Δ can be written in the form

$$\Delta = \frac{\lambda_1 \varkappa_2}{\lambda_2 \varkappa_1} = \frac{c_1 \varkappa_2}{c_2 \varkappa_1} = \frac{c_1 \varkappa_2 p_0}{c_2 \varkappa_1 p_0} = \frac{c_2 \rho_2}{c_1 \rho_1} \tag{145.8}$$

The product ρc is called the *acoustic wave resistance* of the medium. We see that the relationship between the phases of the incident and the reflected wave depends on which of the two media has a greater wave resistance. If the wave resistance of the second medium is less than that of the first medium the reflection does not change the phase of the wave; if, conversely, the wave resistance of the second medium exceeds the resistance of the first medium the incident and reflected waves have opposite phases.

The relationship between the amplitudes of the reflected and transmitted waves depends on Δ, that is on the ratio of the acoustic wave resistances of the media. For instance, we have $\rho c \approx 43$ g/(cm^2·s) for the air and $\rho c \approx 14.2 \times 10^4$ g/(cm^2·s) for the water; consequently, when a sound wave passes from the air into the water we have $\Delta = \frac{14.2}{43} 10^4 \approx 3300$. It follows that in this case $a_2 = a_1 \frac{2}{1+\Delta} \approx \approx \frac{1}{1650} a_1$, that is the amplitude of the transmitted wave is about 1/1500 of the amplitude of the incident wave. For a sound wave passing from the water into the air we obtain

$$a_3 = a_1 \frac{2}{1 + \frac{1}{3300}} \approx 2a_1$$

This means that the amplitude of the transmitted wave is approximately twice as large as the amplitude of the incident wave.

When both media possess the same wave resistance we have $\Delta = 1$, that is $a_2 = 0$ and $a_1 = a_3$. Thus, in this case there is no reflection and the whole energy is transferred from one medium to the other.

We now consider the transfer of energy through the interface between two media for the general case. The variation of the amplitude of the wave passing from one medium into another which we have already investigated does not specify completely the energy flux across the interface and an additional analysis is therefore needed.

As was shown earlier, the average (mean) energy flux in a (harmonic) plane wave is equal to

$$\frac{1}{2}\rho c a^2\omega^2$$

where a is the amplitude of the displacements of the particles of the medium (see Sec. 139). Therefore, taking into account (145.6) and (145.8), we can write the expression

$$E_2 = \frac{1}{2}\rho_2 c_2 a_2^2\omega^2 = \frac{\rho_2 c_2 \omega^2 4a_1^2}{2(1+\Delta)^2}\cdot\frac{\rho_1 c_1}{\rho_1 c_1} = \frac{4\Delta}{(1+\Delta)^2}\,\frac{1}{2}\rho_1 c_1 a_1^2\omega^2 = \frac{4\Delta}{(1+\Delta)^2}E_1$$

for the mean value of the energy flux corresponding to the transmitted wave where E_1 is the mean energy flux of the incident wave. We see that the relative value of the energy passing through the interface does not depend on the direction of the propagation of the waves because

$$\frac{E_2}{E_1} = \frac{4\Delta}{(1+\Delta)^2} = \frac{4\Delta'}{(1+\Delta')^2}$$

where $\Delta' = \frac{1}{\Delta}$.

As an example, let us consider a wave passing from the water into the air (or in the backward direction) perpendicularly to the interface between the media. On substituting the quantity $\Delta' \approx \frac{1}{3300}$ in the formula expressing the ratio E_2/E_1 we find that only $\approx \frac{1}{825}$ of the energy passes through the interface, all the rest being reflected.

146. Propagation of Sound Waves

The propagation of a sound wave in real conditions can be a rather complex phenomenon. A sound wave incident on an obstacle can easily go around it when the wave length is much greater than the dimensions of the obstacle and it is reflected from the obstacle (in approximately the same way in which the light is reflected) when the wave length is much less than the dimensions of the obstacle.

The observation of the reflection of sound from a rock, a wall, and other obstacles of considerable dimensions shows that the angle of incidence is equal to the angle of reflection. In case the dimensions of the obstacle are of the order of the wave length the propagation of the sound wave near the obstacle becomes more complicated and the phenomenon is accompanied both by reflection and by diffraction (diffraction is a process in which a wave goes around an obstacle). It should be noted that any surface on which the magnitude ρc of the wave resistance of the substance changes is an obstacle for a sound wave. For instance, a wave can be reflected from a layer of the air of higher temperature, from a cloud, and so on.

The refraction and the reflection of sound on the interfaces between the layers of the air having different temperatures make the phenomenon of the propagation of sound in the real atmosphere very complex. For instance, the investigation

of the propagation of explosion sound waves shows that the sound is reflected from the layer of the atmosphere lying at an altitude of 4050 km because of which the explosion can sometimes be audible at a very large distance and completely unaudible in a place located not far from the explosion since the direct explosion wave does not achieve that place. This phenomenon gave rise to the hypothesis that the temperature of the air starts growing at the altitude of 40-50 km. Direct measurements carried out in recent years completely confirmed this hypothesis.

147. Hearing

A normal human ear perceives the oscillations of the air of definite frequency and intensity. By the *sound intensity* is meant the amount of energy passing across an area of 1 m^2 in unit time. The sound intensity is usually measured either in W/m^2 (in the SI system) or in $erg/(cm^2 \cdot s)$ (in the cgs system).

For a given frequency there is a minimum sound intensity for which the sound can be perceived by the human ear. This minimum sound intensity (generally speaking, it is different for different frequencies) is called the *threshold of audibility*. The normal human ear does not hear a sound whose intensity is less than the threshold of audibility.

In Fig. 420 the curve representing the variation of the threshold of audibility is shown for the whole audio-frequency range. The human ear is most sensible to the frequencies close to 3000 Hz; for this frequency the sound intensity of 10^{-12} W/m^2 is sufficient for the sound to be audible. For the frequency of 50 Hz the threshold of audibility is 10^{-7} W/m^2; we see that in this case for the sound to be audible the intensity should be increased about 100 000 times.

When the sound intensity exceeds a certain maximum value the ear can no longer perceive it normally and the feeling of pain appears. This maximum intensity is called the *threshold of pain*. As is seen from Fig. 420, the threshold of pain corresponds to the sound intensity which is approximately equal to 1 W/m^2 for all the audio frequencies.

The range of the sound intensity which can be perceived by the ear is very large: from 10^{-12} W/m^2 to 1 W/m^2. That is why it is practically preferable to characterize the sound intensity by means of another unit, the *decibel* (denoted dB; see the left part of Fig. 420) which measures the *levels of sound intensity* instead of measuring the energy flux (as is shown in the right part of Fig. 420).

If a sound intensity I_0 is taken as an initial level then to some other intensity I_1 corresponds the *difference between the levels of these intensities* measured by the quantity $10 \log \frac{I_1}{I_0}$ dB. In other words, the number of decibels measuring the difference between the levels of sound intensities divided by 10 equals the common logarithm of the ratio of the sound intensities. As I_0 is usually taken the sound intensity corresponding to the threshold of audibility for the frequency of 1000 Hz; then the level of the intensity corresponding to the threshold of pain is about 120-130 dB. When the levels of sound intensity are measured in decibels the threshold of audibility for the frequency of 50 Hz corresponds, approximately, to 50 dB. As was said, the threshold of audibility varies with frequency. The *loudness* (or *volume*) *of sound* also varies when the frequency changes; the curves representing the sound intensities corresponding to equal values of the loudness for different frequencies are shown in continuous lines in Fig. 420.

For instance, a sound whose level of sound intensity is equal to 20 dB for the frequency of 1000 Hz is of the same loudness as a sound of 50 dB for the frequency

of 100 Hz. The measurement of the levels of sound intensity in decibels is widely used in engineering and physics because it is simpler. But the curves of equal loudness correspond to different levels of sound intensity measured in decibels and it is therefore more convenient to characterize the loudness of a given sound by means of the *loudness level* for which a new unit called the *phon* is used. If the loudness level of the given sound is N phons this means that the sound of frequency 1000 Hz whose level of sound intensity exceeds by N dB the level of sound intensity corresponding to the threshold of audibility (for the same fre-

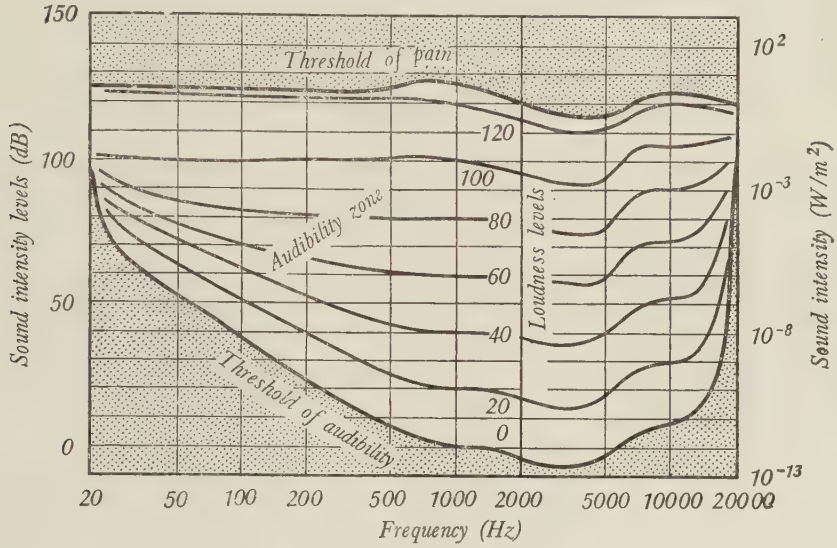

FIG. 420

quency of 1000 Hz) is as loud as the given sound. For the same loudness, low frequency sounds (with frequencies less than 1000 Hz) are of greater intensity than high frequency sounds (of 1000-3000 Hz). As examples, we give here the loudness levels for some sounds: the noise in an underground car moving with a high velocity—approximately 90-95 phons, a loud conversation heard at a distance of 0.5 m—approximately 30 phons, a whisper—about 10 phons.

148. Ultrasonic Vibrations

A mechanical vibration of a body or of a medium whose frequency exceeds 20 000 Hz is referred to as an *ultrasound*. Ultrasounds are not heard by a human ear. Mechanical vibrations of such a high frequency are usually obtained by using the *piezoelectric effect* or the *magnetostriction*.

The piezoelectric effect is based on the property of a crystal to undergo a deformation in the presence of an electric field and, conversely, to become electrified under the action of a deformation. The magnetostriction is an analogous phenomenon arising under the action of a magnetic field.

A plate oscillating with a high frequency under the action of an alternating electric or magnetic field of the same frequency can also serve as a source of

ultrasound (an alternating electric or magnetic field can be obtained with the use of an alternating electric current from an electromagnetic generator).

When analysing the character of the propagation of a wave generated by an ultrasound source one should take into account that the length of such a wave is very small. For instance, to a frequency of 350 kHz there corresponds an ultrasonic wave whose length is of the order of 1 mm and to a frequency of 3 MHz there corresponds a wave length of about 0.1 mm. When the dimensions of an oscillating plate serving as an ultrasound source are large in comparison with the wave length the plate generates a plane wave propagating as if there were parallel rays issued from the surface of the plate (as in the case of a light propagating from a projector). This makes it possible to use ultrasonic rays for measuring distances in the water.

To this end an ultrasound source placed on a ship sends at a certain instant an ultrasonic wave pulse into the water which propagates in a definite direction. When this pulse is incident on an obstacle it is reflected and in some time the reflected wave achieves a receiver on the ship. The measurement of the time interval between the instants when the pulse is sent and received makes it possible to determine the distance to the obstacle from which the ultrasound has been reflected.

This method of measuring the distances in the water with the aid of ultrasonic pulses is used in the construction of a fathometer meant for measuring the depth of the sea and also in various hydrolocation devices measuring the distances to submarines, icebergs, etc. Ultrasound oscillations are also widely used in modern fields of engineering, medicine, etc.

CHAPTER 17

Fundamentals of Special Theory of Relativity

149. Galileo's Principle of Relativity

A reference system (reference frame) for which Newton's laws of dynamics hold is called *inertial*. All the reference systems which are in uniform motion relative to each other and relative to an inertial system are also called inertial. Newton's laws of classical dynamics are valid for all the inertial reference systems. This assumption is *Galileo's relativity principle* which is the fundamental principle of classical mechanics. Inertial reference systems are also called the *Galilean systems*; from the point of view of dynamics they all are equivalent although the kinematic description of a motion is obviously different for different inertial systems.

Knowing the law of motion of a point in a reference system A we can readily derive the law of motion of the same point in an inertial system B provided that the velocity of the motion of system B relative to system A is known.

Suppose that system B is in a translatory motion relative to system A with a constant velocity v. Then the relationship between the coordinates x, y, and z of an arbitrary point in system A and the coordinates x', y', and z' of the same point in system B is written as

$$\boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{r}_0 \tag{149.1}$$

where $\boldsymbol{r}$ (x, y, z) and $\boldsymbol{r}'$ (x', y', z') are the corresponding radius vectors of the point (see Fig. 421). The relationship between the velocities is

$$\boldsymbol{u}' = \boldsymbol{u} - \boldsymbol{v}$$

where $\boldsymbol{u} = \frac{d\boldsymbol{r}}{dt}$, $\boldsymbol{u}' = \frac{d\boldsymbol{r}'}{dt}$, and $\boldsymbol{v} = \frac{d\boldsymbol{r}_0}{dt}$. If $\boldsymbol{r}_0 = 0$ at the initial instant $t = 0$ then $\boldsymbol{r}_0 = \boldsymbol{v}t$ and the relation connecting the coordinates takes the form

$$\boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{v}t \tag{149.2}$$

It is evident that the accelerations in both systems are equal:

$$\frac{d\boldsymbol{u}'}{dt} = \frac{d\boldsymbol{u}}{dt}$$

To simplify the further calculations let us assume that system B moves with velocity v relative to system A so that the axes x and x' are in one straight line, have the same (positive) directions and besides, at the instant $t = 0$, the origins of the systems coincide (Fig. 422). Then a point whose coordinates in system A at time t are x, y, and z has coordinates x', y', and z' in system B expressed by

the formulas

$$\left.\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \end{aligned}\right\} \tag{149.3}$$

In the derivation of formulas (149.3) it is tacitly implied that the time t is *absolute*, that is it is the same for all reference systems. This intuitive assumption is taken for granted in classical mechanics. It is considered natural and not requiring any proof. All the experiments and phenomena in mechanics and engineering in which the velocities of the bodies are much less than the speed of light c ($c = 3 \times 10^8$ m/s) do not contradict this assumption.

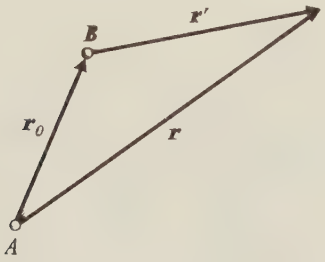

FIG. 421

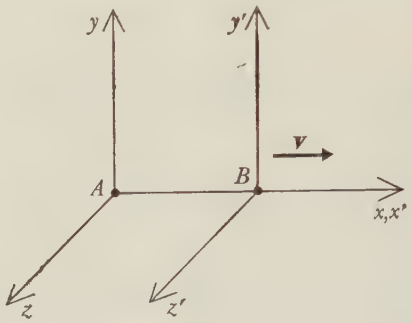

FIG. 422

If we denote (quite formally, at present) the time in system B by t' then we can write the equality

$$t' = t \tag{149.4}$$

which always holds in the Newtonian mechanics. Formulas (149.1) and (149.2) express the *Galilean transformation*. When a point has coordinates x, y and z at time instant t we can regard this fact as an *event* (x, y, z, t). Using this terminology we can say that the Galilean transformation establishes the relationship between the coordinates x, y, z, t of an event in system A and the coordinates x', y', z', t' of the same event in system B.

The Galilean transformation implies the well-known law of addition of velocities in the classical mechanics:

$$\left.\begin{aligned} \frac{dx'}{dt} &= \frac{dx}{dt} - v \\ \frac{dy'}{dt} &= \frac{dy}{dt} \\ \frac{dz'}{dt} &= \frac{dz}{dt} \end{aligned}\right\} \tag{149.5}$$

Thus, the velocity of a body along the x-axis relative to system A is equal to the sum of the projection of the velocity of the same body on the x'-axis in system B (as was assumed, $x \parallel x'$) and the velocity v of system B itself.

All these simple relations are well known from the classical mechanics. As will be shown, Einstein's relativity principle establishes a more general rela-

tionship between the coordinates of an event in different inertial reference systems from which the formulas expressing the Galilean transformation follow for $v \to 0$ ($v \ll c$) as a special case.

When coordinates are transformed there are some quantities depending on them which remain invariable under the transformation. Such quantities are termed *invariants*. For instance, the distance between any two points is an invariant of the Galilean transformation. Among the mechanical quantities there are some which vary under the Galilean transformation (for instance, the coordinates, the velocity, the momentum, the kinetic energy) while some other quantities are invariants.

Let us consider the variations of the fundamental mechanical quantities under the (Galilean) transformation from one inertial reference system to another and the connection between these variations with the laws of dynamics and with the conservation laws for the momentum and the energy of a system of particles (or bodies).

When we pass from an inertial system A to another inertial system B both the momentum and the kinetic energy of a system of moving particles (bodies) change.

Let $\boldsymbol{u}_i'$ denote the velocity of the ith particle in system B and $\boldsymbol{u}_i$ denote the velocity of the same particle in system A. Then, by (149.1), we have

$$\boldsymbol{u}_i = \boldsymbol{u}_i' + \boldsymbol{v} \tag{149.6}$$

The total momentum of the system of particles relative to system A can be written as

$$\boldsymbol{K} = \sum m_i \boldsymbol{u}_i = \sum m_i \boldsymbol{u}_i' + \sum m_i \boldsymbol{v} = \boldsymbol{K}' + m\boldsymbol{v} \tag{149.7}$$

where $\boldsymbol{K}' = \Sigma\, m_i \boldsymbol{u}_i'$ is the momentum of the system of particles relative to B and $m = \Sigma\, m_i$ is the total mass of the system of the particles. Equality (149.7) holds in all the circumstances. It implies that if the momentum in system A does not depend on time it also remains constant in system B because m and $\boldsymbol{v}$ are invariable. This applies to any other inertial system as well. Thus, *the inertia law is valid for all the inertial systems of reference.*

The kinetic energy of the system of particles is expressed in the reference system A by the formula

$$T = \frac{1}{2} \sum m_i u_i^2 = \frac{1}{2} \sum m_i (\boldsymbol{u}_i' + \boldsymbol{v})^2 =$$

$$= \frac{1}{2} \sum m_i u_i'^2 + \sum m_i \boldsymbol{u}_i' \boldsymbol{v} + \frac{1}{2} v^2 \sum m_i = T' + \boldsymbol{K}' \boldsymbol{v} + \frac{1}{2} m v^2 \tag{149.8}$$

where T' is the kinetic energy in system B. Equality (149.8) describes the variation of the kinetic energy under the transformation from one inertial system to another; it holds in all circumstances.

When a system of particles or bodies is isolated (closed), that is it does not interact with the bodies not entering into the system, the momentum $\boldsymbol{K}'$ does not depend on time and remains constant. If there are only elastic interactions between the particles of the system (for instance, elastic impacts) the kinetic energy T' also remains constant and, consequently, the kinetic energy T in the other inertial reference system A is also constant. Hence, *if the law of conservation of kinetic energy holds for a given system of particles or bodies in one inertial reference frame it holds for all the other inertial reference frames as well.*

The momentum of an isolated system of particles remains constant in all cases even when the interaction is inelastic, that is when the particles undergo

plastic impacts. When there are plastic impacts the kinetic energy decreases; however, as it follows from equality (149.8), the quantities T and T' decrease by the same amount for a closed system of particles in all the inertial reference frames. Thus, this decrement of the kinetic energy is an invariant.

It should be noted that it is also possible that a system of particles is not isolated but nevertheless $\boldsymbol{K}'$ remains constant. This case is realized when the resultant (the vector sum) of all the external forces (that is of the forces exerted on the given system by the bodies not entering into the system) is equal to zero. In such a case the kinetic energy can vary in the course of time and, as it follows from equality (149.8), the decrease of the energy is the same for all the inertial reference systems and is an invariant.

There are systems in which the interaction forces between the particles depend solely on the distances between them and are directed along the lines joining the particles (see Sec. 36). Then to every configuration of the particles of the system there corresponds a definite value of the potential energy U. The distances between the particles are not changed under the transformation from one inertial reference frame to another and therefore the potential energy U does not change either. On adding the term U to both members of equality (149.8) we obtain

$$T+U=T'+U+\boldsymbol{K}'\boldsymbol{v}+\frac{1}{2}mv^2 \tag{149.9}$$

We see that if the interaction between the particles of a closed system is such that the sum of the kinetic and potential energies is constant in one inertial reference system it is also constant in all the other inertial reference systems. Hence, *if the mechanical energy is conserved for a given closed system of particles in one inertial reference frame the law of conservation of energy is valid in all the inertial frames.*

The interaction forces between the particles or between the bodies may depend on their mutual positions and on their relative velocities. Since the distances between the particles or the bodies and the relative velocities are invariants we conclude that *a force is also an invariant.*

Consequently, in the general case the kinetic energy and the momentum change under the transformation from one inertial reference system to another while the potential energy, the mass, and the forces are invariants. In the special cases indicated above the variations of the energy are also invariants.

It is obvious that *all the three fundamental laws of dynamics are valid for all the inertial reference systems.*

150. Constancy of Speed of Light

The study of the nature of light and the investigation of the laws governing the propagation of light were the most important problems of physics in the nineteenth century. As early as the times of Newton the Danish astronomer O. Roemer determined approximately the speed of light proceeding from the data obtained in his observation of the eclipses of the Jupiter satellite. According to the wave theory, the propagation of light is a wave process resembling the propagation of sound waves in a medium. In this connection in the second half of the nineteenth century there arose the question concerning the medium in which the light propagates and the reference system with which this medium is connected.

Suppose that there is a special medium, the *ether* (connected with the Sun) in which the light propagates (this assumption is known as the *ether hypothesis*

which proved to be incorrect). Then the velocity of light on the Earth must depend on the velocity of the motion of the Earth relative to a reference system connected with the Sun. If the velocity of the Earth in this system is v and the speed of light relative to the medium is c then, according to the classical rule for the addition of velocities, the speed of light must be equal to $c - v$ in the direction of the Earth's motion and to $c + v$ in the opposite direction.

Although the velocity of the motion of the Earth ($v = 30$ km/s) is very large in comparison with the velocities we deal with in everyday life it is negligibly small in comparison with the speed of light ($c = 3 \times 10^5$ km/s). Therefore, the observation and the measurement of the influence of the Earth's motion on the speed of light encounters many experimental difficulties. That is why the realization of such an experiment requires high precision devices in order to verify the addition rule for the velocities of the light and of the Earth. First experiments of this kind were carried out by A. A. Michelson and E. W. Morley in 1887. Later the results of their famous experiments were checked and confirmed in many analogous modified and more precise experiments. It turned out that none of these experiments detected any influence of the velocity of the Earth's motion on the magnitude of the velocity of propagation of light.

Since the assumption that the speed of light remains constant only relative to the Earth contradicts astronomical and many other observations the scientists came to the conclusion at the beginning of the twentieth century that *the speed of light in vacuum is constant and is independent of the motion of the light source or receiver.* This conclusion was drawn after a thorough analysis and lengthy discussion of all the results of the experiments and observations.

This conclusion obviously contradicts Galileo's relativity principle from which the ordinary addition rule for the velocities follows. To overcome these difficulties A. Einstein suggested in 1905 a completely new hypothesis that the experimental failure in the detection of a uniform motion relative to an "absolute reference system" simply meant that there were no such system at all. According to this hypothesis all the inertial reference systems are completely equivalent in the sense that the courses of all natural phenomena (and not only of the mechanical ones) are the same in all the inertial reference systems and are described in the same manner in all such systems. This new approach to the problem makes it necessary to revise the basic concepts of space and time on the basis of the following postulates expressing Einstein's hypothesis which is universally recognized in modern science:

1. *All the inertial reference systems are equivalent.*
2. *The speed of light in vacuum is independent of the motion of the light source or receiver.*

These postulates should be understood as consequences of the known experimental facts and of the laws established in the theory of the propagation of light. The first postulate expresses the fundamental principle of *Einstein's special relativity theory* according to which all the laws governing the physical phenomena are invariant in all the inertial reference systems. From the first and the second postulate it follows that the light propagates in vacuum with the same constant velocity in all the directions.

151. Simultaneity of Events

Before proceeding to the analysis of the corollaries following from Einstein's postulates we must consider in more detail the definition of the simultaneity of two events occurring in defferent places. When two events occur in approxi-

mately the same place (that is close to each other or, as we say, "at one point") their simultaneity can be detected by means of a simple observation (for instance, by taking photographs and the like). However, when two events take place at different points their simultaneity can only be established by using a signal transmitting information from one place to the other. Since the speed of light is very large (in nature there are no signals whose speed exceeds the speed of light, at least modern science knows nothing about such signals) and is constant A. Einstein suggested that light signals should be used for this purpose.

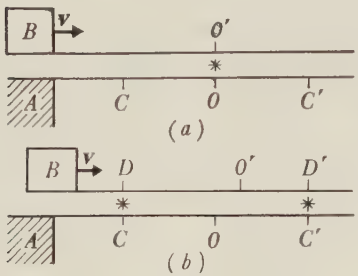

FIG. 423

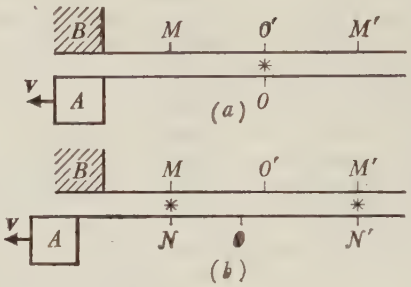

FIG. 424

If the speed of light were infinite the simultaneity of events occurring in different places could be determined, in principle, in the same way as it is done for events taking place at one point. This approach is in fact used in non-relativistic physics where time intervals are considered absolute. When the velocities we deal with are small relative to the speed of light which is very large this approach does not contradict experimental data. However, since the speed of light, although very large, is finite it is necessary to take into account the finiteness of the speed of light for bodies moving with very large velocities when the general definition of the simultaneity of events is stated.

Let us consider two identical reference systems A and B moving with constant velocity v relative to each other (Fig. 423*a*). In the figure these systems are shown as two parallel straight lines placed very close to each other; the systems move in the direction of these lines, system B moving to the right relative to system A. Let a light pulse be emitted by a light source placed at the point O in system A at a certain instant and let this event be observed at the point O' in system B, the points O and O' being very close to each other or even coincide the moment the light flashes. The light signal emitted from the point O' simultaneously achieves the two points C and C' lying at equal distances from the point O in system A. These two events (the observations of the light flash at the points C and C') are simultaneous in system A because the speed of the propagation of light is the same in both directions (see Fig. 423*b*). But it can readily be seen that these two events which are simultaneous in system A are observed at different instants at the corresponding points D and D' in the moving system B because the speed of light has the same value relative to system B as well.

Indeed, it takes the light signal some time Δt to propagate from the point O to the point C (or C') in system A. During this time system B receives a displacement to the right and the point O' undergoes the same displacement. Therefore,

the light signals achieving the points C and C' are observed at the corresponding points D and D' in system B at different instants (relative to system B) because the points D and D' are at different distances from the point O' in system B, the speed of light being constant and equal to c in both systems. Hence, because of the constancy of the speed of light we must conclude that relative to system B the event at D' occurs earlier than the event at D. In other words, *the simultaneity of events is relative* and, consequently, the time intervals between two events depend on the reference system in which they are observed.

When we pass from one reference system to another it is necessary to take into account the variation of time because there is no absolute time. It can readily be seen that the time lag of the events at the point D in system B relative to the events at the point D' depends on the distance CC', that is on the difference between the coordinates of the events in system A. Therefore, when the events occurring in system A are described by an observer moving together with system B (the observer is at rest relative to B and moves relative to system A) it is necessary to take into account the variation of time and its dependence on the coordinates.

It is also important to note that the argument we have presented is reversible: the events which are simultaneous in system B are not such in A. Indeed, let us take two points M and M' in system B lying at equal distances from O' (Fig. 424). Then the light signal emitted at an instant from a light source at point O' simultaneously achieves the points M and M' in system B but these events are not simultaneous at the points N and N' in system A which are not at equal distances from O (the points N and N' correspond to M and M', respectively). When speaking about the simultaneity of the events occurring at the points O and O', M and N, M' and N', etc. belonging to different reference systems (these points are close to each other or even coincide at certain instants) it is necessary to take into account that the distances between the points which are equal in system B (e.g. $MO' = O'M'$) are not equal in system A (e.g. $NO < ON'$).

It is readily seen that in case $v \ll c$ the points O and O' practically gain no displacements during a finite time Δt and then the events which are observed as simultaneous in A are also simultaneous in system B. This corresponds to the assumption of the existence of absolute time (which is the same for the moving reference systems) in the Newtonian mechanics. We see that the concept of absolute time can be regarded as an adequate approximation to reality for the motions whose velocities are much less than the speed of light.

To visualize the variation of time we consider the following instructive example. Suppose that in each reference system there are sets of equidistant clocks placed sufficiently close to one another. Using light signals we can synchronize the clocks in each of the systems. The synchronization is carried out by emitting a light pulse from a light source placed at the midpoint between two clocks. The rates of the synchronized clocks must be such that the light emitted by the light source placed at the midpoint achieves the two clocks simultaneously, that is both clocks must read the same time the moments the light signals achieve them. It is evident that if a clock a is synchronized with a clock b and the clock b is synchronized with a clock c then the clocks a and c are also synchronized with each other, and so on. In this way it is possible to synchronize all the clocks in each of the systems. It is of course supposed that the distances are measured with the aid of the same standard in each system. We can now state the principle of the relativity of simultaneity in the following manner: the clocks which are synchronized in system A do not read the same time when they are watched by

observers who move together with system B and are at rest relative to B at different points belonging to B. Similarly, the clocks which are synchronized in system B do not run synchronously when observed from system A. What has been said will be discussed in more detail after the general relativistic transformation laws from one inertial system to another have been established.

It should also be noted that if the clocks are placed on a straight line perpendicular to the velocity v of the relative motion of the systems then the clocks synchronized in A are also synchronized relative to B and vice versa. Indeed,

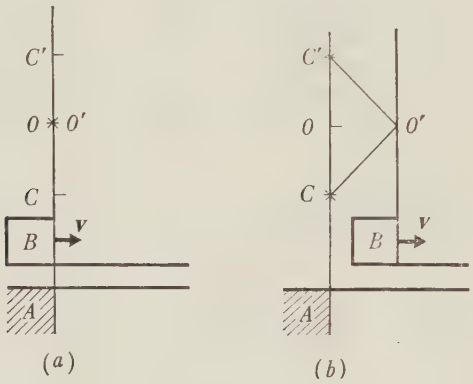

FIG. 425

during the time interval Δt (it takes the light signals emitted from O the time Δt to achieve the points C and C', see Fig. 425) system B passes a distance equal to $v\,\Delta t$ in the direction of v, which does not affect the distances from the point O' to the points C and C' which remain equal to each other. Therefore, these events (the observations of the light flash at C and C') are simultaneous in system B as well. Thus, the change of time under the transformation from one inertial reference system to another inertial system moving relative to the former with velocity v depends solely on the coordinate reckoned along the direction of v.

152. Lorentz Transformation

In this section we state and solve the basic problem concerning the changes of the coordinates and of the time under the transformation from one inertial reference system to another. As before, we consider two identical reference systems A and B in which the standards for measuring distances and time are the same. System B is supposed to move with constant velocity v relative to system A along the directions of the axes x and x' (the x-axis and the x'-axis are in one line and are directed similarly) so that at the initial instants $t = t' = 0$ the origins of the reference systems are at one point.

Suppose that at the initial instants $t = t' = 0$ a light source placed at the origin emits a light pulse. Then during the time interval from $t = 0$ to $t > 0$ the light signal passes a distance of ct in system A and, similarly, in system B during the corresponding time interval from $t' = 0$ to $t' > 0$ the light signal

passes a distance of $c't'$. In both systems the points which are achieved by the light pulse at the instants $t > 0$ and $t' > 0$ respectively lie on spherical surfaces of radii ct and $c't'$. The wave surface in system A is described by the equation

$$x^2 + y^2 + z^2 = c^2t^2 \tag{152.1}$$

and in system B by the equation

$$x'^2 + y'^2 + z'^2 = c'^2t'^2 \tag{152.2}$$

This follows from Einstein's postulates.

The space and the time are considered homogeneous, which is equivalent to the assumption that between the coordinates and the time in different systems there are linear relations. Therefore, the coordinates x and x' are connected by a linear relation which can be written in the form

$$x' = \gamma (x - vt) \tag{152.3}$$

where γ is a constant (yet unknown). Relation (152.3) follows from the fact that the point $x' = 0$ (the origin of system B) moves with the velocity v relative to system A and that the points $x = 0$ and $x' = 0$ coincide for $t = 0$. The unknown constant coefficient γ is dependent on v and on c and it must turn into unity for $v \to 0$ ($v \ll c$), which is the case in the Galilean transformation.

Since the systems only move in the direction of the x-axis (or the x'-axis) the coordinates y, y' and z, z' must not change, that is in this case, as in the Galilean transformation, we have

$$y' = y \text{ and } z' = z \tag{152.4}$$

The time t' in system B depends linearly on the time t and the coordinate x in system A; therefore, we put

$$t' = at + bx \tag{152.5}$$

where a and b are unknown constants which must satisfy the conditions $a \to 1$ and $b \to 0$ for $v \to 0$ ($v \ll c$).

On substituting (152.3), (152.4), and (152.5) in (152.2) we obtain

$$\gamma^2 (x - vt)^2 + y^2 + z^2 = c^2 (at + bx)^2 \tag{152.6}$$

The values of the coefficients γ, a, and b should be chosen so that relations (152.1) and (152.6) coincide identically. Therefore, the following equalities must hold:

$$\gamma^2 - c^2b^2 = 1, \quad \gamma^2 v + c^2ab = 0 \text{ and } c^2a^2 - \gamma^2v^2 = c^2$$

Expressing b from the second equality and substituting the result in the first one we obtain, taking into account the third equality, the relation

$$a^2 = \gamma^2$$

Therefore, the third equality yields

$$\gamma^2 = \frac{c^2}{c^2 - v^2}$$

We now deduce from the second equality the relation

$$b = -\gamma \frac{v}{c^2}$$

The physical meaning of γ and a indicates that these quantities are positive and therefore (see Fig. 426)

$$\gamma = a = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \tag{152.7}$$

We have thus arrived at the following transformation from system A to system B satisfying Einstein's postulates :

$$\left.\begin{aligned} t' &= \gamma\left(t - \frac{vx}{c^2}\right) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}\right\} \tag{152.8}$$

where $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$. Relations (152.8) express the famous *Lorentz transformation*. H. Lorentz derived this transformation formulas in connection with the study of electromagnetic phenomena. It was A. Einstein who pointed out that this transformation has a universal character because it involves only spacial coordinates and time. As is readily seen, for $v \to c$ ($v \ll c$) the Lorentz transformation goes into the Galilean transformation.

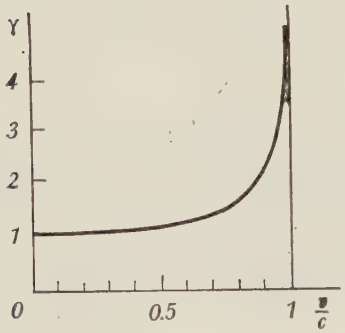

FIG. 426

Equalities (152.8) can be resolved with respect to the quantities t, x, y, and z characterizing system A to find the inverse transformation from B to A which has the form

$$\left.\begin{aligned} t &= \gamma\left(t' + \frac{vx'}{c^2}\right) \\ x &= \gamma(x' + vt') \\ y &= y' \\ z &= z' \end{aligned}\right\} \tag{152.9}$$

We see that the inverse transformation expressed by (152.9) only differs from (152.8) in the sign in front of v. Formulas (152.9) also express a Lorentz transformation (of type (152.8)) because $-v$ is nothing but the velocity with which system A moves relative to system B (A moves relative to B in the direction of the negative x'-axis).

Expression (152.7) of the coefficient γ shows that the velocity v is always less than c. Since reference systems are connected with some material bodies it follows that relative velocities of any bodies cannot exceed the speed of light. We have thus come to one of the fundamental inferences of the relativity theory: the speed of light is the maximum (limiting) velocity of motion. That is why c is said to be the *ultimate speed*.

153. Analysis of Lorentz Transformation

The analysis of the Lorentz transformation makes it possible to draw a number of important consequences following from the relationship between the coordinates and the time.

For our further aims it is more convenient to express the Lorentz transformation (formula (152.8)) and its inverse (formula (152.9)) in terms of the increments of the variables involved. Since all the relations in (152.8) and (152.9) are linear they retain the linear character for the increments as well and we thus come to the equalities

$$\left.\begin{aligned} \Delta t' &= \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) \\ \Delta t &= \gamma\left(\Delta t' + \frac{v}{c^2}\Delta x'\right) \\ \Delta x' &= \gamma(\Delta x - v\,\Delta t) \\ \Delta x &= \gamma(\Delta x' + v\,\Delta t') \\ \Delta y' &= \Delta y \\ \Delta z' &= \Delta z \end{aligned}\right\} \tag{153.1}$$

1. Relativity of simultaneity. Earlier we presented some qualitative considerations concerning the relativity of simultaneity. Now, using (153.1), we can estimate quantitatively the difference between time intervals measured in different inertial reference systems. For the sake of simplicity we again consider the two reference systems A and B which move along the parallel coordinate axes x and x'; in this case the course of time is connected with the coordinates x and x' solely.

Suppose that two events occur simultaneously in system A at a distance of Δx from each other; $\Delta t = 0$ for these events. We now pose the following question; what value of $\Delta t'$ corresponds to these events in system B, that is what is the time interval between the two events occurring simultaneously in system A when they are recorded physically by an observer who is at rest relative to system B and moves together with the latter?

The substitution of $\Delta t = 0$ in the first equality (153.1) yields

$$\Delta t' = -\gamma\frac{v}{c^2}\Delta x \tag{153.2}$$

Consequently, the events which are simultaneous in A are not simultaneous in B and are observed from system B as occurring at different instants. It also evidently follows that the event which takes place at a point with a greater coordinate x in system A (as before, we reckon x along the direction of motion of B relative to A) is observed in system B as occurring earlier. In other words, the event at a point in A which lies farther in the direction of motion of B relative to A occurs earlier in B, the time difference increasing together with the velocity v and with the distance Δx.

Similarly, two events occurring simultaneously in system B ($\Delta t' = 0$ for them) for which $\Delta x' \neq 0$ are not observed simultaneously from A and have a time separation

$$\Delta t = \gamma\frac{v}{c^2}\Delta x' \tag{153.3}$$

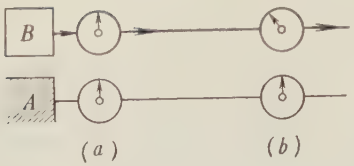

FIG. 427

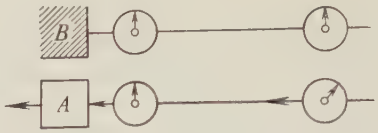

FIG. 428

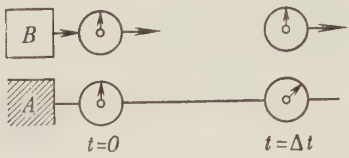

FIG. 429

This conclusion follows from the second equality (153.1). We again see a quite similar situation: the event in B at a point which lies farther in the direction of motion of system A relative to system B occurs earlier in A.

It can readily be shown that clocks synchronized in system B and moving together with B are observed from system A as running non-synchronously. Suppose that in each of the systems from system A there are identical synchronized clocks placed sufficiently close to each other along the axes x and x' respectively. The synchronization of the clocks in each system can be performed with the aid of light signals as was indicated in Sec. 151. Let us consider two clocks a and b in system A placed at a distance of Δx from each other. At a given time registered by the synchronous clocks a and b in system A there are two corresponding clocks moving together with system B which go past the clocks a and b. According to (153.2), these moving clocks show different time; the clock placed farther in the direction of motion of B advances more slowly (see Fig. 427).

The same situation takes place for two clocks in system B lying at a distance $\Delta x'$ from each other if we register with the aid of these clocks the data shown by the corresponding clocks in system A (Fig. 428).

Suppose now that we take two clocks in system A and register the time shown by the corresponding clocks in system B passing by these clocks in A at different instants $t = 0$ and $t = \Delta t$ (see Fig. 429). The time difference Δt can always be chosen so that the clocks in system B going past the clocks in system A show the same time (that is $\Delta t' = 0$). From the first equality (153.1) it follows that this time difference is equal to

$$\Delta t = \frac{v}{c^2} \Delta x$$

This means that for the clock in system A placed farther along the direction of motion of system B relative to system A the registration of the time shown by the corresponding clock moving together with system B should be made later, the time lag Δt being expressed by the above formula. Hence, the events which are simultaneous in system B are recorded physically at different instants in system A. We see that the relativity of simultaneity is completely similar for both systems A and B.

2. Length contraction. Let us consider a definite line segment $l_0 = \Delta x'$ in system B. It can be understood as the length of a rod lying along the x'-axis which is at rest in the reference frame B. The length of the rod in system B (the *rest length*) can simply be determined by finding the coordinates of the ends of the rod and computing the absolute value of their difference. As to the measurement of the length of the same rod in system A relative to which the rod moves, it can be performed as follows. Let us measure the coordinates x of the ends of the rod $l_0 = \Delta x'$ in system A at the same instant shown by the synchronized clocks in that system. Then the length $l_m = \Delta x$ of the rod in system A is equal to the absolute value of the difference between these coordinates. It is now required to find the relationship between $l_0 = \Delta x'$ and $l_m = \Delta x$ for $\Delta t = 0$. From the third equality (153.1) it follows that

$$\Delta x' = \gamma \Delta x, \quad \text{that is} \quad l_m = \frac{l_0}{\gamma} \tag{153.4}$$

where l_0 is the rest length of the rod (measured in system B) and l_m is the length of the same rod in system A relative to which it moves. Since $\gamma > 1$ it follows that the rest length of the rod is greater than its length as viewed from system A relative to which the rod moves. In other words, the rod moving parallel to its length undergoes length contraction. We see that *the length is relative* and depends on the reference frame in which it is measured.

Analogous considerations show that a rod of length $\Delta x = l_0$ which is at rest in system A is viewed from system B (relative to which the rod moves) as having a shorter length. In this case the length $\Delta x = l_0$ is determined for $\Delta t' = 0$ and the fourth equality (153.1) implies

$$\Delta x = \gamma \Delta x', \quad \text{that is} \quad l_m = \frac{l_0}{\gamma} \tag{153.5}$$

We again see the length contraction of the moving rod.

There is no contradiction between formulas (153.4) and (153.5) because they correspond to different measurements performed in each system although the measurement procedure is completely the same in every system. For each of the systems we obtain the same result: like the time, the length is relative and this property is reciprocal for both systems.

It is evident that the lengths measured along the axes y and z (or along y' and z'-axes) do not change under the transformation from system A to system B and under the inverse transformation from B to A. Only the differences between the coordinates along the axes x and x' going in the direction of the relative motion of the reference systems change under these transformations.

3. Time dilation of moving clocks. Let us consider a clock which is at rest relative to reference frame B and moves together with B relative to reference frame A. Suppose that the clock in system B measures a time interval $\Delta t'$ while another clock which is at rest relative to system A measures the corresponding time interval Δt. Let us determine the relation between the time intervals Δt and $\Delta t'$ measured by the clock which is at rest in system A and by the clock which moves relative to A respectively.

During time Δt the moving clock gains a displacement of $\Delta x = v \Delta t$ along the x-axis and $\Delta x' = 0$ for this clock. Therefore, the first equality (153.1) implies

$$\Delta t' = \gamma \left(\Delta t - \frac{v^2}{c^2} \Delta t \right) = \gamma \left(1 - \frac{v^2}{c^2} \right) \Delta t = \frac{\Delta t}{\gamma} \tag{153.6}$$

because $\gamma^2 \left(1 - \frac{v^2}{c^2}\right) = 1$. We see that the time $\Delta t'$ shown by the moving clock is less than the corresponding time Δt shown by the clock which is at rest. Hence, the moving timepieces lag behind the timepieces in a state of rest. This is what we call the *time dilation* of the moving clocks.

It should be noted that relation (153.6) can be obtained directly from the second equality (153.1) by substituting $\Delta x' = 0$ in it.

Analogous considerations concerning a clock which is at rest relative to system A and moves relative to system B together with A imply that

$$\Delta t = \frac{\Delta t'}{\gamma} \qquad (153.7)$$

where Δt is the time read by the moving clock. We again see that the moving clock runs more slowly.

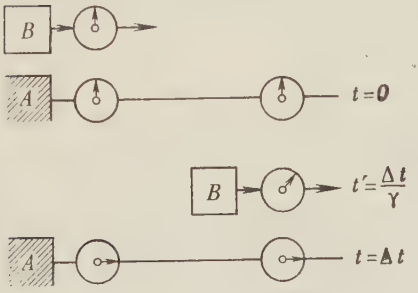

FIG. 430

The dilation of time can also be illustrated by the following (imaginary) experiments with clocks. Let us consider a clock moving relative to A and compare the time shown by it with the time shown by synchronized clocks which are at rest in system A (see Fig. 430) at different instants. Suppose that at instant $t = 0$ the clocks in A and in B read the same time. During a time interval Δt the moving clock undergoes a displacement and shows time $\Delta t' = \frac{\Delta t}{\gamma}$ different from that shown by the clock which is at rest and is placed opposite the former.

In Fig. 431 we see a unified picture showing the data read by several synchronized clocks moving together with system B which are observed from system A. The clocks are depicted for equal time intervals $\Delta t = 2$ s. The numbers in the circles indicate the number of seconds for each of the clocks at definite instants of time measured by the clocks in A. The time and the distances are calculated for $\gamma = 2$, which corresponds to $v \approx 0.87$ s. Every inclined line joins the same clock moving together with system B. The computations are carried out according to the formulas $\Delta x = v\,\Delta t$, $\Delta t' = \frac{\Delta t}{\gamma}$, and $\Delta t'_1 = -\gamma \frac{v}{c^2} \Delta x = (1 - \gamma^2)\,\Delta t'$. Consequently,

$$\Delta t' = 1 \text{ s}, \quad \Delta t = 2 \text{ s}, \quad \Delta t'_1 = -3 \text{ s}, \quad \text{and} \quad \Delta x \approx 5.2 \times 10^5 \text{ km}$$

Figure 431 demonstrates both the dilation of time and the non-simultaneity shown by the moving clocks. The reader should pay attention to the magnitudes of the distances and of the variations of time. If we depicted an analogous scheme for $\Delta t = 2$ μs we would obtain $\Delta x \approx 520$ m.

The dilation of time accounts for a comparatively long lifetime of moving mesons. Mesons are elementary particles generated when fast particles collide with a substance. They are observed in cosmic rays and in experiments with charged-particle accelerators. It was found that a meson (we mean the American or mu meson) is an unstable particle whose mean proper lifetime is of the order of 10^{-6} s (more precisely, 2.2×10^{-6} s).

Mesons are observed at an altitude of 10-20 km and at the same time they are found in cosmic radiation laboratories at the Earth's surface. If the mean lifetime remained invariable a meson would travel a distance of the order of 300 m during that time even if it moved with a velocity very close to the speed of light. The existence of mesons near the Earth's surface is accounted for by the dilation of the lifetime of a moving meson. To the mean lifetime $\Delta t' \sim 2 \times 10^{-6}$ s

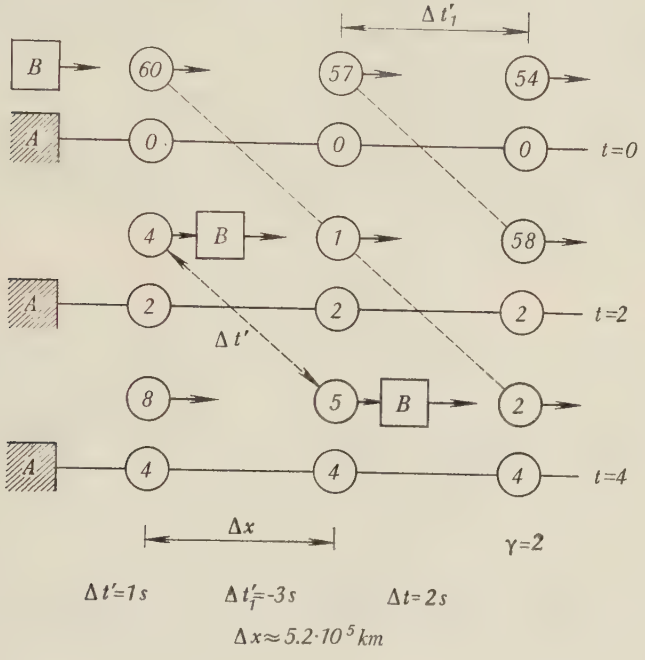

FIG. 431

of a meson which is at rest in system B there corresponds a time interval Δt measured by a clock which is at rest relative to the Earth (here the Earth plays the role of system A). These time intervals are connected by the equality $\Delta t = \gamma \Delta t'$ (see (153.6)).

During time Δt the meson can reach the Earth's surface provided that γ is sufficiently large. For mesons in cosmic rays we have $\gamma \sim 50$ and therefore during time Δt the meson can travel a distance of the order of

$$s \sim c\,\Delta t = c\gamma\,\Delta t' = 3 \times 10^8 \times 50 \times 2 \times 10^{-6} = 30 \text{ km}$$

It should be noted that for $\gamma \sim 50$ the particle moves with a velocity which is approximately by 60 km/s less than the speed of light.

Physicists can judge on the phenomenon of time dilation for moving particles by the results obtained from the analysis of the observation data concerning the mesonic decay in the accelerators. These observations also indicate the increase (approximately 100 times increase) of the path of the particles which can only be explained by the time dilation. The results of the experiments with accelerators and of some other experiments with fast particles (whose velocity

is close to c) confirm the validity of the inferences following from the fundamental principles of the special relativity theory.

4. Transformation of velocities. Suppose that a particle moves with a velocity $\boldsymbol{u}'(u'_x,\ u'_y,\ u'_z)$ in system B*. The velocity in system B is expressed by the usual formula

$$\boldsymbol{u}' = \frac{d\boldsymbol{r}'}{dt'}\left(\frac{dx'}{dt'},\quad \frac{dy'}{dt'},\quad \frac{dz'}{dt'}\right)$$

where $\boldsymbol{r}'(x',\ y',\ z')$ is the radius vector of the particle in system B. Let us determine the velocity of the same particle relative to system A. It is obviously given by the relation

$$\boldsymbol{u} = \frac{d\boldsymbol{r}}{dt}\left(\frac{dx}{dt},\quad \frac{dy}{dt},\quad \frac{dz}{dt}\right)$$

where $\boldsymbol{r}\ (x,\ y,\ z)$ is the radius vector of the particle in system A. Equations (153.1) imply the relationship between the differentials of the quantities involved for different inertial reference frames:

$$\left.\begin{aligned} dt' &= \gamma\left(dt - \frac{v}{c^2}\,dx\right)\\ dx' &= \gamma\,(dx - v\,dt)\\ dy' &= dy\\ dz' &= dz \end{aligned}\right\} \tag{153.8}$$

Taking into account these relations we can write

$$\left.\begin{aligned} u'_x &= \frac{dx'}{dt'} = \frac{\gamma\,(dx - v\,dt)}{\gamma\left(dt - \frac{v}{c^2}\,dx\right)} = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = \delta\,(u_x - v)\\ u'_y &= \frac{dy'}{dt'} = \frac{dy}{\gamma\left(dt - \frac{v}{c^2}\,dx\right)} = \frac{u_y}{\gamma\left(1 - \frac{u_x v}{c^2}\right)} = \frac{\delta}{\gamma}\,u_y\\ u'_z &= \frac{dz'}{dt'} = \frac{\delta}{\gamma}\,u_z \end{aligned}\right\} \tag{153.9}$$

where

$$\delta = \left(1 - \frac{u_x v}{c^2}\right)^{-1} \tag{153.10}$$

We remind the reader that $\gamma^2\left(1 - \frac{v^2}{c^2}\right) = 1$. It can similarly be shown that

$$u_x = \delta'\,(u'_x + v), \quad u_y = \frac{\delta'}{\gamma}\,u'_y \text{ and } u_z = \frac{\delta'}{\gamma}\,u'_z \tag{153.11}$$

where

$$\delta' = \left(1 + \frac{u'_x v}{c^2}\right)^{-1} \tag{153.12}$$

The comparison of (153.9) and (153.11) shows that

$$\delta\delta' = \gamma^2 \tag{153.13}$$

---

* Here and henceforth we write the components of a vector in the brackets after the letter denoting that vector.

It should first of all be noted that the transformation law for the velocities we have derived essentially differs from the ordinary addition rule for the velocities known from the Newtonian mechanics. Indeed, for the Newtonian mechanics we have the Galilean transformation (valid for $v \ll c$) according to which

$$u_x = u'_x + v, \quad u_y = u'_y \text{ and } u_z = u'_z$$

while in the relativistic case (when $v \sim c$), as is shown by formulas (153.11), the sum of the components of the velocity along the x-axis is multiplied by the quantity δ' dependent on u'_x, v, and c. The components of the velocity along the y- and z-axes also change because, although the differences between the coordinates y and z remain invariable, the time intervals change and $dt \neq dt'$.

It should also be noted that the Newtonian addition rule for the velocities does not apply to the speed of light. If we have $u'_x = c$ for the speed of light in system B then in system A we also have $u_x = c$. Indeed, according to (153.11), in this case

$$u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}} = \frac{c + v}{1 + \frac{v}{c}} = c$$

Of course, this result simply coincides with what is asserted by the basic principles of the relativity theory.

By the way, when a light signal in system B propagates along the y'-axis then, as it follows from formulas (153.11) if we put $u'_y = c$, $u'_x = 0$, and $u'_z = 0$ in them, the components of the velocity of light in system A are

$$u_y = \frac{c}{\gamma}, \quad u_x = v \text{ and } u_z = 0$$

Consequently, a ray of light which is normal to the x'-axis in system B has another direction in system A. As to the magnitude of the speed of light, it remains of course constant and is equal to c:

$$\sqrt{u_x^2 + u_y^2} = c$$

Generally, using formulas (153.11) we can readily show that the angle φ formed by a ray of light with the x-axis in system A is determined by the formula

$$\tan \varphi = \frac{\sin \varphi'}{\gamma \left(\cos \varphi' + \frac{v}{c} \right)}$$

where φ' is the angle between that ray and the x'-axis in system B. This change of the direction of the rays of light accounts for the *aberration of starlight* (which is an observed seasonal change in the position of stars) appearing owing to the orbital motion of the Earth. For a star at the zenith the angular deviation is of the order of 40″ per half a year.

All these facts, unusual as they might appear at first glance, are a natural consequence of the equivalence of all the inertial reference systems and of the constancy of the speed of light in all of them.

Let us also establish a relation connecting the absolute values of the velocities in systems A and B which is important for our further aims. Since

$$u^2 = u_x^2 + u_y^2 + u_z^2 \text{ and } u'^2 = u_x'^2 + u_y'^2 + u_z'^2$$

it follows from (153.9) that

$$u'^2 = \delta^2 \left[(u_x - v)^2 + \frac{1}{\gamma^2} (u_y^2 + u_z^2) \right] \tag{153.14}$$

We calculate the quantity $1 - \frac{u'^2}{c^2}$ (some simple transformations are omitted below):

$$1 - \frac{u'^2}{c^2} = \delta^2 \left[\frac{1}{\delta^2} - \frac{1}{c^2} (u_x - v)^2 - \frac{1}{c^2 \gamma^2} (u_y^2 + u_z^2) \right] =$$

$$= \delta^2 \left[\left(1 - \frac{u_x v}{c^2} \right)^2 - \frac{1}{c^2} (u_x - v)^2 - \frac{1}{c^2} \left(1 - \frac{v^2}{c^2} \right) (u_y^2 + u_z^2) \right] =$$

$$= \delta^2 \left(1 - \frac{v^2}{c^2} \right) \left(1 - \frac{u^2}{c^2} \right) = \frac{\delta^2}{\gamma^2} \left(1 - \frac{u^2}{c^2} \right) \tag{153.15}$$

We will frequently use this relation; it can be written in the shortened form

$$\delta \alpha' = \gamma \alpha \tag{153.16}$$

where

$$\alpha = \left(1 - \frac{u^2}{c^2} \right)^{-1/2} \quad \text{and} \quad \alpha' = \left(1 - \frac{u'^2}{c^2} \right)^{-1/2} \tag{153.17}$$

(all the quantities in (153.16) are positive). Taking into account (153.13) we can also show that

$$\delta' \alpha = \gamma \alpha' \tag{153.18}$$

We conclude this section by presenting the Lorentz transformation in the vector form. Denoting the unit vectors along the axes x, y, and z as $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ respectively we can write the expressions $\boldsymbol{r} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2 + z\boldsymbol{e}_3$ and $\boldsymbol{r}' = x'\boldsymbol{e}_1 + y'\boldsymbol{e}_2 + z'\boldsymbol{e}_3$ for the radius vectors in systems A and B, respectively. Using this notation we obtain the relation

$$\boldsymbol{r}' = \gamma (x - vt)\, \boldsymbol{e}_1 + y\boldsymbol{e}_2 + z\boldsymbol{e}_3 + x\boldsymbol{e}_1 - x\boldsymbol{e}_1 = \boldsymbol{r} + (\gamma - 1)\, x\boldsymbol{e}_1 - \gamma v t \boldsymbol{e}_1 \tag{153.19}$$

which expresses the transformation of coordinates by one vector formula. Let us write the following obvious relations

$$\boldsymbol{e}_1 = \frac{\boldsymbol{v}}{v}, \quad x = \boldsymbol{r}\boldsymbol{e}_1 = \frac{\boldsymbol{r}\boldsymbol{v}}{v} \quad \text{and} \quad v x = \boldsymbol{v}\boldsymbol{r}$$

We can now rewrite (153.19) in the general form

$$\boldsymbol{r}' = \boldsymbol{r} + (\gamma - 1) \frac{\boldsymbol{r}\boldsymbol{v}}{v^2} \boldsymbol{v} - \gamma \boldsymbol{v} t \tag{153.20}$$

The transformation formula for time can be written as

$$t' = \gamma \left(t - \frac{\boldsymbol{v}\boldsymbol{r}}{c^2} \right) \tag{153.21}$$

Formulas (153.20) and (153.21) represent the Lorentz transformation for any direction of the vector $\boldsymbol{v}$ and for any point $\boldsymbol{r}$; it should only be remembered that system B is supposed to be in a uniform translatory motion relative to system A and that $\boldsymbol{r}' = \boldsymbol{r} = 0$ for $t = t' = 0$. The transformation law is completely specified by the direction and the magnitude of the velocity vector $\boldsymbol{v}$. The components of the vector $\boldsymbol{r}$ normal to $\boldsymbol{v}$ do not change and only the projection of the vector $\boldsymbol{r}$ on $\boldsymbol{v}$ and the vector $\boldsymbol{v}$ itself are involved in the relationship connecting the spatial coordinates and the time. For $\gamma \to 1$ we obtain from (153.20)

and (153.21) the Galilean transformation in the vector form:

$$t' \to t \quad \text{and} \quad r' \to r - vt$$

The Lorentz transformation shows that time and space are inseparable. All the physical phenomena and all the processes in the world take place in space and time which form a unified complex of conditions for any motion of the substance.

154. Momentum

A motion with a velocity comparable with the speed of light was first encountered in physics in the investigation of a flow of charged particles (electrons) emitted by a radioactive substance. The investigation makes it possible to determine the velocity and the mass of an electron because the laws describing the action of electric and magnetic fields on a moving charged particle are known.

The experiments carried out at the beginning of the twentieth century showed that there is a dependence of the inertial mass on the velocity or, more precisely, on the ratio of the velocity of motion to the speed of light. First it was assumed that this was a property characteristic of charged particles only because of the inertia of the electromagnetic field generated by an electron (in this connection the notion of the electromagnetic mass was introduced). However, as was shown later by A. Einstein, the dependence of the mass on the velocity is a property of all material bodies and the mass-variability is one of the corollaries following from the postulates of the theory of relativity.

In mechanics every body (a solid, a fluid or a gas) is thought of as a collection of interacting moving particles or, in other words, as a system of particles. For this approach it is of no importance whether these particles are sufficiently small parts of the body or molecules and the only important assumption is that the dimensions of each particle are sufficiently small in comparison with the dimensions of the body (of the whole system of particles).

The law of conservation of the momentum of a closed (isolated) system of particles (of an isolated body) can be interpreted as a consequence of the inertia law. Irrespective of the variation of the velocities of some particles (entering into the system) because of their interaction the velocity of the isolated system regarded as a whole (the velocity of the translatory motion) remains constant.

The momentum of a system of particles is equal to the sum of the momenta of the particles, that is

$$K = \sum m_i u_i \tag{154.1}$$

This definition is used in both non-relativistic and relativistic mechanics although in the latter the mass m_i of a particle is assumed to be variable. Using this definition we can introduce the notion of the mean velocity of motion u of a system of particles by putting

$$u = \frac{\sum m_i u_i}{\sum m_i} \tag{154.2}$$

The velocity u can be interpreted as the velocity of motion of the system of particles as a whole and is equal to the mean momentum per unit mass.

In the Newtonian mechanics (in which $m_i = \text{const}$) definition (154.2) implies that u is the velocity of the centre of mass of the system of particles. In relativistic mechanics this is not necessarily so because in this case m_i are variable quan-

tities (dependent on time). That is why relation (154.2) will merely be regarded as a formal definition of the mean velocity $\boldsymbol{u}$. The quantity $\boldsymbol{u}\,dt$ will be interpreted as the translatory displacement of the system of particles as a whole (the displacement of the body) during time dt. In what follows we conditionally call $\boldsymbol{u}$ "the velocity of the centre of mass of the system".

For an isolated system the conservation law for the momentum can be written in the form

$$\boldsymbol{K} = \sum m_i \boldsymbol{u}_i = \boldsymbol{u} \sum m_i = m\boldsymbol{u} = \text{const} \,. \tag{154.3}$$

where $m = \sum m_i$.

It should be stressed that for an isolated system $m = \sum m_i$ is always invariable although in the general case each of the masses m_i can vary. Only for the constant quantities K and m is the velocity $\boldsymbol{u}$ constant; in other words, for the inertia law to hold for the isolated systems it is necessary to assume that the total masses of these systems should be invariable.

155. Dependence of Mass on Velocity of Motion

The aim of this section is to investigate the dependence of the mass on the velocity of motion. To this end we use the natural requirement that under the transformation from reference system B to reference system A the equality

$$\sum m_i' \boldsymbol{u}_i' = \boldsymbol{u}' \sum m_i'$$

should go into the equality

$$\sum m_i \boldsymbol{u}_i = \boldsymbol{u} \sum m_i \tag{155.1}$$

where m_i' is the magnitude of the mass of the ith particle in system B. It is of course assumed that $\boldsymbol{u}_i'$ and $\boldsymbol{u}'$ are transformed into $\boldsymbol{u}_i$ and $\boldsymbol{u}$, respectively, in accordance with formulas (153.9) expressing the Lorentz transformation. The conditions we have stated make it possible to determine the dependence of the mass on the velocity and then to find the transformation laws for the mass and the momentum corresponding to the passage from system B to A.

To simplify the calculations we consider a system consisting of only two particles. For the same reason we assume that the momentum relative to B is equal to zero, that is

$$\left.\begin{aligned} m_1' u_{1x}' + m_2' u_{2x}' = 0 \\ m_1' u_{1y}' + m_2' u_{2y}' = 0 \\ m_1' u_{1z}' + m_2' u_{2z}' = 0 \end{aligned}\right\} \tag{155.2}$$

This assumption means that the velocity of the centre of mass in reference frame B is equal to zero and therefore the velocity of the centre of mass relative to reference frame A is equal to the velocity v of motion of B relative to A. This can also be readily verified using formulas (153.11): since $u_x' = 0$ and $u_y' = u_z' = 0$ it follows that $u_x = v$ and $u_y = u_z = 0$.

Let us consider the component of the momentum of the system of particles along the x-axis. In system A this component is equal to

$$m_1 u_{1x} + m_2 u_{2x} = (m_1 + m_2)\, v \tag{155.3}$$

On substituting the components of the velocity relative to system A given by formulas (153.9) in the first equality (155.2) we find that

$$\delta_1 m_1' (u_{1x} - v) + \delta_2 m_2' (u_{2x} - v) = 0$$

Equality (155.3) can be rewritten as

$$m_1(u_{1x}-v)+m_2(u_{2x}-v)=0 \tag{155.4}$$

We have thus obtained two homogeneous simultaneous equations in two unknowns (the unknowns are the expressions in the brackets). For these equations to possess non-zero solutions it is necessary and sufficient that

$$\delta_1 m_1' m_2 - \delta_2 m_2' m_1 = 0 \tag{155.5}$$

The substitution of the values of δ_1 and δ_2 given by formula (153.16) in (155.5) results in

$$\frac{\gamma\alpha_1}{\alpha_1'} m_1' m_2 = \frac{\gamma\alpha_2}{\alpha_2'} m_2' m_1$$

On cancelling the last equality by γ we rewrite it in the form

$$\frac{\alpha_1 m_1'}{\alpha_1' m_1} = \frac{\alpha_2 m_2'}{\alpha_2' m_2} \tag{155.6}$$

The left-hand member of (155.6) involves quantities which depend on the properties and characteristics of the first particle solely while the right-hand member is completely specified by the corresponding properties and characteristics of the other particle. Consequently, for equality (155.6) to hold it is necessary that each of the members should be equal to the same constant. On denoting this constant as D we arrive at the relations

$$\frac{m_1'}{m_1} = D\frac{\alpha_1'}{\alpha_1} \quad \text{and} \quad \frac{m_2'}{m_2} = D\frac{\alpha_2'}{\alpha_2}$$

Using expressions (153.17) for α_1 and α_1' we can write the first of these relations in the form

$$\frac{m_1'}{m_1} = D\frac{\sqrt{1-\frac{u_1^2}{c^2}}}{\sqrt{1-\frac{u_1'^2}{c^2}}} \tag{155.7}$$

whence the dependence of the mass of the particle on the absolute value of the velocity can be found.

Suppose that for system B we have $u_1' = 0$, that is the particle is at rest in B, its mass m_1' being equal to m_{10} (m_{10} is the *rest mass* of the particle). Then it follows from (155.7) that the mass of the first particle in system A is given by the formula

$$m_1 = \frac{1}{D}\frac{m_{10}}{\sqrt{1-\frac{u_1^2}{c^2}}} \tag{155.8}$$

This general relation must hold for the case $u_1 \ll c$ as well (that is for the Newtonian mechanics) and therefore $D = 1$ because in the limiting case of the Newtonian mechanics we have $m_1 = m_{10}$. For the other particle we also obtain an analogous equality.

Consequently, any particle moving with a velocity u relative to A possesses a mass of magnitude

$$m = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}} = m_0\alpha \tag{155.9}$$

where m_0 is the rest mass of the particle (which coincides with the mass of the particle when it moves with a velocity $u \ll c$) and u is the absolute value of the velocity.

In what follows we shall simply use the letter m (instead of the symbol m_0) for denoting the rest mass of a particle. The mass of a particle moving with a velocity u relative to A can be written as the product

$$m\alpha = \frac{m}{\sqrt{1-\frac{u^2}{c^2}}} \tag{155.10}$$

while the mass of the same particle in system B is equal to the product

$$m\alpha' = \frac{m}{\sqrt{1-\frac{u'^2}{c^2}}} \tag{155.11}$$

The dependence of the mass on the velocity is one of the fundamental facts of Einstein's relativistic mechanics. According to this principle, the inertial mass depends on the absolute value of the velocity or, more precisely, on the ratio of the absolute value of the velocity to the speed of light. The inertial mass of a body increases together with the velocity and tends to infinity as $u \to c$. It follows that none of the bodies with $m > 0$ can reach the speed of light c.

The experiments with accelerators in which the motion of fast particles (whose velocity is very close to the speed of light) is studied confirm the existence of the dependence of the mass on the velocity and the validity of formula (155.9).

156. Transformation Laws for Momentum and Mass

The dependence of the mass on the velocity is a consequence of the transformation law for the projection of the momentum of a system consisting of two particles on the x-axis along which the reference systems move relative to each other. Knowing this dependence we can easily investigate the change of the momentum under the transformation from one inertial reference frame to another. To this end we must take into account that, in the first place, the components of the velocity of the centre of mass along the y'- and z'-axes are equal to zero in system B and retain the zero values in system A and, in the second place, the mass depends on the velocity in accordance with formula (155.10). These properties, as will be shown later, imply that the components of the momentum of each particle along the axes y' and z' remain invariable under the transformation from system B to system A.

Since the dependence of the mass on the velocity is described by formula (155.10) the momentum of a particle with rest mass m in system A is expressed by the formula

$$\boldsymbol{K} = m\alpha\boldsymbol{u} = \frac{m\boldsymbol{u}}{\sqrt{1-\frac{u^2}{c^2}}} \tag{156.1}$$

In system B the momentum of the same particle is expressed as

$$\boldsymbol{K}' = m\alpha'\boldsymbol{u}' \tag{156.2}$$

Now, using the transformation formulas for the velocity we can find the transformation law for the components of the momentum $\boldsymbol{K}$ as corresponding to the passage from system A to system B. The component K_x' of the momentum in sys-

tem B is expressed by the formula

$$K'_x = m\alpha' u'_x$$

Using the expressions $u'_x = \delta (u_x - v)$ and $\delta = \frac{\gamma\alpha}{\alpha'}$ and taking into account (153.16) we obtain

$$K'_x = m\alpha' \delta (u_x - v) = \gamma m\alpha (u_x - v)$$

that is

$$K'_x = \gamma (K_x - vm\alpha) \tag{156.3}$$

We see that the transformation formula for the component K'_x is analogous to the formula for the coordinate x' and can be obtained from the latter by substituting K_x for x and $m\alpha$ for t.

Let us proceed to the transformation of the components K'_y and K'_z. Since their roles are quite equivalent it suffices to investigate one of them, for instance, K'_y. By definition, we have

$$K'_y = m\alpha' u'_y$$

Remembering that $u'_y = \frac{\delta}{\gamma} u_y$ and taking into account (153.16) we find

$$K'_y = m\alpha' \frac{\delta}{\gamma} u_y = m\alpha u_y = K_y \tag{156.4}$$

We see that, like the coordinate y (z), the projection of the momentum of the particle on the y-axis (z-axis) does not change. Consequently, the projections of the momentum are transformed like the coordinates when we pass from system B to system A (of course, the same applies to the inverse transformation from A to B). The only distinction between the formulas is that the transformation of the momentum involves the mass $m\alpha$ instead of time t.

We can now investigate the relationship between $m\alpha'$ and $m\alpha$ and derive the transformation law for the mass. This can readily be done by replacing the corresponding quantities in accordance with formulas (153.10) and (153.16), which yields

$$m\alpha' = m \frac{\gamma\alpha}{\delta} = \gamma m\alpha \left(1 - \frac{u_x v}{c^2}\right)$$

that is

$$m\alpha' = \gamma \left(m\alpha - \frac{K_x v}{c^2}\right) \tag{156.5}$$

Here, as should be expected, the same analogy takes place, that is t and x are replaced by $m\alpha$ and K_x, respectively.

Thus, both *the mass $m\alpha$ and the momentum K are relative* and depend on the reference system with respect to which they are considered. This is a new result of the theory of relativity which was unknown in prerelativistic physics.

Let us write down all the transformation formulas for the mass and for the momentum in one group:

$$\left.\begin{aligned} m\alpha' &= \gamma \left(m\alpha - \frac{vK_x}{c^2}\right) \\ K'_x &= \gamma (K_x - vm\alpha) \\ K'_y &= K_y \\ K'_z &= K_z \end{aligned}\right\} \tag{156.6}$$

The time t and the position vector r (x, y, z) change under the transformation from system A to system B and the mass $m\alpha$ and the momentum K of every particle also change. It is evident that the formulas expressing the inverse transfor-

mation from B to A have the same form with the only distinction that $-v$ should be replaced by $+v$.

By analogy with (153.20), we can write (156.6) in vector form:

$$\left.\begin{aligned} m\alpha' &= \gamma\left(m\alpha - \frac{\boldsymbol{K}v}{c^2}\right) \\ \boldsymbol{K}' &= \boldsymbol{K} + (\gamma - 1)\frac{\boldsymbol{K}v}{v^2}\boldsymbol{v} - \gamma m\alpha\boldsymbol{v} \end{aligned}\right\} \tag{156.7}$$

When considering a translatory motion of an arbitrary body we can regard the body as a particle; accordingly, the momentum of the body is equal to

$$\boldsymbol{K} = m\alpha\boldsymbol{u}$$

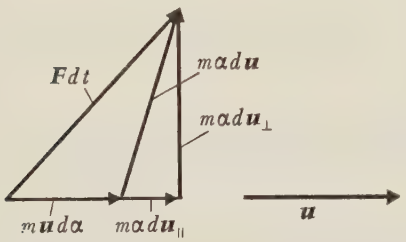

FIG. 432

where m is its rest mass (that is the mass of the body for $u \ll c$) and $\boldsymbol{u}$ is its velocity. For a motion of a body relative to an inertial reference frame the second law of dynamics is stated in relativistic mechanics in the same way as it was done by Newton in classical mechanics (see Sec. 19): the force is equal to the derivative of the momentum with respect to time, that is

$$\boldsymbol{F} = \frac{d\boldsymbol{K}}{dt} = \frac{d}{dt}(m\alpha\boldsymbol{u}) = m\frac{d}{dt}(\alpha\boldsymbol{u}) \tag{156.8}$$

Using this law we can study a motion of a body moving with a large velocity provided that the force acting on the body is known. As it follows from (156.8), in the general case the directions of the force $\boldsymbol{F}$ and of the acceleration $\frac{d\boldsymbol{u}}{dt}$ do not coincide because

$$\boldsymbol{F} = m\boldsymbol{u}\frac{d\alpha}{dt} + m\alpha\frac{d\boldsymbol{u}}{dt} \tag{156.9}$$

From (156.9) it follows that the vectors representing the force, the acceleration and the velocity lie in one plane. To examine the connection between the force and the acceleration it is expedient to resolve these vector quantities into components (lying in this plane) perpendicular (or parallel) to the velocity $\boldsymbol{u}$ (see Fig. 432; the subscripts $\perp$ and $\parallel$ denote the components perpendicular and those parallel to $\boldsymbol{u}$, respectively). We obviously have

$$F_\perp\, dt = m\alpha\, du_\perp \quad \text{and} \quad F_\parallel\, dt = mu\, d\alpha + m\alpha\, du_\parallel \tag{156.10}$$

Since $du_\parallel$ is equal to du (that is to the increment of the absolute value of $\boldsymbol{u}$) and

$$d\alpha = \left(1 - \frac{u^2}{c^2}\right)^{-3/2}\frac{u\,du}{c^2} = \frac{\alpha^3}{c^2}u\,du \tag{156.11}$$

The second equality (156.10) can be written in the form

$$F_\parallel\, dt = m\alpha\left(1 + \alpha^2\frac{u^2}{c^2}\right)du_\parallel = m\alpha^3\, du_\parallel$$

We see that the directions of the force and of the acceleration coincide only when the force is perpendicular to the velocity or parallel to it.

Let us introduce the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & \alpha^2 \end{pmatrix}$$

With the aid of this matrix the relationship between the components of the force and of the acceleration can be written in the form

$$\begin{pmatrix} F_\perp \\ F_\parallel \end{pmatrix} = m\alpha \begin{pmatrix} 1 & 0 \\ 0 & \alpha^2 \end{pmatrix} \begin{pmatrix} \frac{du_\perp}{dt} \\ \frac{du_\parallel}{dt} \end{pmatrix} \tag{156.12}$$

Relation (156.12) proves useful when it is required to determine the force for given values of the mass and of the acceleration and, conversely, when the latter quantities are found for a given force.

157. Energy

From the general expression

$$m\alpha = m\left(1 - \frac{u^2}{c^2}\right)^{-1/2}$$

of the mass of a body moving with a velocity u we can draw some important conclusions concerning the law connecting the mass and the energy. Since the mass increases together with the velocity it is natural to expect that there is a connection between the mass and the kinetic energy.

We start with the investigation of this connection for small values of u/c. To this end we expand α in series in powers of u/c:

$$m\alpha = m + \frac{1}{c^2}\frac{mu^2}{2} + \frac{3}{8}\frac{mu^4}{c^4} + \ldots \tag{157.1}$$

It follows that for small values of the ratio u/c we have

$$m\alpha \approx m + \frac{1}{c^2}\frac{mu^2}{2} = m + \frac{1}{c^2}T \tag{157.2}$$

where $T = \frac{mu^2}{2}$ is the kinetic energy of the body with mass m moving with velocity u (here is meant the notion of the kinetic energy as it was defined in the Newtonian mechanics). According to the relativity principle, any law holding in one of the inertial reference systems must hold for all the other inertial systems moving with arbitrary velocities, however large. Therefore, the kinetic energy for large velocities of motion must satisfy a relation analogous to formula (157.2) which we have derived for the case $u/c \ll 1$ when the Newtonian mechanics is applicable. This means that in the general case

$$T = m\alpha c^2 - mc^2 = mc^2(\alpha - 1) \tag{157.3}$$

Taking into account expansion (157.1) we can write the kinetic energy for any value of u/c (of course, it is assumed that $u < c$) in the form

$$T = \frac{mu^2}{2}\left[1 + \frac{3}{4}\left(\frac{u}{c}\right)^2 + \ldots\right] \tag{157.4}$$

These expressions for T show that $T \to \infty$ for $u \to c$ because $m\alpha \to \infty$.

The quantity $m\alpha c^2 = E$ is called the *total relativistic energy* of the moving body and $mc^2 = E_0$ is referred to as the *rest energy* of the body. Equality (157.3) can be written as

$$E = E_0 + T, \tag{157.5}$$

Thus, the total relativistic energy is equal to the sum of the kinetic energy T and the rest energy E_0.

The notion of the rest energy was unknown in prerelativistic physics. In modern physics it plays a fundamental role. It follows that a body having an inertial rest mass m (coinciding with its mass for $u \ll c$) possesses a definite amount of energy proportional to m, this amount being by far not small in comparison with the kinetic energy when $u \ll c$.

Before proceeding to the discussion of the physical significance of the fundamental formula

$$E = m\alpha c^2 \tag{157.6}$$

we present an alternative derivation of the basic relation for T.

Let us assume that, as in the Newtonian mechanics, the work of a force applied to a free body is expended on the increase of its kinetic energy. Taking into account that the force satisfies the equation

$$\boldsymbol{F} = \frac{d\boldsymbol{K}}{dt} = m\frac{d}{dt}(\alpha \boldsymbol{u}) \tag{157.7}$$

where α is the factor appearing because of the relativistic change of the mass we obtain

$$dT = \boldsymbol{F}\, d\boldsymbol{r} = \frac{d\boldsymbol{K}}{dt}\, d\boldsymbol{r} = d\boldsymbol{K}\, \boldsymbol{u}$$

because $\boldsymbol{u} = \frac{d\boldsymbol{r}}{dt}$. On substituting $d\boldsymbol{K} = m\, d(\alpha \boldsymbol{u})$ in this relation we get

$$dT = m\, d(\alpha \boldsymbol{u})\, \boldsymbol{u} = m\, d\alpha\, \boldsymbol{u}\boldsymbol{u} + m\alpha \boldsymbol{u}\, d\boldsymbol{u} = mu^2\, d\alpha + m\alpha\, d\left(\frac{u^2}{2}\right) \tag{157.8}$$

According to equality (156.11), we have $d\alpha = \frac{\alpha^3}{c^2}\, d\left(\frac{u^2}{2}\right)$ whence it is seen that

$$\alpha d\left(\frac{u^2}{2}\right) = \frac{c^2}{\alpha^2}\, d\alpha = c^2\left(1 - \frac{u^2}{c^2}\right) d\alpha = (c^2 - u^2)\, d\alpha$$

The substitution of the last expression in (157.8) results in the following simple relation:

$$dT = mc^2\, d\alpha$$

The integration of this relation yields

$$T = mc^2\alpha + C$$

where C is an arbitrary constant. Now, assuming that for the state of rest we must have $T = 0$ and $\alpha = 1$ we find from the last relation the value of C: $C = -mc^2$. Hence, we have again arrived at expression (157.3).

The relation $E = m\alpha c^2$ shows that there is an invariable simple directly proportional dependence between the energy E of a body and its inertial mass $m\alpha$. The energy and the inertial mass are different quantities: the former characterizes the ability of the body to perform work and the latter is a measure of the inertia of the body. However, there is a universal relationship between these quantities. If it is known that the inertial mass gains an increment of magnitude $\Delta(m\alpha)$ then the energy increases by $c^2\Delta(m\alpha)$ and, conversely, an increase of

the energy of a physical object by ΔE is accompained by the increase of its inertial mass by $\Delta E/c^2$.

It should be noted that this principle must apply to any form of energy, for instance, to the kinetic, potential, electromagnetic forms of energy and the like. As early as 1905 A. Einstein showed by means of a simple example that an amount E of the electromagnetic radiant energy possesses an inertial mass of magnitude E/c^2. This principle is sometimes referred to as the *equivalence of mass and energy*. The equations describing the transformation of the mass and of the momentum (formulas (156.6)) can be written in an alternative form if we replace the mass m by the quantity E/c^2 equal to the former. Then the transformation formulas take the form

$$E' = \gamma (E - vK_x) \quad \text{and} \quad K'_x = \gamma \left(K_x - \frac{vE}{c^2} \right) \tag{157.9}$$

The general transformation of the energy and of the momentum can obviously be written in vector form:

$$E' = \gamma (E - \boldsymbol{v}\boldsymbol{K}) \quad \text{and} \quad \boldsymbol{K}' = \boldsymbol{K} + (\gamma - 1) \frac{\boldsymbol{K}\boldsymbol{v}}{v^2} \boldsymbol{v} - \gamma \frac{E}{c^2} \boldsymbol{v} \tag{157.10}$$

Equalities (157.10) have been obtained from (156.7) by substituting E/c^2 for $m\alpha$. These equalities show that the conservation laws for the energy and for the momentum are interrelated. If the energy E and the momentum $\boldsymbol{K}$ are constant in system A they are also constant in system B.

The rest energy E_0 (this quantity was unknown in prerelativistic physics) is a particularly important notion. It follows that a heated body possesses a mass greater than the mass of the same body when it is cooled, the mass of a spring increases when it is contracted, after a chemical reaction accompanied by energy loss the substances taking place in the reaction have a smaller total mass than before the reaction, and so on. However, in everyday practice these mass variations can never be registered because they are very small since the quantity $\Delta E/c^2$ (here ΔE is the increment of the energy) is usually negligibly small in comparison with the mass m of the bodies involved. In other words, the measurement precision is usually insufficient for the detection of such variations.

On the contrary, in nuclear physics in which are studied subnuclear processes and phenomena taking place in the collision of fast nuclear particles the corresponding mass variations turn out to be sufficiently large and therefore measurable, which makes it possible to obtain reliable estimates for the energy released or absorbed in such processes and phenomena. The variation of mass is particularly evident in the annihilation (or birth) of a pair of particles of equal masses and opposite electric charges. When two such particles (for instance, an electron and a positron) collide their mass goes into the electromagnetic radiant energy. More precisely, according to the conservation law for the energy of interacting particles, the energy of the electron-positron pair goes (after the collision) into an amount of radiant energy possessing a mass equal to the total mass of the colliding particles. Experiments carried out in atomic and nuclear physics confirm the inferences of the theory of relativity; many such experiments were designed and performed on the basis of the facts established in this theory.

Let us discuss an example of this kind. As is known, in a perfectly inelastic (plastic) impact of two bodies, in accordance with the laws of classical (prerelativistic) mechanics, the momentum is preserved while the mechanical energy is not. It turns out that in relativistic mechanics both the momentum and the mass remain invariable in such an impact although the rest mass after the im-

pact exceeds the sum of the rest masses of the colliding bodies before the impact. In this phenomenon a part of the kinetic energy (or the whole energy) goes into the rest mass. Suppose that two identical particles move with equal velocities towards each other and undergo a perfectly inelastic impact. Then a mass whose magnitude is equivalent to the kinetic energy of the particles is transformed into the rest mass of one particle appearing after the impact. That is why the resultant rest mass exceeds the sum of the rest masses of the particles which they possess before the collision.

There are a number of extremely important practical conclusions following from the equivalence of mass and energy established in the relativity theory. The factor $c^2 = 9 \times 10^{16}$ m^2/s^2 is very large and therefore it follows that ordinary substances store huge amounts of energy: calculations show that 1 mole of a substance contains an amount of energy of 9×10^{13} J. At present only a very small part of these amounts of energy contained in substances is used practically in atomic power stations which have many important advantages over the ordinary steam electric stations. In particular, it is highly important that the former consume much less fuel.

158. Momentum and Energy of System of Particles

In mechanics it is in most cases allowable to consider a given body as a system of particles which interact only when they get into direct contact or collide. In this approach the gravitational forces between the particles of the body are neglected. Of course, in the study of cosmic phenomena where the gravitational interaction plays an essential role this approach is inapplicable. However, these questions are a subject of the general relativity theory which we do not consider here; the study of these questions falls outside the frames of the special relativity theory we deal with in our course. So, in what follows we shall neglect the gravitational interaction. It should also be mentioned that in nuclear physics are studied collisions of fast charged particles which interact at a distance from one another but it turns out that in ordinary circumstances the energy of this interaction is small in comparison with the total energy $m\alpha c^2$ and it is therefore allowable to neglect the former and to regard the particles as interacting only when they get in direct contact (that is when they collide). As will be seen, in relativistic mechanics the total energy of colliding particles remains invariable (in contrast to classical mechanics where the mechanical energy decreases during a plastic impact).

Since the total energy of every particle consists of the energy localized within the particle itself (the rest energy) and of the kinetic energy, the total energy of a system of particles can be written as the sum of the energies of the constituent particles in the form

$$c^2 \sum m_i \alpha_i = c^2 m \alpha \tag{158.1}$$

where m is the mass of the system of particles as viewed from the centre-of-mass reference frame which moves with the velocity

$$\boldsymbol{u} = \frac{\sum m_i \alpha_i \boldsymbol{u}_i}{\sum m_i \alpha_i}$$

We remind the reader that $\alpha^2 \left(1 - \frac{u^2}{c^2}\right) = 1$. Therefore, the total momentum of a system of particles (of a body) can also be represented as a momentum of one

particle:

$$K = m\alpha u = \sum m_i \alpha_i u_i \tag{158.2}$$

All the equations expressing the transformation laws for energy and momentum corresponding to the passage from system A to another inertial system B are of the same form as in the case of a single particle (see formulas (156.6) and (156.7)).

Let us consider the conservation laws for a closed (isolated) system of particles. Transformation formulas (156.6) indicate that the conservation laws for the momentum of an isolated system of particles and for its energy (mass) are inseparably connected with each other.

Suppose that an isolated system of particles has a constant (time-independent) momentum relative to an inertial reference frame A. Then the same system of particles retains a constant momentum relative to another inertial reference frame B only on condition that both the momentum and the energy (mass) of the system is constant in A: $\boldsymbol{K} = \text{const}$ and $E = \text{const}$. Thus, the conservation law for the energy (mass) of an isolated system and the law of conservation of its momentum are inseparable.

In the classical Newtonian mechanics the mass is considered constant in all circumstances; that is why the importance of the law of conservation of mass of an isolated system of particles was not stipulated in prerelativistic mechanics.

To demonstrate the difference between the approaches of relativistic mechanics and of the Newtonian mechanics let us consider the following simple example. Suppose that two particles collide. Then the law of conservation of momentum implies

$$m_1\alpha_1 u_1 + m_2\alpha_2 u_2 = m_1^*\alpha_1^* u_1^* + m_2^*\alpha_2^* u_2^*$$

where the asterisk marks the quantities after the impact. This relation is valid both in classical mechanics and in relativistic mechanics. At the same time, in prerelativistic mechanics the mass of each of the particles is invariable while in relativistic mechanics only the sum of the masses is preserved:

$$m_1\alpha_1 + m_2\alpha_2 = m_1^*\alpha_1^* + m_2^*\alpha_2^*$$

In the former case each of the particles retains its mass while in the latter case only the total mass of the system of particles is invariable. From the relativistic point of view each particle taken separately is not isolated and only the whole closed system does not interact with other bodies and therefore its mass remains constant.

In other words, an isolated body (or system of particles) or a single isolated particle can only move by inertia with a constant velocity and its momentum and mass are constant. For these isolated objects the inertia law holds both in prerelativistic mechanics and in relativistic mechanics. As to a non-isolated system, in relativistic mechanics (in contrast to prerelativistic mechanics) not only the momentum but also the mass of the system may change.

The same conclusion can be drawn in the general form from the definition of the mass of a system of particles expressed by the relation

$$m\alpha = \sum m_i \alpha_i$$

where m is the rest mass of the system and m_i is the rest mass of the ith particle. If we pass to another reference system moving with a constant velocity $\boldsymbol{u}$ with respect to the former ($\boldsymbol{u}$ is the velocity of the centre of mass of the system of

particles) then we have $\boldsymbol{K}' = 0$ and $\boldsymbol{u}' = 0$ for the momentum and for the velocity in the centre-of-mass reference system. Consequently, $\alpha' = 1$ and

$$m = \sum m_i \alpha_i$$

Therefore, $m \geqslant \sum m_i$ because $\alpha_i' \geqslant 1$. Thus, the rest mass of a system of particles is greater than or equal to the sum of the rest masses of the constituent particles. The equality of these quantities takes place only when $\alpha_i' = 1$, that is when $\alpha_i' = \alpha'$ for all the particles. Since this can only be the case when $\boldsymbol{u}_i' = \boldsymbol{u}'$ for all particles we see that the rest mass of the system equals the sum of the rest masses of the particles only when all particles are at rest relative to one another.

It should be noted that a collision of two particles may result in a change of the rest masses of each of the particles or in the appearance of one or more than two particles instead of the two original particles. It can also happen that the rest masses of each of the particles remain invariable after the impact. For this case we speak in relativistic mechanics of an *elastic impact*; in all the other cases the collision phenomenon is spoken of as an *inelastic impact*. When two particles merge and form a single particle in the collision the impact is referred to as *perfectly* (*completely*) *inelastic*. However, we see that, as was mentioned at the beginning of this section, in relativistic mechanics (in contrast to the classical, prerelativistic case) the total energy is preserved for all possible kinds of impact.

159. Invariants

According to the Lorentz transformation, the coordinates and time are relative and change under the transformation from one inertial reference frame to another. However, there also exist physical quantities which do not change under such a transformation and remain invariable and which play an extremely important role in the theory; as was mentioned, these quantities are called invariants. Therefore, one of the primary tasks of the relativity theory is to study the invariants under the transformation from one reference system to another. That is why the relativity theory can also be spoken of as the theory of invariants.

The relativity theory is based on the fundamental fact that the speed of light is an invariant. Another important invariant is the *interval* (we denote it s). For an event (x, y, z, t) the square of the interval is defined as

$$s^2 = c^2t^2 - (x^2 + y^2 + z^2) = c^2t^2 - r^2 \tag{159.1}$$

The quantities y^2 and z^2 do not change when we pass from system A to system B (we mean the reference systems with which we deal throughout this chapter). Therefore, for these reference systems we have

$$c^2t^2 - x^2 = c^2t'^2 - x'^2 \tag{159.2}$$

The last relation can readily be verified using formulas (152.8) for the transformation of time and of the coordinate x:

$$c^2t'^2 - x'^2 = \gamma^2 \left[c^2 \left(t - \frac{vx}{c^2} \right)^2 - (x - vt)^2 \right] = \gamma^2 \left(1 - \frac{v^2}{c^2} \right) (c^2t^2 - x^2) = c^2t^2 - x^2$$

For a given event the interval s remains invariable in all the inertial reference systems.

It can similarly be readily checked that, according to formulas (157.9) expressing the transformation laws for the energy and for the momentum, the quantity

$$\frac{E^2}{c^2} - K^2 = \frac{E^2}{c^2} - (K_x^2 + K_y^2 + K_z^2) \tag{159.3}$$

remains invariable. Indeed, we see that (159.3) can be obtained formally from the invariance relation for expression (159.1) by substituting E/c^2 for t, K_x for x, K_y for y, and K_z for z. As we know, this very substitution brings the transformation formulas for time and spatial coordinates to the transformation formulas for energy and momentum (see Secs. 156 and 157). That is why to equality (159.2) there corresponds the relation

$$c^2 \frac{E^2}{c^4} - K_x^2 = c^2 \frac{E'^2}{c^4} - K_x'^2$$

whence it follows that the quantity

$$\frac{E^2}{c^2} - K^2 = \frac{E'^2}{c^2} - K'^2 \tag{159.4}$$

is an invariant of the transformation from one inertial frame of reference to another.

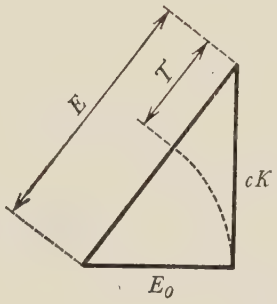

FIG. 433

From what has been established it follows that there is an important relationship between the energy E and the momentum $\boldsymbol{K}$ which we are going to derive now. Suppose that a given body is at rest in system B; then $\boldsymbol{K}' = 0$ and $E' = E_0$ where E_0 is the rest energy of the body. Therefore (159.4) implies

$$E^2 - c^2K^2 = E_0^2$$

whence

$$E^2 = c^2K^2 + E_0^2 \tag{159.5}$$

Thus, the square of the total energy of a body is equal to the square of the rest energy of the body plus the product of c by the absolute value K of the momentum (see Fig. 433). The expression $E_0 = mc^2$ of the rest energy makes it possible to rewrite equality (159.5) in the form

$$E^2 - c^2K^2 = m^2c^4 \tag{159.6}$$

whence it is seen that the rest mass m and the rest energy E_0 are invariants, which is in a complete agreement with the physical meaning of these quantities.

Equality (159.6) implies the following connection between the total energy of a body moving with a velocity v and the rest mass of the body:

$$E = m\gamma c^2 \tag{159.7}$$

Indeed, for the momentum we have the expression $K = \frac{E}{c^2} v$. On substituting it in (159.6) and taking into account that $\gamma^2 \left(1 - \frac{v^2}{c^2}\right) = 1$ we arrive at (159.7).

The kinetic energy

$$T = E - mc^2$$

is expressed in terms of the absolute value K of the momentum by means of the formula

$$T = mc^2 \left(\sqrt{1 - \frac{K^2}{m^2c^2}} - 1 \right)$$

which can readily be established using (159.6).

For $K \ll mc$ we can expand the square root in the last formula into series in powers of $\frac{K^2}{m^2c^2}$ and pass to the limit for $c \to \infty$, which results in the well-known non-relativistic equality

$$T \approx \frac{K^2}{2m} \tag{159.8}$$

160. Four-vector. Space-like, Time-like, and Light-like Intervals

It is convenient (particularly in the study of dynamical problems concerning particles moving with very large velocities) to pass in relativistic mechanics from the SI system of units to a special system in which the speed of light is taken as a basic unit: $c = 1$. The matter is that most of the formulas of the relativity theory involve c and therefore the elimination of this factor considerably simplifies the formulas. Since we put $c = 1$ it only remains to choose two more basic units. As these units we can choose units of length and of energy. For the sake of simplicity, let us choose 1 metre (1 m) and 1 joule (1 J) (of course, we can also take 1 metre and 1 newton or 1 second and 1 joule, and the like). Then time is measured in derived units; let us choose as unit time the time interval during which a light signal (propagating with speed $c = 1$) travels a distance of 1 m; it is evident that this new unit of time is equal to $1/3 \times 10^{-8}$ s.

Let us again write down the basic relations connecting the mass $m\alpha$ of a moving body, the energy E (as we know, $m\alpha = E/c^2$), and the momentum $\boldsymbol{K} = m\alpha\boldsymbol{u} = E\boldsymbol{u}/c^2$:

$$E^2 - c^2K^2 = m^2c^4, \quad T = E - mc^2 \quad \text{and} \quad \boldsymbol{F} = m\frac{d(\alpha\boldsymbol{u})}{dt}$$

Putting $c = 1$ in these relations we obtain the corresponding formulas in the system with basic units 1 m, 1 J, and $c = 1$.

Marking temporarily the values of the corresponding quantities in this system of units by an asterisk we compile the following table giving the relationship between the values of the quantities in the SI system and in the units of the new system (for which $c = 1$):

System of units \ Dimension	m		J			m^{-1}	$J \cdot m^{-1}$	Dimensionless quantities		
SI	$\boldsymbol{r}$	ct	mc^2	$c\boldsymbol{K}$	E	$\boldsymbol{w}c^{-1}$	F	$\boldsymbol{u}c^{-1}$	α	γ
m, J, $c=1$	$\boldsymbol{r}$	t^*	m^*	$\boldsymbol{K}^*$	E	$\boldsymbol{w}^*$	F	$\boldsymbol{u}^*$	α	γ

The symbol w denotes the acceleration $\frac{du}{dt}$; the notation of all the other quantities in the table is already familiar to the reader. From the table it is seen that the energy E, the length, the force, and the dimensionless quantities α and γ do not change their values. The rest mass and the momentum are measured in the new system in the units of energy while the velocity is dimensionless (its magnitude in the new system equals the ratio of the absolute value of the velocity to the speed of light).

For a given relation involving quantities measured in the new system of units it is quite easy to pass to the corresponding quantities measured in the SI system: to this end we must simply replace the quantities with the asterisk by the corresponding quantities measured in the SI system which are placed in the table above the quantities marked by the asterisk. As to the transition from the SI system to the new system of units, this is performed quite elementarily: it simply suffices to put $c = 1$. In what follows we shall omit the asterisks for the quantities in the new system of units and retain the previous symbols bearing in mind that all the quantities are meant to be measured in the new system with the basic units 1 m, 1 J, and $c = 1$.

It should be noted that, as in the SI system, for the momentum we have $\boldsymbol{K} = m\alpha\boldsymbol{u}$ but at the same time in the new system $\boldsymbol{K} = E\boldsymbol{u}$ and $E = m\alpha$ (we see that in the last two cases the relations in the new system differ from those in the SI system in the factor c^2).

As an example, let us write down the Lorentz transformation and the energy-momentum transformation in the new system of units:

$$\left.\begin{aligned} t' &= \gamma(t - vx) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ E' &= \gamma(E - vK_x) \\ K'_x &= \gamma(K_x - vE) \\ K'_y &= K_y \\ K'_z &= K_z \end{aligned}\right\} \tag{160.1}$$

where $\gamma^2(1 - v^2) = 1$.

In the new system of units the invariants expressed by (159.1) and (159.4) go into the following expressions:

$$s^2 = t^2 - r^2 \quad \text{and} \quad m^2 = E^2 - K^2 \tag{160.2}$$

We see that for the new system of units the transformation formula for the coordinate x (for the component K_x of the momentum) is quite analogous to the transformation formula for the time t (for the energy E) and that these formulas go one into another under the change $t' \to x'$, $t \to x$, and $x \to t$ ($E' \to K'_x$, $E \to K_x$, and $K_x \to E$); of course, as before, we assume that the positive x-axis and the velocity v are parallel and go in one direction.

Every event is characterized by four numbers: the time t and the spatial coordinates x, y, and z (this simply means that the event takes place at a point with the radius vector $\boldsymbol{r}$ (x, y, z) at instant t). An ordered collection of four numbers (coordinates) can be regarded as a point belonging to a four-dimensional space of all such ordered 4-tuples. We therefore interpret every event as a point in

four-dimensional space-time. When we pass from one inertial reference frame to another the coordinates of an event change according to the Lorentz transformation formulas which express a mapping of space-time onto itself.

As before, we consider a simplified model of two inertial reference systems A and B whose x- and x'-axes go along the velocity v of the relative motion of the systems; as we know, in this case the coordinates x and x' and the quantities t and t' are in relations described by the formulas established in the previous sections. The coordinate x' depends on x and t in the same way as the coordinate t' depends on t and x (this dependence is similar to the one which takes place when Cartesian coordinates in a plane are rotated through an angle). The coordinates along the axes normal to v do not change.

Let us introduce the radius vector R (in four-dimensional space-time) whose components are t, x, y, and z (or, in the contracted notation, t and r). We write the vector R as $R\ (t,\ x,\ y,\ z)$ or $R\ (t,\ r)$. The vector R is called a *four-vector* (*4-vector*). When Cartesian coordinates in the ordinary three-dimensional (geometrical) space are rotated the length of the radius vector r, that is the square root of the scalar product rr (of the scalar square of the vector r) remains invariable. More generally, under the rotation of Cartesian coordinates the scalar product of any two vectors remains invariable. As was shown in Sec. 158, under the Lorentz transformation of space-time (which is a four-dimensional space) the magnitude of the interval does not change. The square of the interval can be written in the form

$$s^2 = t^2 - r^2 = t^2 - rr = t^2 - (x^2 + y^2 + z^2) \tag{160.3}$$

The interval s between two events $R\ (0,\ 0)$ and $R\ (t,\ r)$ (the former occurs at the origin at time $t = 0$ and the latter at the point r at instant t) retains the same value under the transformation from t and r to t' and r'. The square of the interval can be (formally) interpreted as the scalar product in space-time of the vector R by itself; hence, we simply put $s^2 = RR = t^2 - rr$. In the ordinary three-dimensional space we have

$$rr = x^2 + y^2 + z^2$$

The scalar product of r by itself is simply equal to the sum of the squares of the components of the vector r in the Cartesian coordinate system. The square of the interval is also an algebraic sum involving the squares of the components of the vector R but the signs in front of t^2 and in front of x^2, y^2, and z^2 are different. This fact reflects a significant physical distinction between time and spatial coordinates and shows that space-time differs from an ordinary four-dimensional Eucledian space considered in mathematics.

The interval $s = \sqrt{RR}$ between $R\ (0,\ 0)$ and $R\ (t,\ r)$ is a (positive) real number when $t^2 > r^2$ and is a pure imaginary number when $t^2 < r^2$.

Let us denote by s_0 the modulus of the interval s when s is an imaginary number; then

$$s = is_0 \quad \text{and} \quad s_0^2 = t^2 - r^2$$

When the interval s between two events $R\ (0,\ 0)$ (in what follows we often use the simplified notation $R\ (0,\ 0) = 0$) and $R\ (t,\ r)$ is a real number these events can be in a causal relationship with each other, that is one of them can affect the other. Indeed, a body (or a signal) can pass from point 0 (here we mean the point $r = 0$) to the point r (or from the point r to the point 0) moving with a constant velocity of magnitude

$$u = r/t < 1$$

along the radius vector $\boldsymbol{r}$. That is why events occurring at these points can affect each other. In this case (for a real s) we can always choose a reference frame B moving with velocity $\boldsymbol{u}$ ($|\boldsymbol{u}| = u$) parallel to the radius vector $\boldsymbol{r}$ such that the origin of B coincides with the point $\boldsymbol{r} = 0$ for $t = 0$ in the case $t > 0$ and with the point $\boldsymbol{r}$ in the case $t < 0$. In this frame both events 0 and R occur in one place at different instants $t = 0$ and $t \neq 0$. For $t > 0$ the directions of the velocity $\boldsymbol{u}$ and of the vector $\boldsymbol{r}$ coincide and for $t < 0$ the directions of $\boldsymbol{u}$ and $\boldsymbol{r}$ are opposite. In the former case the origin of the reference system B passes during time t from $\boldsymbol{r} = 0$ to $\boldsymbol{r}$, which means that both events 0 and $\boldsymbol{R}$ occur at one place in system B characterized by the radius vector $\boldsymbol{r}' = 0$. In the latter case the origin of B goes from the point $\boldsymbol{r}$ to the point $r = 0$ during time t and coincides with the origin of system A at instant $t = 0$. The time interval t' (in system B) separating these two events can be found from the constancy condition for the interval which is written as

$$t'^2 - r'^2 = t^2 - r^2$$

Since $r' = 0$ we obtain

$$t' = \sqrt{t^2 - r^2} = t\sqrt{1 - u^2} = \frac{t}{\alpha}$$

because $r = ut$.

An interval whose value is a real number is said to be *time-like*; this term is accounted for by the fact that for such an interval between two events it is always possible to choose a reference frame in which the interval between the two given events reduces to a pure time interval.

When the interval between two events $\boldsymbol{R} = 0$ (that is $\boldsymbol{R} = \boldsymbol{R}(0, 0)$) and $\boldsymbol{R}(t, \boldsymbol{r})$ has an imaginary value, that is when $t^2 < r^2$ then there can be no causal relationship between these events because none of the bodies (and none of the signals) can have a velocity of magnitude $u = r/t > 1$. In other words, the event $\boldsymbol{R} = 0$ cannot affect the event $\boldsymbol{R}(t, \boldsymbol{r})$ and vice versa. In this case it is always possible to choose a reference frame in which both events occur simultaneously at different places. This possibility is a consequence of the constancy of the interval whose square of modulus is

$$s_0^2 = r^2 - t^2 = r'^2 - t'^2 > 0$$

This relation shows that there always exists a reference system B for which $t' = 0$, which exactly means that in system B both events occur simultaneously at a distance of

$$r' = \sqrt{r^2 - t^2}$$

from each other. Indeed, let us take a reference system B moving along $\boldsymbol{r}$ with a velocity v. Then the time in system B is expressed as $t' = \gamma(t - rv)$. If we choose the velocity v of system B so that

$$v = \frac{t}{r} < 1$$

(it should be taken into account that in the system of units we are using here the velocity v is dimensionless; in the SI system we would have $v = c(ct/r)$ and $r^2 > c^2 t^2$) then $t' = 0$. In other words, in this reference system B both events occur simultaneously. The distance between the events in this system is

$$r' = \sqrt{r^2 - t^2} = \sqrt{r^2 - r^2 v^2} = r\sqrt{1 - v^2} = \frac{r}{\gamma}$$

(it should be stressed again that $v < 1$). An interval between two events whose value is an imaginary number is called a *space-like interval*; this term is accounted for by the fact that in this case there is always a reference frame in which these events occur simultaneously at different places.

A zero interval $s = 0$ between two events $R = 0$ and $R = R(t, r)$ (in this case $\sqrt{RR} = 0$; for such an interval we have $t^2 = r^2$) is referred to as a *light-like interval*. This term is accounted for by the fact that any two events with a zero interval between them can be connected by a light signal. The points in space-time representing all the events with zero (light-like) interval (relative to the event $R = 0$) lie in a surface specified by the equation

$$t^2 - r^2 = t^2 - x^2 - y^2 - z^2 = 0 \quad (160.4)$$

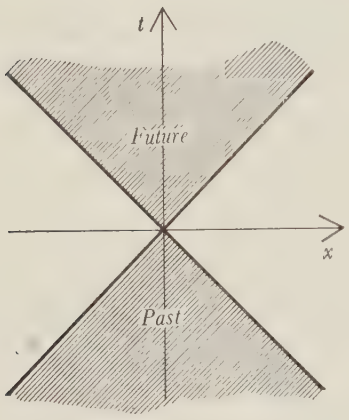

FIG. 434

This is a conical surface which divides space-time into the regions of events with space-like and time-like intervals. The vertex of this cone in space-time lies at the origin $R = 0$ and its axis coincides with the t-axis. This conical surface is spoken of as the *light-cone*.

In Fig. 434 is shown the section of the light cone by the coordinate plane (x, t) (whose equations are $y = 0$, $x = 0$). A section of the same shape is formed in the intersection of the light cone with any other plane passing through the t-axis. The events occurring at instants $t < 0$ (the points corresponding to them lie within the lower shaded part of the light cone in Fig. 434; this part is called the *past light cone*) may, in principle, affect the event at the point $R = 0$, that is there may exist a causal relationship between these events. This part of the cone is the *absolute past* (relative to the event $R = 0$). The event at the point $R = 0$ may affect the events lying within the upper shaded part of the light cone corresponding to $t > 0$. This part is the *future light cone* and the events within it are the *absolute future* for the event $R = 0$. Finally, the events whose representing points are outside the light cone lie in the region of space-like intervals and can neither be affected by what occurs at $R = 0$ nor have an effect on the event $R = 0$. They are *absolutely separated* from $R = 0$.

Up till now we considered an interval between the event $R = 0$ and an arbitrary event $R = R(t, r)$. However, in just the same sense we can speak of the interval between any two events R_1 and R_2. To this end we put, by definition, that the interval s between R_1 and R_2 or, more precisely, its square is given by the relation $s^2 = (R_2 - R_1)^2$:

$$s^2 = (R_2 - R_1)^2 = (t_2 - t_1)^2 - (r_2 - r_1, r_2 - r_1) = \\ = (t_2 - t_1)^2 - [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]$$

It can easily be shown with the aid of simple calculations that

$$s^2 = (R_2 - R_1)^2 = (R_2' - R_1')^2 \quad (160.5)$$

This means that s^2 is an invariant of the Lorentz transformation and so is the interval s itself. The interval between $R = 0$ and $R(t, r)$ is a special case of

the general interval defined by (160.5) (in this special case $R_1 = 0$ and $R_2 = R(t, r)$). Similarly, the square of the infinitesimal interval ds between two events R and $R + dR$ is equal to

$$ds^2 = dR\,dR = dt^2 - dr\,dr = dt^2 - dr^2$$

Using the general definition of an interval we can consider the light cone with vertex at an arbitrary point R_1 lying in space-time. The surface of the cone divides all the events in the world (relative to the event R_1) into the region of the past, the region of the future and the region of events absolutely separated from R_1.

161. Relativistic Mechanics

We call R a 4-vector (four-vector) in order to distinguish between the vector R in space-time and the 3-vector (the three-dimensional vetor) r lying in the ordinary geometrical space. Generally, any physical quantity completely specified by three numbers which change under a rotation of Cartesian coordinates in the same way as the components of the vector r is called, by definition, a vector quantity in a three-dimensional space or, simply, a three-dimensional vector (3-vector). Similarly, by a 4-vector in space-time is meant any ordered 4-tuple of numbers representing some physical quantities which change according to the Lorentz formulas under the transformation from one inertial reference frame to another. The scalar product of two 4-vectors $R_1(t_1, r_1)$ and $R_2(t_2, r_2)$ is defined as

$$R_1R_2 = t_1t_2 - r_1r_2 \tag{161.1}$$

The scalar product is an invarient of the Lorentz transformation, which can easily be checked directly by means of calculations according to formulas (160.1).

The vector R is the *position vector* of a point in space-time relative to an inertial reference system (for definiteness, relative to system A).

Let us proceed to the definition of the *velocity 4-vector*. When a particle moves in space with a velocity $u = \frac{dr}{dt}$ the 4-vector R varies as function of time. It is evident that during time dt it gains an increment dR (dt, dr) which is a 4-vector, the length of this vector by which is meant the corresponding interval ds is invariable under the Lorentz transformation. However, the ratio of dR to dt, that is the quantity $\left(1, \frac{dr}{dt}\right)$ is not a 4-vector (see the general definition above) because the numbers 1, u_x, u_y, and u_z do not change like the numbers t, x, y, and z under the transformation from system A to system B.

That is why in relativity theory instead of the increment of time dt is taken the increment of another quantity called *proper time* and denoted τ. By definition, the differential $d\tau$ is defined by means of the equality

$$d\tau^2 = dt^2 - dr^2 \tag{161.2}$$

The quantity $d\tau^2$ is nothing but the squared interval between the events (t, r) and $(t + dt, r + dr)$. Below we present an interpretation of the quantity $d\tau$.

Let us consider an inertial reference system moving with a constant velocity $u = \frac{dr}{dt}$ relative to system A. Suppose that there is a clock in this system which is at rest relative to the system and moves together with it and that this clock reads time $d\tau$ while the corresponding time interval in system A is equal to dt. Then the position radius vector r' in the moving system is constant and $dr' = 0$.

Let us write down the invariance condition for the elementary interval in these two reference systems:

$$d\tau^2 - dr'^2 = dt^2 - dr^2$$

Now, putting $dr' = 0$ we obtain

$$d\tau = dt \sqrt{1 - \left(\frac{dr}{dt}\right)^2} = dt \sqrt{1 - u^2}$$

For the sake of brevity, we can rewrite the last relation in the form

$$\alpha \, d\tau = dt \tag{161.3}$$

where, as before, $\alpha = (1 - u^2)^{-1/2}$. It is obvious that, like the interval ds corresponding to dR the quantity $d\tau$ is an invariant of the Lorentz transformation.

The increment of the proper time τ is the time interval measured by a clock at rest relative to a reference system moving together with the body with a constant instantaneous velocity $\boldsymbol{u}$. When the velocity $\boldsymbol{u}$ varies the relationship between $d\tau$ and dt changes since the quantity α changes. It follows that the proper time should be defined in the general case by means of the formula

$$\tau_2 - \tau_1 = \int_{t_1}^{t_2} d\tau = \int_{t_1}^{t_2} \frac{dt}{\alpha} = \int_{t_1}^{t_2} \sqrt{1 - u^2}\, dt \tag{161.4}$$

It is obvious that the proper time $\tau_2 - \tau_1$ is always less than $t_2 - t_1$, that is the time interval measured in the moving frame of reference is longer than the time interval in reference frame A (the time dilation).

We can now define the velocity four-vector in space-time by putting

$$\boldsymbol{U} = \frac{d\boldsymbol{R}}{d\tau} \tag{161.5}$$

The components of the 4-vector $\boldsymbol{U}$ change according to the Lorentz transformation formulas when we pass from one inertial reference system to another, which can readily be verified. To find the expressions of the components of the 4-vector $\boldsymbol{U}$ we use the relation

$$\boldsymbol{U} = \frac{d\boldsymbol{R}\,(dt,\ d\boldsymbol{r})}{d\tau} = \boldsymbol{U}\left(\frac{dt}{d\tau},\ \frac{d\boldsymbol{r}}{d\tau}\right) = \boldsymbol{U}\,(\alpha,\ \alpha\boldsymbol{u}) \tag{161.6}$$

which is valid because $\frac{dt}{d\tau} = \alpha$ and $\frac{d\boldsymbol{r}}{d\tau} = \frac{d\boldsymbol{r}}{dt}\frac{dt}{d\tau} = \alpha\frac{d\boldsymbol{r}}{dt} = \alpha\boldsymbol{u}$. Consequently, the velocity 4-vector of a particle (of a body) in space time $(t,\ x,\ y,\ z)$ has the first (time) coordinate equal to α and its other three components form the (spatial) vector $\alpha\boldsymbol{u}$ where $\boldsymbol{u}$ is the ordinary 3-vector representing the velocity. It should be noted that the length of the velocity 4-vector $\boldsymbol{U}$ (understood as the square root of the scalar product of $\boldsymbol{U}$ by itself) is equal to unity. Indeed,

$$U^2 = \boldsymbol{U}\boldsymbol{U} = \alpha^2 - \alpha^2 u^2 = \alpha^2\,(1 - u^2) = 1 \tag{161.7}$$

Equality (161.7) is quite evident because $d\tau$ is equal to the interval ds corresponding to $d\boldsymbol{R}$.

It is also possible to introduce the *acceleration 4-vector* (although it is used very rarely) $\boldsymbol{W}$:

$$\boldsymbol{W} = \frac{d\boldsymbol{U}}{d\tau} = \boldsymbol{W}\left(\frac{d\alpha}{d\tau},\ \frac{d}{d\tau}(\alpha\boldsymbol{u})\right) = \boldsymbol{W}\left(\alpha\frac{d\alpha}{dt},\ \alpha\frac{d\,(\alpha\boldsymbol{u})}{dt}\right) \tag{161.8}$$

In the theory of relativity is more frequently used the *momentum 4-vector* which is naturally defined as the product of the mass by the velocity 4-vector:

$$\boldsymbol{P} = m\boldsymbol{U} = \boldsymbol{P}\,(m\alpha,\ m\alpha\boldsymbol{u}) = \boldsymbol{P}\,(E,\ \boldsymbol{K}) \tag{161.9}$$

The first (time) component of the momentum 4-vector $\boldsymbol{P}$ in space-time is equal to the energy and its other three spatial components form the ordinary momentum (3-vector) $\boldsymbol{K}$. That is why the vector $\boldsymbol{P}$ is referred to as the *energy-momentum vector* in space-time. The square of the length (the length is understood here as in the case of the vector U; see (161.7)) is obviously equal to

$$P^2 = \boldsymbol{P}\boldsymbol{P} = E^2 - K^2$$

It is convenient to rewrite this equality in the form

$$\boldsymbol{P}\boldsymbol{P} = E^2 - K^2 = m^2 \tag{161.10}$$

because for a reference frame relative to which the body in question is at rest we have $\boldsymbol{K}' = 0$ and the energy E is equal to the rest mass m. Formula (161.10) can also be obtained by using the relation $\boldsymbol{U}\boldsymbol{U} = 1$:

$$\boldsymbol{P}\boldsymbol{P} = m^2\boldsymbol{U}\boldsymbol{U} = m^2$$

From the universal relationship between the energy, the mass, and the momentum expressed by formula (161.10) follows the relationship between the energy and the work of a force. On differentiating (161.10) with respect to time t (the same result can readily be obtained by differentiating with respect to τ) we obtain

$$E\frac{dE}{dt} = \boldsymbol{K}\frac{d\boldsymbol{K}}{dt}$$

that is

$$E\frac{dE}{dt} = \boldsymbol{K}\boldsymbol{F}$$

since $\frac{d\boldsymbol{K}}{dt} = \boldsymbol{F}$.

Now, taking into account that $E = m\alpha$ and $\boldsymbol{K} = m\alpha\boldsymbol{u}$, we arrive at the familiar equality

$$\frac{dE}{dt} = \boldsymbol{F}\boldsymbol{u} \tag{161.11}$$

In the SI system of units equality (161.11) retains the same form for on replacing t by ct and $\boldsymbol{u}$ by $\boldsymbol{u}/c$ and cancelling by c we obtain the same relation because E and $\boldsymbol{F}$ do not change. Equality (161.11) can also be written as

$$dE = \boldsymbol{F}\,d\boldsymbol{r}$$

that is, as in the Newtonian mechanics, the increment of the (total relativistic) energy is equal to the work of the external force acting on the particle (on the body). We used the last equality in order to determine the value of the relativistic energy of a moving body; to this end we substituted

$$\boldsymbol{F} = \frac{d\boldsymbol{K}}{dt} = m\frac{d}{dt}(\alpha\boldsymbol{u})$$

into the equality.

Now let us form the 4-vector

$$\boldsymbol{f} = \frac{d\boldsymbol{P}}{d\tau} \tag{161.12}$$

which can naturally be interpreted as the *force four-vector* in space-time. To determine the values of the components of the 4-vector $\boldsymbol{f}$ we use the relation

$$\boldsymbol{f}=f\left(\frac{dE}{d\tau}, \frac{d\boldsymbol{K}}{d\tau}\right)=f\left(\alpha\frac{dE}{dt}, \alpha\frac{d\boldsymbol{K}}{dt}\right)=f\left(\alpha\frac{dE}{dt}, \alpha\boldsymbol{F}\right)$$

We see that the first (time) component of the 4-vector $\boldsymbol{f}$ in space time (t, x, y, z) is the derivative of the energy with respect to time (the power) times α and that the other three (spatial) components are equal to the components of the ordinary 3-vector of force multiplied by α. We can also write the relation

$$\boldsymbol{f}=m\frac{d\boldsymbol{U}}{d\tau}=m\boldsymbol{W} \tag{161.13}$$

whose form is analogous to the form of Newton's second law stating that the force equals the mass times the acceleration.

It should be noted that the vectors $\boldsymbol{P}$ and $\boldsymbol{f}$ are orthogonal in space-time (in the sense that their scalar product is equal to zero). Indeed, by (160.10), we have

$$\boldsymbol{P}\boldsymbol{P}=m^2$$

whence

$$\boldsymbol{P}\frac{d\boldsymbol{P}}{d\tau}=\boldsymbol{P}\boldsymbol{f}=0$$

From the last relation we can directly derive equality (161.11).

The general conservation law for an isolated svstem of particles (for an isolated body) can now be written in the form

$$\sum \boldsymbol{P}_i=\text{const} \tag{161.14}$$

This general relation includes the law of conservation of energy

$$\sum E_i=\text{const}$$

and the conservation law for the momentum

$$\sum \boldsymbol{K}_i=\text{const}$$

as special cases.

When operating on 4-vectors one should take into account that the scalar product of two 4-vectors is an invariant of the Lorentz transformation. For instance, such are the expressions

$$\boldsymbol{P}\boldsymbol{R}=Et-\boldsymbol{K}\boldsymbol{r}, \quad \boldsymbol{f}\boldsymbol{R}=\alpha\frac{dE}{dt}t-\alpha\boldsymbol{F}\boldsymbol{r}$$

and the like.

To sum up what has been said, let us write down the basic quantities involved in relativistic kinematics and dynamics of a particle described in the four-dimensional space (space-time) with coordinates t, x, y, z:

$$\left.\begin{aligned}
&\text{the position 4-vector } \boldsymbol{R}\\
&\text{the velocity 4-vector } \boldsymbol{U}=\frac{d\boldsymbol{R}}{d\tau}\\
&\text{the acceleration 4-vector } \boldsymbol{W}=\frac{d\boldsymbol{U}}{d\tau}\\
&\text{the momentum 4-vector } \boldsymbol{P}=m\boldsymbol{U}\\
&\text{the force 4-vector } \boldsymbol{f}=\frac{d\boldsymbol{P}}{d\tau}=m\frac{d\boldsymbol{U}}{d\tau}=m\boldsymbol{W}
\end{aligned}\right\} \tag{161.15}$$

All these quantities are obviously defined in a complete formal agreement with the Newtonian mechanics; the corresponding relations of the Newtonian mechanics can be obtained from formulas (161.15) by replacing the 4-vectors by the corresponding 3-vectors, substituting t for the proper time τ and retaining the mass m.

In connection with what we have studied it is expedient to note that the components of the 4-vectors involved have been written in the special system of units with the basic units 1 m, 1 J, and $c = 1$; in the SI system these vectors have a slightly different form. For the comparison we give the following table:

System of units \ Vector	R	U	W	P	f
SI	$ct,\ \boldsymbol{r}$	$\alpha c,\ \alpha \boldsymbol{u}$	$\alpha c \frac{d\alpha}{dt},\ \alpha \frac{d(\alpha \boldsymbol{u})}{dt}$	$c^{-1},\ E,\ \boldsymbol{K}$	$c^{-1}\ \alpha \frac{dE}{dt},\ \alpha \boldsymbol{F}$
m, J, $c=1$	$t,\ \boldsymbol{r}$	$\alpha,\ \alpha \boldsymbol{u}$	$\alpha \frac{d\alpha}{dt},\ \alpha \frac{d(\alpha \boldsymbol{u})}{dt}$	$E,\ \boldsymbol{K}$	$\alpha \frac{dE}{dt},\ \alpha \boldsymbol{F}$

We see that in the passage from one of these systems of units to another only the first (time) components of the 4-vectors change while the other three (spatial) components remain the same; however, when using this table one should take into account that, with the exclusion of R and f, the dimensions of the components of the 4-vectors are different in these systems of units.

As was mentioned, the basic notions used in relativistic mechanics and in the Newtonian mechanics have only a formal analogy. Only in the case $u \ll 1$ we can pass from relativistic mechanics to prerelativistic (Newtonian) mechanics by passing to the limit for $c \to \infty$. Hence, the classical mechanics is an approximation to the more precise relativistic mechanics, the latter being based on the postulate of the independence of the speed of light of the relative motion of the light source and the light detector and on the postulate of relativity (the equivalence of all the inertial reference systems). As was shown, these basic assumptions lead to the unified notion of space-time, to the mass-variability, to the dependence of the length and time on the choice of the inertial reference frame and to the inseparable relationship between the energy and the mass. Because of the small values of the velocity in the everyday phenomena and experiments in comparison with the speed of light all these fundamental inferences of the relativity theory were unknown and could not be detected until human experience encountered processes in which velocities close to the speed of light were attained. That is why in the limiting case $c \to \infty$ leading to the Newtonian mechanics it is legitimate to deal with invariable masses, lengths, and time and to consider the energy as being in no connection with the mass of the body. In the limiting case $c \to \infty$ the Lorentz transformation goes into the Galilean transformation, and space and time become independent, mass length and time are invariable in all inertial reference systems and the energy is not connected with the mass of the body.

162. Theory of Elastic Impact of Two Particles

As was mentioned (see Sec. 158), in relativistic mechanics by an elastic impact of two particles is meant a collision of the particles after which (or, more precisely, after a very short period of their interaction following the moment the particles get in contact) the total rest mass of the system of the particles remains invariable. All the other cases are inelastic collisions in which the rest energy of the particles may change and even new particles may appear after the impact (decay of particles). The latter class of collisions also includes the classical case of a perfectly plastic impact in which the rest mass of the newly formed particle exceeds the sum of the rest masses of the colliding particles.

To investigate the phenomenon of an elastic impact (and also an inelastic impact) we can choose either a laboratory reference frame or a centre-of-mass reference frame relative to which the motion of the particles is considered. Since in the latter system the total momentum is equal to zero it is preferable first to consider the motion of the particles relative to the centre-of-mass reference frame and then pass to the laboratory frame. To give a complete description of the phenomenon in question we begin with studying an elastic impact in the laboratory reference frame, after which we solve the same problem in the centre-of-mass reference frame and then again pass from the latter frame to the former.

Let the first particle (particle *1*) of mass m_1 move and collide with another particle (particle *2*) of mass m_2 which is at rest. Such a situation corresponds to the conditions of experiments in which a flow of accelerated charged particles is incident on target particles (atomic nuclei at rest).

Let before the collision the momentum of particle *1* be equal to K and its total energy be equal to E_{1n}; the initial energy E_{1n} and the momentum K of this particle satisfy the relation

$$E_{1n}^2 = K^2 + m_1^2 \tag{162.1}$$

The total energy of the other particle before the impact is equal to m_2.

Let the momenta and the energies of particles *1* and *2* after the impact be K_1, E_1 and K_2, E_2, respectively. For this case the conservation laws imply

$$K = K_1 + K_2 \tag{162.2}$$

and

$$E = E_{1n} + m_2 = E_1 + E_2 \tag{162.3}$$

Let us suppose that after the impact the velocity of particle *2* is directed at an angle of ϑ to the velocity which particle *1* had before the collision (that is the vectors K_2 and K form an angle of magnitude ϑ as shown in Fig. 435). Then, as it follows from conservation law (162.2), see Fig. 435, we have

$$K_1^2 = K^2 + K_2^2 - 2KK_2 \cos \vartheta \tag{162.4}$$

We now express K_1^2 in terms of the known quantities E, m_2 and K and substitute this expression in (162.4). For the square of the momentum of particle *1* we have

$$K_1^2 = E_1^2 - m_1^2 \tag{162.5}$$

Taking into account equalities (162.1) and (162.3) we obtain

$$K_1^2 = E_1^2 + K^2 - (E - m_2)^2 \tag{162.6}$$

Using the relation

$$E_2^2 = K^2 + m_2^2$$

we bring conservation law (162.3) to the form

$$E_1 = E - \sqrt{K^2 + m_2^2} \tag{162.7}$$

On substituting this expression in (162.6) and then in (162.4) we arrive at the equality

$$\left(E - \sqrt{K_2^2 + m_2^2}\right)^2 = (E - m_2)^2 + K_2^2 - 2KK_2 \cos \vartheta$$

Finally, on rationalizing the last equality we obtain the equation

$$E^2 K_2^2 = K^2 K_2^2 \cos^2 \vartheta + 2Em_2 KK_2 \cos \vartheta \tag{162.8}$$

connecting the unknown K_2 with the known quantities K, E, and m_2 for any value of the angle ϑ. This equation makes it possible to find all possible values of the momentum of particle *2* after the impact.

Let us introduce the notation

$$K_2 \cos \vartheta = x \quad \text{and} \quad K_2 \sin \vartheta = y$$

and substitute these expressions in (162.8). This results in the equation

$$E^2 (x^2 + y^2) = K^2 x^2 + 2EKm_2 x \tag{162.9}$$

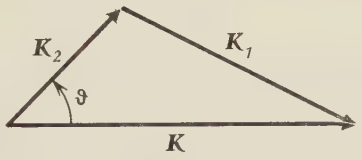

FIG. 435

describing a curve traced by the tip of the vector $\boldsymbol{K}$ as it passes through all its possible positions. Equation (162.9) describes an ellipse passing through the origin $x = y = 0$ from which the vector $\boldsymbol{K}$ is issued. The semi-axis of this ellipse along the x-axis is

$$a = \frac{m_2 E}{E^2 - K^2} K \tag{162.10}$$

and the semi-axis along the y-axis is

$$b = \frac{m_2}{\sqrt{E^2 - K^2}} K \tag{162.11}$$

It should be noted that $b < a$ because

$$b = a \sqrt{1 - \frac{K^2}{E^2}} \tag{162.12}$$

Hence, a is the major semi-axis of the ellipse and b is the minor one.

On carrying out simple transformations of the expression for a we find that $2a < K$ when $m_1 > m_2$. Similarly, for $m_1 < m_2$ we must have $2a > K$ and for $m_1 = m_2$ we of course have $2a = K$. Indeed, taking into account (162.1) and (162.3) we find

$$E^2 - K^2 = 2E_{1n} m_2 + m_1^2 + m_2^2 = 2Em_2 + m_1^2 - m_2^2 \tag{162.13}$$

Finally, the substitution of this expression in (162.10) results in

$$a = \frac{K}{2 + \frac{m_1^2 - m_2^2}{Em_2}} \tag{162.14}$$

whence follows what has been stated.

Thus, for $m_1 > m_2$ the ellipse described by the end of the vector $\boldsymbol{K}_2$ is of the shape shown in Fig. 436. In this case after the impact both particles always move forwards (relative to the direction of the vector $\boldsymbol{K}$). The angle ϑ can assume any

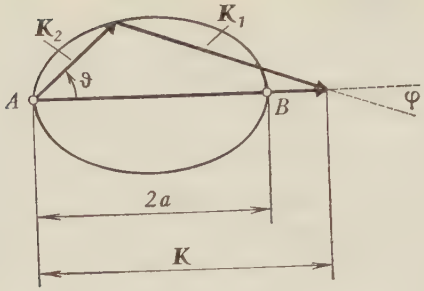

FIG. 436

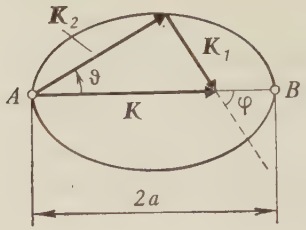

FIG. 437

value from 0 to $\pi/2$ and the angle φ is less than a definite maximum value φ_{max} which is less than $\pi/2$. To a given admissible value of φ there correspond, in the general case, two values of the angle ϑ. The point B corresponds to the case of a central impact after which both particles move in the direction of the vector K. The point A corresponds to a "miss", that is in this case particle *1* goes past particle *2* without getting into contact with it, that is without interacting with it; this simply means that $K_1 = K$ for the point A.

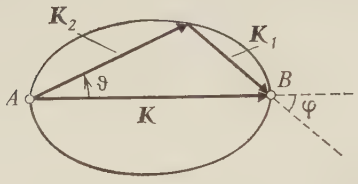

FIG. 438

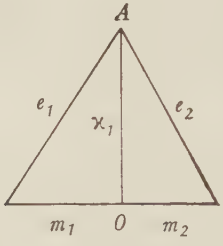

FIG. 439

In the case $m_1 < m_2$ the possible distributions of the momenta after the impact are shown in Fig. 437 which in some way differs from Fig. 436. In this case the angle of incidence φ of particle *1* may vary from 0 to π and it is possible that after the impact particle *1* moves backwards (relative to the direction of the vector K), that is it can be reflected after the interaction with particle *2*. To the point B in Fig. 437 there corresponds the case of a central impact after which particle *1* is repelled backwards while particle *2* proceeds to move in the direction of K and $K_2 > K$. The point A again corresponds to a "miss", that is in this case there is no interaction between the particles. From Fig. 437 it is seen that to every admissible value of the angle ϑ there corresponds a single value of the angle φ.

For the case $m_1 = m_2$ all possible types of a collision are represented in Fig. 438. In this case φ varies from 0 to $\pi/2$. In the case of a central impact (the point B in the figure) particle *1* stops moving while particle *2* starts moving after the impact with a velocity equal to the velocity of particle *1* which it had before the collision, that is $K_2 = K$ and $\varphi = 0$.

We now investigate the phenomenon of an elastic collision of two particles in the centre-of-mass reference frame. In this frame the total momentum of the system of the two particles is equal to zero and therefore before the impact the particles move towards each other and after the impact they go apart with momenta of the same magnitude and opposite directions. The velocity v of the centre of mass of a system of particles is expressed by the well-known equality which takes the form

$$v = \frac{\sum K_i}{\sum E_i}$$

in the system of units we are using here. In the case under consideration the velocity of the centre-of-mass reference frame relative to the laboratory reference frame is

$$v = \frac{K}{E} \tag{162.15}$$

In the centre-of-mass reference frame the conservation laws for the momentum and for the energy have the form

$$\varkappa_1 + \varkappa_2 = 0, \quad e_1 + e_2 = E' \tag{162.16}$$

before the impact and the form

$$\varkappa_1^* + \varkappa_2^* = 0, \quad e_1^* + e_2^* = E$$

after the impact where the letters $\varkappa$ and e with the subscripts 1 and 2 denote the momenta and the energies of the corresponding particles, the asterisk marking the quantities after the impact.

From the conservation laws and from the constancy of the rest masses it follows that

$$\varkappa_1 = \varkappa_2 = \varkappa_1^* = \varkappa_2^* = \varkappa \tag{162.17}$$

Indeed, let us consider Fig. 439 which represents geometrically the law $e^2 = \varkappa^2 + m^2$ for each particle before the impact together with the condition $\varkappa_1 = \varkappa_2$. Since $e_1^* + e_2^* = e_1 + e_2$ the distance OA must be equal to $\varkappa_1^* = \varkappa_2^* = \varkappa$ whence it follows that

$$e_1 = e_1^* \quad \text{and} \quad e_2 = e_2^* \tag{162.18}$$

The absolute values of the momenta of each particle before and after the impact are equal to each other; this fact is the major simplification in the study of the collision in the centre-of-mass reference frame. It is only the directions of the momenta of the particles that can change during the collision (see. Fig. 440); the angle ψ can take on an arbitrary value.

Since the centre-of-mass reference frame moves with the velocity v in the direction of K relative to the laboratory reference frame, particle *2* which is at rest in the latter system before the impact has a momentum after the impact equal to

$$\varkappa_2 = -\gamma m_2 v$$

in the former reference frame where, as usual, $\gamma = (1 - v^2)^{-1/2}$. It follows that, according to (162.17), we have

$$\varkappa = \gamma m_2 v \tag{162.19}$$

and the energy of particle *2* in the centre-of-mass reference frame is

$$e_2 = \gamma m_2 = e_2^* \tag{162.20}$$

Using the transformation laws for the momentum and for the energy we can find the magnitude of the energy of particle *1* in the centre-of-mass reference frame but we do not need it for our further aims.

We now pass from the centre-of-mass reference frame to the laboratory reference frame and find the momentum K_2 of the second particle in the latter frame.

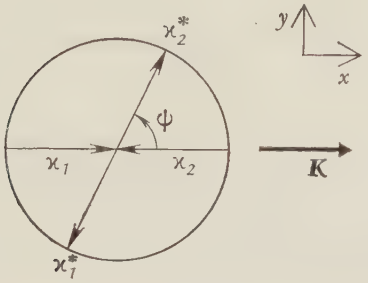

FIG. 440

We denote the projection of the vector K_2 on the direction of K (coinciding with the direction of v) by x and the projection of K_2 on the direction perpendicular to v by y. Using the notation in Fig. 440 and taking into account the transformation formula for the momentum from the moving centre-of-mass reference frame to the laboratory reference frame we can write

$$x = \gamma (\varkappa \cos \psi + ve_2) = \gamma \varkappa (1 + \cos \psi) \tag{162.21}$$

and

$$y = \varkappa \sin \psi \tag{162.22}$$

(see (162.19) and (162.20)). On eliminating ψ from these equations we arrive at the equation

$$\left(\frac{x}{\gamma \varkappa} - 1\right)^2 + \left(\frac{y}{\varkappa}\right)^2 = 1 \tag{162.23}$$

which describes the same ellipse having equation (162.9) which passes through the origin and is traced by the end of the vector representing the momentum of the second particle after the impact as this vector takes on all the possible values. The semi-major and semi-minor axes

$$a = \gamma \varkappa \quad \text{and} \quad b = \varkappa$$

of the ellipse are obviously equal to the corresponding values (162.10) and (162.11) because we have (162.15) and (162.19).

We see that the centre-of-mass reference frame is more convenient for the investigation of the collision phenomenon.

When two particles undergo an inelastic collision and the rest masses of the particles change after the impact the momenta after the impact in the centre-of-mass reference frame are of equal magnitude and opposite directions but they do not remain invariable after the impact. In this case the calculations become more intricate but it is also preferable to begin with the centre-of-mass reference frame.

The laws of impact of particles established in the Newtonian (prerelativistic) mechanics can be obtained from the relativistic relations by putting $u \ll c$ or, in other words, by neglecting K in comparison with m_1. It is readily seen that this condition is written in the SI system of units in the form $m_1 u_1 \ll m_1 c$ where u_1 is the velocity of particle *1* before the impact. From (162.1) and (162.3) it follows that

$$E = m_2 + \sqrt{K^2 + m_1^2}$$

whence, in the approximation we have assumed,

$$E = m_1 + m_2$$

Therefore, we neglect K^2 in comparison with E^2 in formulas (162.10) and (162.11) to obtain

$$a = \frac{m_2 E}{E^2 - K^2} K \approx \frac{m_2}{E} K = \frac{m_2}{m_1 + m_2} K$$

and

$$b = \frac{m_2}{\sqrt{E^2 - K^2}} K \approx \frac{m_2}{E} K = \frac{m_2}{m_1 + m_2} K$$

We see that in the limiting case of the Newtonian mechanics the ellipse degenerates into a circle passing through the origin of the vector $\boldsymbol{K}$ (this agrees with what was established in Sec. 34).

The results of the theory of relativity (of Einstein's mechanics) have not only turned out to be in the complete agreement with the laboratory experiments in different branches of physics (in the first place, in atomic and nuclear physics) but also have become the foundation of many remarkable achievements of physics in recent decades. These new results of physics in their turn have given rise to the development of completely new divisions of engineering. For instance, on the basis of the achievements of modern physics powerful nuclear power-plants and nuclear propulsion plants using nuclear fuel have been constructed and so on. Therefore, we can say that the validity of the laws of Einstein's mechanics and of the laws of the Newtonian mechanics (which is a limiting case of the former) is confirmed by the human technical experience.

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